**The Derivative (89.4)**

Examples: Let \( f(x) = x^2 \).

Let's find \( f'(m) \), which is the slope of the tangent line to \( f(x) \) at \( x = m \).

For now, we use the fact that

\[
f'(m) \approx \frac{f(m+h)-f(m)}{h} \quad \text{when } h \text{ is tiny.}
\]

\[
f(m+h) = (m+h)^2 = m^2 + 2mh + h^2
\]

\[
f(m) = m^2
\]

\[
\frac{f(m+h)-f(m)}{h} = \frac{2mh + h^2}{h} = 2m + h
\]

As \( h \to 0 \), \( 2m + h \to 2m \).

We conclude \( f'(m) = 2m \).

So, \( f'(0) = 2 \cdot 0 = 0 \).

\( f'(1) = 2 \cdot 1 = 2 \)

\( f'(20) = 2 \cdot 20 = 40 \)

\[
\boxed{f'(x) = 2x}
\]

This is a new function derived from the original function \( f(x) = x^2 \).

We call \( f'(x) \) the derivative of \( f(x) \).
Let \( f(x) = x^3 \). Find \( f'(m) \) and \( f'(x) \).

\[
f(m+h) = (m+h)^3 = m^3 + 3m^2h + 3mh^2 + h^3
\]

\[
f(m) = m^3
\]

\[
\frac{f(m+h) - f(m)}{h} = \frac{3m^2h + 3mh^2 + h^3}{h}
\]

\[
= 3m^2 + 3mh + h^2
\]

As \( h \to 0 \), \( 3m^2 + 3mh + h^2 \to 3m^2 \).

We conclude \( f'(m) = 3m^2 \).

and the derivative of \( f(x) \)

is therefore \( f'(x) = 3x^2 \).

We've shown:

\[
\begin{align*}
\text{x}^2 & \quad \text{x}^3 \\
D \downarrow & \quad D \downarrow \\
2x^1 & \quad 3x^2
\end{align*}
\]

We could do:

\[
\begin{align*}
\text{x}^4 & \quad \text{x}^5 \\
D \downarrow & \quad D \downarrow \\
4x^3 & \quad 5x^4
\end{align*}
\]
THE POWER RULE: If \( f(x) = x^n \), then \( f'(x) = nx^{n-1} \).

e.g., \( g(x) = x^{10} \)  
\( g'(x) = 10x^9 \)  

don't do this: \( g(x) = x^{10} = 10x^9 \)

\[
\begin{align*}
\cdot h(x) &= x = x^1 \\
h'(x) &= 1x^0 = 1
\end{align*}
\]

\[
\begin{align*}
\cdot k(x) &= 1 = x^0 \\
k'(x) &= 0x^{-1} = 0
\end{align*}
\]

\[
\begin{align*}
\cdot l(x) &= \sqrt{x^7} = x^{\frac{1}{2}} \\
l'(x) &= \frac{1}{2}x^{\frac{1}{2} - 1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}}
\end{align*}
\]

\[
\begin{align*}
\cdot m(x) &= \frac{1}{x} = x^{-1} \\
m'(x) &= -1x^{-2} = \frac{-1}{x^2}
\end{align*}
\]

IMPORTANT: The derivative of \( k \) is 1.

IMPORTANT: The derivative of \( 1 \) is 0.
The Sum Rule: \((f(x) + g(x))' = f'(x) + g'(x)\)

E.g., \(y = x^{10} + x^9\)
\[y' = 10x^9 + 9x^8\]

\(B(x) = \sqrt{x} + \frac{1}{\sqrt{x}} = x^{\frac{1}{2}} + x^{-\frac{1}{2}}\)
\[B'(x) = \frac{1}{2}x^{-\frac{1}{2}} + \left(-\frac{1}{2}x^{-\frac{3}{2}}\right)\]
\[= \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}\]

\(y = x^{15} + x + 1\)
\[y' = 15x^{14} + 1 + 0 = 15x^{14} + 1\]
The Coefficient Rule: \((c \cdot f(x))' = c \cdot f'(x)\)

e.g., \(y = 4x^3\)
\[ y' = 4(3x^2) = 12x^2 \]

\(G(x) = -14x^{10}\)
\[ G'(x) = -14(10x^9) = -1400x^9 \]

\(H(x) = 3x^{10} - 14x^2\)
\[ H'(x) = 3(10x^9) - 14(2x) = 30x^9 - 28x \]

\(I(x) = 3 = 3(1)\)
\[ I'(x) = 3(0) = 0 \]

\[ y = \frac{3}{2\sqrt{x}} = \frac{3}{2x^{1/2}} = \frac{3}{2}x^{-1/2} \]
\[ y' = \frac{3}{2}(-\frac{1}{2}x^{-3/2}) = -\frac{3}{4x^{3/2}} \]

**IMPORTANT**: The derivative of any constant is 0.
APPLICATION: MARGINAL ANALYSIS

Story: You produce and sell hats:

- Total Revenue: \( TR(q) = -3.75q^2 + 28.5q \)
- Total Cost: \( TC(q) = 2q^3 - 0.4q^2 + 2q + 15 \)

\( q \) in 100's of hats

\( TR \) & \( TC \) in 100's of dollars

Def: (BIG CHANGE FROM MATH III)
Marginal revenue is the derivative of total revenue.
Marginal cost is the derivative of total cost.

\[
MR(q) = TR'(q) = -3.75(2q) + 28.5(1) \\
= -7.5q + 28.5 \quad \text{if } q \text{ is in 100's of hats}
\]

\[
MC(q) = TC'(q) = 6q^2 - 0.8q + 3 \quad \text{MR \& MC in \$/hat}
\]

\[
\begin{array}{c}
\text{Slope: } \frac{100's \text{ of } \$}{100's \text{ of hats}} \\
\text{100's of hats} \\
\text{100's of } \$
\end{array}
\]

Graph showing TR and TC with slope indicated.