AREAS BETWEEN CURVES (§13.3)

Example: Suppose \( MR(q) = -q^2 + 2q + 5 \) and \( MC(q) = \frac{5}{2} q \).

- \( q \) is in hundreds of items, \( MR \) and \( MC \) are in \$/item.

\[ \text{a}) \text{ Compute area under } MR \text{ from 0 to 2. What does that represent?} \]

\[
\text{area under } MR \text{ from 0 to 2} = \int_0^2 MR(q) \, dq = \int_0^2 (-q^2 + 2q + 5) \, dq
\]

\[
= \left[ -\frac{1}{3}q^3 + q^2 + 5q \right]_0^2
\]

\[
= -\frac{8}{3} + 4 + 10 - 0
\]

\[
= \frac{-8 + 12 + 30}{3} = \frac{34}{3}
\]

\[
= 11.33 \text{ hundred dollars}
\]

**TR(2) is $1133.**

\[ \text{b}) \text{ Compute area under } MC \text{ from 0 to 2. What does it represent?} \]

\[
\text{area under } MC \text{ from 0 to 2} = \int_0^2 MC(q) \, dq = \int_0^2 \frac{5}{2} q \, dq = \left[ \frac{5}{4} q^2 \right]_0^2
\]

\[
= \frac{5}{4} \cdot 4 - \frac{5}{4} \cdot 0 = 5 \text{ hundred }$

**VC(2) is $500.**
Suppose FC = 1 hundred dollars. Compute maximum profit.

Profit is maximized when MR = MC — here that occurs at q = 2.

Max profit is:

\[
P(2) = TR(2) - TC(2) \quad TC(2) = VC(2) + FC
\]

\[
= TR(2) - (VC(2) + FC)
\]

\[
= TR(2) - VC(2) - FC
\]

\[
= (\text{area between } MR \text{ and } MC \text{ from 0 to } 2) - FC
\]

\[
= (11.33 - 5) - 1
\]

\[
= 6.33 - 1
\]

\[
= 5.33 \text{ hundred dollars.}
\]

Max profit is $533.

In general, if MR > MC from 0 to q, then:

\[
\text{(area between } MR \text{ and } MC \text{ from 0 to } q) = P(q) + FC
\]
Another: Red Car and Purple Car next to each other at \( t = 0 \).

Instantaneous speeds:

\[
\begin{align*}
    \mathbf{r}(t) &= -\frac{1}{2} t^2 + 4t \\
    \mathbf{p}(t) &= t^{3/2}
\end{align*}
\]

area under \( \mathbf{r} \) from \( 0 \) to \( t \) \(
\text{dist traveled by red} = R(t)
\)

area under \( \mathbf{p} \) from \( 0 \) to \( t \) \(
\text{dist traveled by purple} = P(t)
\)

area between \( \mathbf{r} \) and \( \mathbf{p} \) from \( 0 \) to \( t \) \(
\text{distance between cars at time} t = R(t) - P(t)
\)

**Note**: Cars are farthest apart when they have the same speed: \( t = 4 \).

area between \( \mathbf{r} \) and \( \mathbf{p} \) from \( 0 \) to \( 4 \) \(
\text{largest dist of purple} = \int_0^4 \mathbf{r}(t) \, dt - \int_0^4 \mathbf{p}(t) \, dt
\)

\[
= \int_0^4 \mathbf{r}(t) - \mathbf{p}(t) \, dt \quad \text{SHORT CUT}
\]

\[
= \int_0^4 -\frac{1}{2} t^2 + 4t - t^{3/2} \, dt
\]

\[
= \left[ -\frac{1}{6} t^3 + 2t^2 - \frac{2}{5} t^{5/2} \right]_0^4
\]

\[
= -\frac{4^3}{6} + 2 \cdot 4^2 - \frac{2}{5} \cdot 4^{5/2} - 0 \approx 8.53 \text{ yds}
\]
SHORTCUT: \[ \int_a^b f(x) \, dx - \int_a^b g(x) \, dx = \text{area between } f(x) \text{ and } g(x) \text{ from } a \text{ to } b \]

right

\[ \int_a^b f(x) - g(x) \, dx \]

left

\[ \int_a^b \]

\[ \int_a^b \]

\[ f(x) \]

\[ g(x) \]

\[ a \]

\[ b \]
Example: Compute the area of the shaded region.

\[ y = x^2 - 8x + 24 \]

\[ y = -x^2 + 8x \]

\[ \text{top - bottom} = (-x^2 + 8x) - (x^2 - 8x + 24) \]

\[
\begin{array}{|c|}
\hline
2 & 6 \\
\hline
\end{array}
\]

\[
\int_{2}^{6} -2x^2 + 16x - 24 \quad dx
\]

to find the interval, find where the curves intersect
\[ x^2 - 8x + 24 = -x^2 + 8x \]
\[ 2x^2 - 16x + 24 = 0 \]
\[ x = 2 \quad \text{and} \quad x = 6 \]

Check: \[ \text{area} = \int_{2}^{6} -2x^2 + 16x - 24 \quad dx = 21.33 \]