MATH 136 — Spring 2006
Exam I Solutions

1. \( \frac{\partial z}{\partial x} = f'(x^2 - y^2)(2x) \) and \( \frac{\partial z}{\partial y} = 1 + f'(x^2 - y^2)(-2y) \).

Thus, \( y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = 2xy f'(x^2 - y^2) + x - 2xy f'(x^2 - y^2) = x \).

2. arclength = \( \int_{-b}^{b} ||r'(t)|| \, dt = \int_{-b}^{b} 3 \, dt = 6b \Rightarrow b = \frac{70}{6} \)

3. If \( f(x, y) \) is continuous at \( (0, 0) \), then \( \lim_{(x,y) \to (0,0)} f(x, y) = f(0, 0) = 1 \). But along the path \( x = y \),

\[
\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{x \to 0} \frac{x^2 y^3}{2x^2 + y^2} = \lim_{x \to 0} \frac{x^3}{3} = 0.
\]

This means that, if \( \lim_{(x,y) \to (0,0)} f(x, y) \) exists, then the limit must be 0, which is not equal to \( f(0,0) \). So, \( f(x, y) \) is not continuous at \( (0, 0) \).

4. Since \( N \) and \( B \) are both perpendicular to \( T \), \( T \) is a normal vector to the normal plane. Moreover, \( T \) has the same direction as \( r'(t) = \langle 3t^2, 3, 4t^3 \rangle \). So, we need \( \langle 3t^2, 3, 4t^3 \rangle \) to be parallel to \( (6, 6, -8) \), the normal vector to the plane \( 6x + 6y - 8z = 1 \). This occurs only when \( t = -1 \). (Why?) So, the only point on \( r(t) \) at which the normal plane is parallel to \( 6x + 6y - 8z = 1 \) is the point \((-1, -3, 1)\).

5. We need to maximize \( f(x, y, z) = xyz^2 \), subject to the constraint \( g(x, y, z) = x + y + z - 50 = 0 \).

We use the Lagrange multiplier method.

\[
\nabla f(x, y, z) = \langle yz^2, xz^2, 2xyz \rangle \quad \text{and} \quad \nabla g(x, y, z) = \langle 1, 1, 1 \rangle.
\]

The only points that could give optimum values of \( f \) are those with the property \( \nabla f = \lambda \nabla g \) for some real number \( \lambda \). This gives \( yz^2 = \lambda \) and \( xz^2 = \lambda \), which means that \( x = y \).

We also have \( 2xyz = 2x^2z = \lambda \), which, since \( xz^2 = \lambda \), means that \( z = 2x \). That gives:

\[
x + y + z = x + x + 2x = 4x = 50. \quad \text{So,} \quad x = 12.5, \quad y = 12.5, \quad \text{and} \quad z = 25 \text{ are the only values that could yield an optimum value of } f.
\]

6. Let \( g(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z} - \sqrt{c} \). The normal vector to the plane tangent to this surface at an arbitrary point \((x_0, y_0, z_0)\) is \( \nabla g(x_0, y_0, z_0) = \langle \frac{1}{2\sqrt{x_0}}, \frac{1}{2\sqrt{y_0}}, \frac{1}{2\sqrt{z_0}} \rangle \). So, the equation of the tangent plane is:

\[
\frac{1}{2\sqrt{x_0}}(x - x_0) + \frac{1}{2\sqrt{y_0}}(y - y_0) + \frac{1}{2\sqrt{z_0}}(z - z_0) = 0.
\]

To get the \( x \)-intercept, we set \( y \) and \( z \) equal to 0 and solve for \( x \): \( x = x_0 + \sqrt{x_0y_0} + \sqrt{x_0z_0} \).

Similarly, the \( y \)-intercept is \( y = y_0 + \sqrt{x_0y_0} + \sqrt{y_0z_0} \) and the \( z \)-intercept is \( z = z_0 + \sqrt{x_0z_0} + \sqrt{y_0z_0} \). Adding the intercepts gives:

\[
x + y + z = x_0 + y_0 + z_0 + 2\sqrt{x_0y_0} + 2\sqrt{x_0z_0} + 2\sqrt{y_0z_0} = (\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0})^2 = (\sqrt{c})^2 = c.
\]