Maxes and Mins (§10.3)

Recall: A function \( f(x) \) has a critical value at \( x=a \) if \( f'(a) = 0 \); i.e., \( f(x) \) has a critical value at \( x=a \) if the graph of \( f \) has a horizontal tangent.

- The graph of \( f(x) \) is concave up at \( x=a \) if \( f''(a) > 0 \); \( f(x) \) is concave down at \( x=a \) if \( f''(a) < 0 \).

![Graph showing concavity and inflection points]

If the graph of \( f \) changes concavity at \( x=a \), then \( f \) has a point of inflection at \( x=a \).

If the graph of \( f \) changes concavity at \( x=a \) AND \( f \) has a horizontal tangent at \( x=a \), then \( f \) has a horizontal point of inflection at \( x=a \).
Second Derivative Test:

- If $f'(a) = 0$ and $f''(a) > 0$, then $f$ has a local min at $x = a$.
- If $f'(a) = 0$ and $f''(a) < 0$, then $f$ has a local max at $x = a$.
- If $f'(a) = 0$ and $f''(a) = 0$, then THE TEST FAILS. Anything could happen: you could have a local max, a local min, or a h.p.o.i. at $x = a$. 

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$h.p.o.i.$
Example: \( f(x) = x^3 - \frac{9}{2} x^2 - 12x + 10. \)

(a) Find all critical values and use SOT to determine whether each gives a local max or a local min.

\[
 f'(x) = 3x^2 - 9x - 12 = 0
\]

\[
 x = -1 \text{ and } x = 4 \leftarrow \text{two critical values}
\]

\[
 f''(x) = 6x - 9
\]

\[
 f''(-1) < 0 \leftarrow \text{local max at } x = -1
\]

\[
 f''(4) > 0 \leftarrow \text{local min at } x = 4
\]

(b) Compute the largest and smallest values of \( f(x) \) on the interval from \( x = -2 \) to \( x = 10. \) SKETCH THE GRAPH!

\[
 f(-2) = 8 \quad \text{global max at } (10, 440)
\]

\[
 f(-1) = 16.5 \quad \text{local max at } (-2,5)
\]

\[
 f(0) = 10 \quad \text{point of inflection at } \left(\frac{3}{2}, f\left(\frac{3}{2}\right)\right)
\]

\[
 f(4) = -46 \quad \text{local min at } (4, -46) \text{ global min}
\]
(c) Find the point of inflection.

where $f$ changes concavity
where $f''$ changes sign

$$f''(x) = 6x - 9$$

Set $f'' = 0$

$$6x - 9 = 0$$
$$6x = 9$$
$$x = \frac{9}{6} = \frac{3}{2}$$
2. Daily sales volume of a product after 't' days on the market:

\[ S(t) = \frac{3}{t+3} - \frac{18}{(t+3)^2} + 1 \]

(a) Find time at which daily sales volume is maximized.

\[ S(t) = 3(t+3)^{-1} - 18(t+3)^{-2} + 1 \]

\[ S'(t) = -3(t+3)^{-2} \cdot (1) + 36(t+3)^{-3} \cdot (1) \]

\[ = -\frac{3}{(t+3)^2} + \frac{36}{(t+3)^3} = 0 \]

Solve for 't':

\[ \frac{36}{(t+3)^3} \cdot \left[ \frac{36}{(t+3)^3} \right] = \left[ \frac{3}{(t+3)^2} \right] \]

\[ 36 = 3(t+3) \]

\[ 3t = 27 \]

\[ t = 9 \]

Critical value of \( S(t) \):

Use SDT to show that this gives a maximum.

\[ S''(t) = 6(t+3)^{-3} - 108(t+3)^{-4} \]

\[ = \frac{6}{(t+3)^3} - \frac{108}{(t+3)^4} \]

\[ S''(9) = \frac{6}{12^3} - \frac{108}{12^4} \approx -0.002 < 0 \]

\( S \) is concave down at \( t = 9 \)
\[ S'(t) = -3(t+3)^{-2} + 36(t+3)^{-3} \]
\[ S''(t) = 6(t+3)^{-3} - 108(t+3)^{-4} \]
\[ \frac{6}{(t+3)^3} - \frac{108}{(t+3)^4} \]
\[ S''(9) = \frac{6}{12^3} - \frac{108}{12^4} \approx -0.002 < 0 \]

Since this is the only critical value of \( S(t) \), this must be the place where \( S(t) \) is absolute biggest.

\( t=9 \) is where sales are maximized.
(b) Find time at which the rate of change of sales is minimized.

\[
\text{rate of change of sales} = S'(t) = R(t) \quad \text{minimize} \quad R(t)
\]

We need to know where \( R'(t) = 0 \)

\[
R'(t) = 6(t+3)^{-3} - 108(t+3)^{-4}
\]

\[
= \frac{6}{(t+3)^3} - \frac{108}{(t+3)^4} = 0
\]

Solve for \( t \):

\[
t = 1.5 \quad \text{show this gives the min}
\]