CONCAVITY (§10.2)

Consider the graph of a function \( f(x) \):

We know that \( f \) is increasing when \( f' \) is positive and \( f \) is decreasing when \( f' \) is negative. So, \( f' \) is increasing when \( f'' \) is positive and \( f' \) is decreasing when \( f'' \) is negative.

\[
\begin{align*}
\text{for } x = a & : f'' \text{ decreases} \\
\text{for } x = b & : f'' \text{ is negative} \\
\text{for } x = c & : f'' \text{ increases and } f' \text{ is positive}
\end{align*}
\]

When \( f'' \) is negative, the graph of \( f \) is concave down. When \( f'' \) is positive, the graph of \( f \) is concave up.

A point on the graph where \( f \) changes concavity is a point of inflection.
Examples: ① Let \( f(x) = \frac{1}{2} x^4 - 3x^2 + 5x + 1 \).

Find all intervals on which \( f(x) \) is concave up or concave down and find all points of inflection.

\[ f'(x) = 2x^3 - 6x + 5 \]
\[ f''(x) = 6x^2 - 6 \]

Set \( f''(x) = 0 \) and solve for \( x \):

\[ 6x^2 - 6 = 0 \]
\[ 6x^2 = 6 \]
\[ x^2 = 1 \]
\[ x = -1 \text{ or } x = 1 \]

\[ f'' \]
\[ + + + + \]
\[ - - - - \]
\[ + + + + \]

\[ f \]
\[ \uparrow \]
\[ \text{p.o.i.} \]
\[ \text{p.o.i.} \]

\( f \) is concave up for \( x < -1 \) and \( x > 1 \)
\( f \) is concave down for \( -1 < x < 1 \)
\( f \) has points of inflection at \( x = -1 \) and \( x = 1 \)
2. Let $g(x) = x^3$. Sketch its graph, noting any relative optima or points of inflection.

$g'(x) = 3x^2 = 0$ when $x = 0 \rightarrow$ critical value

$g''(x) = 6x = 0$ when $x = 0 \rightarrow$ possible point of inflection

No relative optima.
3. Suppose $TC(q) = 5000q^2 + 125,000$ dollars for producing $q$ Things.

$$AC(q) = \frac{TC(q)}{q} = \frac{5000q^2 + 125,000}{q}$$

$$= \frac{5000q + 125,000}{q}$$

Find the smallest value of $AC$.

$$AC(q) = 5000q + 125,000q^{-1}$$

$$AC'(q) = 5000 - 125,000q^{-2}$$

$$= 5000 - \frac{125,000}{q^2} = 0$$

Solve for $q$.

$$q^2 \cdot 5000 = 125,000 \cdot q^2$$

$$\frac{5000q^2}{5000} = \frac{125,000}{5000}$$

$$q^2 = 25$$

$$q = 5 \text{ or } -5$$

The only quantity where $AC$ has a horizontal tangent. Let's look at concavity to see if there's a min here.
\[ AC'(q) = 5000 - 125,000q^{-2} \]
\[ AC''(q) = 250,000q^{-3} \]

\[ = \frac{250,000}{q^3} \]

Since \( q \) is a quantity, \( q \) is always positive, which means \( \frac{250,000}{q^3} \) is positive. This means \( AC \) is concave up.

\[ AC(q) = 5000q + \frac{125,000}{q} \]
\[ AC(5) = 50,000 \] $\text{\$ Things}$