Your exam should consist of 5 problems. Check that you have a complete exam.

- You are allowed to use the list of results passed out with this exam. You are not allowed to use any other sources.
- In your proofs, you may use any item on the list of results, including basic algebra that follows from the ordered field axioms for the real numbers. All other claims should be justified.
- If you need more room, use the back of the page. Indicate to the grader that you have done so. DO NOT USE SCRATCH PAPER.
- Turn your cell phone OFF and put it AWAY for the duration of the exam.

GOOD LUCK!
1. (12 points) State whether each of the following is TRUE or FALSE. (No proofs necessary.)

(a) _____If \( \{a_n\} \) converges to \( a \), then \( \{(a_n)^n\} \) must converge to \( a^n \).

(b) _____If \( \{a_n\} \) converges to \( a \), then every subsequence of \( \{a_n\} \) must also converge to \( a \).

(c) _____If \( \{a_n\} \) has a convergent subsequence, then \( \{a_n\} \) must converge.

(d) _____If \( a_n \in \mathbb{Q} \) for every \( n \in \mathbb{N} \), then \( \{a_n\} \) must be bounded.

(e) _____If \( \{a_n\} \) and \( \{b_n\} \) are both increasing, then \( \{a_n + b_n\} \) must also be increasing.

(f) _____If a set \( S \) is dense in \( \mathbb{R} \), then any subset of \( S \) is also dense in \( \mathbb{R} \).

2. (8 points)

(a) Give an example of a monotone sequence that does not converge.

(b) Give an example of a convergent sequence that is not monotone.

(c) Give an example of an increasing sequence \( \{a_n\} \), with \( 0 < a_n < 1 \) for all \( n \in \mathbb{N} \), that converges to a limit that is not in the interval \((0,1)\).

(d) State the Comparison Lemma.
3. (10 points)

(a) Complete the definition: A sequence \( \{a_n\} \) converges to \( a \) if...

(b) Prove that, if \( c < a < d \) and \( \{a_n\} \) converges to \( a \), then there is an \( N \) such that \( c < a_n < d \) for all \( n \geq N \).

(c) You proved in homework that, for all \( x, y \in \mathbb{R} \), \(|x| - |y| \leq |x - y|\). Use this fact to prove that, if \( \{a_n\} \) converges to \( a \), then \( \{|a_n|\} \) converges to \(|a|\).
4. (10 points) Let $a_n = \frac{n + 1}{2n - 1}$.

(a) Is $\{a_n\}$ monotonic? Justify your answer.

(b) Is $\{a_n\}$ bounded? Justify your answer.

(c) Is $\{a_n\}$ Cauchy? Justify your answer.
5. (10 points) Using only the Archimedean Property of $\mathbb{R}$ and basic algebra, give a direct $\epsilon-N$ verification of the following limits.

(a) \[
\lim_{n \to \infty} \left( \frac{4}{n^2} + \frac{1}{\sqrt{n}} \right) = 0.
\]

(b) \[
\lim_{n \to \infty} \left( \frac{6n^5 + n}{3n^5 + 1} \right) = 2.
\]