UNCERTAINTY PRINCIPLES FOR THREE-DIMENSIONAL INVERSE SOURCE PROBLEMS*

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Abstract. Novel uncertainty principles for far field patterns of time-harmonic acoustic or electromagnetic waves radiated by collections of well-separated localized sources in two-dimensional free space have recently been established in [R. Griesmaier and J. Sylvester, *SIAM J. Appl. Math.*, accepted for publication]. These uncertainty principles have been utilized to develop reconstruction algorithms and stability estimates for the restoration of missing data segments and for the recovery of the far fields radiated by each component of a collection of sources from incomplete observations of the far field radiated by the whole ensemble.

In this paper, we present generalizations of these uncertainty principles for a relevant three-dimensional setting. We consider extensions of the reconstruction schemes for data completion and far field splitting, including their stability analysis, and we discuss the sharpness of the corresponding stability estimates. We also comment on the implementation of the reconstruction algorithms and include a numerical example.

Key words. Inverse source problem, Helmholtz equation, uncertainty principles, far field splitting, data completion, stable recovery

AMS subject classifications. 35R30, 65N21

1. Introduction.

If we model the propagation of time-harmonic acoustic or electromagnetic waves by the Helmholtz equation, then the far field of such a wave radiated by a source f coincides up to a constant with the Fourier transform of the source restricted to a sphere. It is not possible to uniquely recover a function f from its restricted Fourier transform, but it is possible to recover information about its support [9, 13, 14, 18]. Unique continuation of solutions to the Helmholtz equation implies that no two sources with disjoint supports can radiate the same far fields, so the subspaces of far fields radiated from disjoint compact sets intersect only at the origin, and therefore a sum of sources with disjoint supports has a unique splitting into a sum of far fields, each of which is radiated by an individual source. Similarly, because the Fourier transform of a compactly supported function is analytic, observations of the far field on any open subset of the sphere uniquely determine the far field on the entire sphere. This implies that far field splitting and data completion are theoretically possible, but, without further assumptions both are severely ill-posed inverse problems.

We recently investigated both data completion and source splitting in two dimensions [8]. Based on the singular value decomposition of the operator that maps sources supported in a ball to the far fields they radiate, we developed a *regularized Picard criterion*, which characterized the subspaces of nonevanescent far fields radiated by L^2 sources supported in a ball. These are the far fields that can be radiated by a limited power source, and at the same time have enough power to be detected by a sensor with limited sensitivity. We combined the regularized Picard criterion with an *uncertainty principle for the far field translation operator* to develop reconstruction algorithms and stability results for far field splitting. The *far field translation operator*

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* Last modified February 2, 2017.

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maps the restricted Fourier transform of a compactly supported source $f(x)$ to the restricted Fourier transform of its translate $f(x+c)$, and our *uncertainty principle* is a sharp upper bound on the cosine of the angle between two different subspaces of nonevanescent far fields, each of which is radiated from a different ball. The bound on the cosine implies a bound on the cosecant, and the cosecant is exactly the condition number of the linear splitting operator.

We also combined the regularized Picard criterion with another uncertainty principle for the operator that maps far fields to their Fourier components, and obtained reconstruction algorithms and stability estimates for recovering missing data segments of a far field radiated by a localized source. Both results can be combined to simultaneously complete far fields and split them into the components radiated by well-separated localized sources. In both cases, the bounds depend simply on wavelength, diameter, and (in the case of splitting) distance between the sources.

In this work we establish the analogous uncertainty principles, reconstruction schemes and stability results in three dimensions. The basic structure is similar to the two-dimensional case, with the Fourier decomposition of functions on the circle replaced by the decomposition of functions on the sphere into spherical harmonics. The new ingredient here is the analysis of the three dimensional far field translation operator, which is considerably more complicated than in the two dimensional case. Much of the two dimensional analysis was facilitated by the facts that the product of two exponentials (i.e. $e^{\mathrm{i}n\phi}$'s) is again an exponential, and the ratio of the L^2 and L^∞ norms of $e^{\mathrm{i}n\phi}$ is independent of n. Neither fact remains true in higher dimensions.

Once the uncertainty principles have been established, it is relatively straightforward to carry over the stability results from [8] to the three-dimensional setting. Therefore, we only state three basic applications of the uncertainty principles here, and refer the reader to [8] for proofs and for more results.

Our main results are based on estimates of condition numbers for linear operators that split a vector into two or more components. The estimates are all of the same general form. Define the cosine of the angle θ between two subspaces V_1 and V_2 as

$$\cos\theta = \sup_{v_1\in V_1,\, v_2\in V_2}\left|\frac{\langle v_1, v_2\rangle}{\|v_1\|\|v_2\|}\right| .$$

As long as $|\cos\theta| < 1$, the following result is straightforward to check (and we will carry this out explicitly in special cases in section 5).

THEOREM 1.1. *Suppose that $v = v_1 + v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$, then, for $i = 1, 2$,*

$$\|v_i\|^2 \leq \frac{1}{1-\cos^2\theta}\|v\|^2 . \tag{1.1}$$

The inequality (1.1) asserts that the splitting operator that maps v to the pair (v_1, v_2) exists and its condition number is bounded by $\csc\theta$. In fact it is exactly $\csc\theta$. For us, V_1 and V_2 will be finite dimensional subspaces of $L^2(S^2)$. In section 3, we will use the *regularized Picard criterion* to define the subspaces V_R^c of *nonevanescent far fields* radiated by sources supported in the ball of radius R centered at a point c (at wavenumber k). We will show that, in $\mathbb{R}^3$, the subspace V_R^c is the approximately $(kR)^2$ dimensional space spanned by $e^{\mathrm{i}kc\cdot\phi}$ times the spherical harmonics of degree $n \leq kR$ (alternatively, the spherical harmonics with eigenvalues $n(n+1) \leq (kR)^2$). For comparison, in $\mathbb{R}^2$, V_R^c has dimension $2kR+1$ and is spanned by $e^{\mathrm{i}kc\cdot\phi}$ times the complex exponentials $e^{\mathrm{i}n\phi}$ with $|n| \leq R$ (alternatively, the exponentials with eigenvalues $n^2 \leq (kR)^2$) [8].

It is worth noting that the dimension of the nonevanescent subspace increases with wavenumber in a way that is consistent with the Rayleigh criterion, which limits resolution to the scale of a wavelength. If we combine the fact that the far field radiated by a source can also be radiated by a superposition of single- and double-layer sources on any surface enclosing the source, and the expectation that we cannot resolve on scales smaller than a wavelength, which we denote by λ, we expect one dimension[1] for each $\lambda \times \lambda$ patch of surface area on the surface of smallest area that encloses the source. Thus the dimension of V_R^c should be proportional to the surface area in units of squared wavelengths, i.e., $(kR)^2$ in $\mathbb{R}^3$ and kR in $\mathbb{R}^2$.

In section 4, we establish estimates for the cosine of the angle between nonevanescent far fields radiated from well-separated balls. Specifically, if $|kc_2 - kc_1| > 2(kR_1 + kR_2 + \frac{3}{2})$, the cosine of the angle θ between the subspaces $V_{R_1}^{c_1}$ and $V_{R_2}^{c_2}$ satisfies

$$|\cos\theta| \lesssim \frac{(kR_1)^{\frac{3}{2}}(kR_2)^{\frac{3}{2}}}{|kc_2 - kc_1|}\,, \tag{1.2}$$

where the symbol $\lesssim$ means that there is a nonspecified constant, independent of k, R_i, and c_i, $i = 1, 2$. In $\mathbb{R}^2$, the analogous inequality is

$$|\cos\theta| \lesssim \frac{(kR_1)^{\frac{1}{2}}(kR_2)^{\frac{1}{2}}}{|kc_2 - kc_1|^{\frac{1}{2}}}\,, \tag{1.3}$$

and an example using a line source shows that the dependence on k, R_i, and c_i, $i = 1, 2$, in (1.3) is sharp. The example in section 6, which calculates the inner product of the far fields radiated by constant sources supported on translated disks in $\mathbb{R}^3$, gives a cosine estimate

$$\cos\theta \gtrsim \frac{kR_1 kR_2}{|kc_2 - kc_1|} \sin(|kc_2 - kc_1|)$$

so we don't yet know if the dependence in (1.2) is sharp, or if it is possible to replace the $\frac{3}{2}$ power in (1.2) by the first power.

Also in section 4, we show that the cosine of the angle θ between the subspaces V_R^c and $L^2(\Omega)$, the subspace of functions in $L^2(S^2)$ supported in $\Omega \subseteq S^2$, satisfies the inequality

$$|\cos\theta| \lesssim \sqrt{\frac{|\Omega|}{4\pi}(kR)^2}\,, \tag{1.4}$$

where $|\Omega|$ is the area of Ω. An example in section 6 shows that the dependence on k, R, and $|\Omega|$ is sharp. The analogy in two dimensions, with $|\Omega|$ equal to the length of Ω, is

$$|\cos\theta| \lesssim \sqrt{\frac{|\Omega|}{2\pi}kR}\,, \tag{1.5}$$

which is also sharp.

These estimates are used in section 5 to establish stability estimates for least squares algorithms for far field splitting and data completion, and in section 7 we

[1] Not two because roughly half (the Cauchy data of free solutions to the Helmholtz equation) are nonradiating.

describe a numerical implementation of the reconstruction algorithms from section 5 and give a numerical example.

Note that all the estimates in (1.2), (1.3), (1.4), and (1.5) are unitless, and all reveal that the conditioning worsens (recall that the condition number is $\csc\theta$) as the wavenumber increases. This is different from the conclusions suggested by studying the point spread function, which treats the resolution of point sources. In fact, setting $kR = 1$ in the estimates above recovers the resolution results for point sources.

We end this section by explaining why we refer to the inequalities (1.2), (1.3), (1.4), and (1.5) as uncertainty principles. Let V_T denote the L^2 functions supported in T and $V_{\widehat{W}}$ denote the L^2 functions whose transforms are supported in W. By *transforms*, we mean either the Fourier transform on the line, the Fourier series on the circle, or the N-point discrete Fourier transform[2]

THEOREM 1.2. *If there is a nonzero f that belongs to both $V_T \cap V_{\widehat{W}}$, then*

$$C \;\le\; |T||W|\,,$$

where the constant C is 2π for the Fourier transform on the line and the circle, and $C = N$ for the N-point DFT.

The contrapositive of theorem 1.2 is the following.

THEOREM 1.3. *If $|T||W| < C$, then $V_T \cap V_{\widehat{W}} = \{0\}$.*

As we did in [8], we reformulate this as follows.

THEOREM 1.4. *If $f \in V_T$ and $g \in V_{\widehat{W}}$, then*

$$|\langle f, g\rangle| \;\le\; \sqrt{\frac{|T||W|}{C}}\,\|f\|\|g\|\,. \tag{1.6}$$

Setting $f = g$ in (1.6) recovers theorems 1.3 and 1.2. Where theorem 1.3 guarantees the existence of a splitting operator, theorem 1.4 explicitly estimates its norm, which is $\csc\theta$. In the case of the Fourier transform, theorem 1.2 as stated is vacuous, because a compactly supported function cannot have a compactly supported Fourier transform[3]. Theorem 1.4 does not suffer this inconvenience. Finally, theorem 1.4 admits a straightforward generalization.

THEOREM 1.5. *Suppose that $A : L^2 \to L^2$ and $A^{-1} : L^1 \to L^\infty$, and that $f \in V_T$ and $Ag \in V_W$, then*

$$|\langle f, g\rangle| \;\le\; \sqrt{|T||W|}\,\|A\|\|A^{-1}\|\|f\|\|g\|\,.$$

In theorem 1.5 we have been intentionally vague about which L^p spaces we are using. Our main applications in section 4 will be with A equal to the far field translation operator. We will work with the sequence of spherical harmonic components of a far field, each component of which is a function in $L^2(S^2)$. In this case, our norms will be $l^p(L^2(S^2))$ norms with $p = 1, 2, \infty$, and we will in fact need weighted versions of these norms, with the weights related to the dimension of the spaces of spherical harmonics of each degree.

2. Far fields radiated by compactly supported sources. Let $g \in L^2(\mathbb{R}^3)$ be a compactly supported function representing an acoustic or electromagnetic *source.*

[2]This example in [4] motivated this work, and the corollaries in section 5 are analogous to those in [4].

[3]Theorem 1.2 can easily be modified to a useful statement about functions *essentially supported* in certain subsets [4].

Using the Helmholtz equation as our model for the propagation of time-harmonic waves, the wave radiated by g at *wavenumber* $k > 0$ satisfies

$$-\Delta v - k^2 v \;=\; k^2 g \qquad \text{in } \mathbb{R}^3\,, \tag{2.1a}$$

and the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\Bigl(\frac{\partial v}{\partial r} - \mathrm{i}kv\Bigr) \;=\; 0\,, \qquad r = |x|\,. \tag{2.1b}$$

After rescaling $u(x) := v(kx)$ and $f(x) := g(kx)$, i.e., measuring distances in wavelengths[4], the system (2.1) is equivalent to

$$-\Delta u - u \;=\; f \qquad \text{in } \mathbb{R}^3 \qquad \text{and} \qquad \lim_{r\to\infty} r\Bigl(\frac{\partial u}{\partial r} - \mathrm{i}u\Bigr) \;=\; 0\,. \tag{2.2}$$

Thus we may assume in our calculations below that $k = 1$, and easily restore the dependence on wavelength by simply rescaling the spatial variable when we are done.

For $k = 1$ the fundamental solution of the three-dimensional Helmholtz equation is

$$\Phi(x) \;:=\; \frac{1}{4\pi}\frac{e^{\mathrm{i}|x|}}{|x|}\,, \qquad x \in \mathbb{R}^3 \setminus \{0\}\,,$$

and accordingly the unique solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ to (2.2) can be written as a volume potential

$$u(x) \;=\; \int_{\mathbb{R}^3} \Phi(x-y) f(y)\,\mathrm{d}y\,, \qquad x \in \mathbb{R}^3\,.$$

For large arguments,

$$u(x) \;=\; \frac{e^{\mathrm{i}r}}{r}\alpha(\phi_x) + O(r^{-2}) \qquad \text{as } r \to \infty\,,$$

where $x = r\phi_x$ with $\phi_x \in S^2$, and $\alpha \in L^2(S^2)$ is given by

$$\alpha(\theta) \;=\; \int_{\mathbb{R}^3} e^{-\mathrm{i}\theta\cdot y} f(y)\,\mathrm{d}y\,, \qquad \theta \in S^2\,. \tag{2.3}$$

The function α is known as the *far field* radiated by the source f. In particular, equation (2.3) shows that the *far field operator* $\mathcal{F}$, which maps sources to far fields, is a *restricted Fourier transform*,

$$\mathcal{F} :\, L^2_0(\mathbb{R}^3) \to L^2(S^2)\,, \quad \mathcal{F}f \;:=\; \widehat{f}\big|_{S^2}\,.$$

3. A regularized Picard criterion. We begin by characterizing far fields radiated by sources supported in a ball $B_R(0)$ of radius $R > 0$ centered at the origin, i.e., far fields in the range of the *restricted far field operator*

$$\mathcal{F}_{B_R(0)} :\, L^2(B_R(0)) \to L^2(S^2)\,, \quad \mathcal{F}_{B_R(0)}f \;:=\; \widehat{f}\big|_{S^2}\,.$$

We will describe the singular value decomposition of this compact operator in terms of spaces of spherical harmonics. Let $\mathbb{Y}_n(\mathbb{R}^3)$ denote the space of harmonic homogeneous

[4]One unit represents 2π wavelengths.

polynomials of degree n on $\mathbb{R}^3$, and let $\mathbb{Y}_n := \mathbb{Y}_n(\mathbb{R}^3)|_{S^2}$ denote their restrictions to the unit sphere S^2, called the space of *spherical harmonics* of degree n. Both, $\mathbb{Y}_n(\mathbb{R}^3)$ and $\mathbb{Y}_n$ have dimension $2n+1$. The subspaces $\mathbb{Y}_n$ are mutually orthogonal and span $L^2(S^2)$, i.e.,

$$L^2(S^2) \,=\, \bigoplus_{n=0}^{\infty} \mathbb{Y}_n \,. \tag{3.1}$$

The Funk-Hecke formula tells us that

$$\int_{S^2} e^{\pm \mathrm{i}\theta\cdot x} Y_n(\theta)\,\mathrm{d}s(\theta) \,=\, 4\pi(\pm\mathrm{i})^n j_n(r) Y_n(\phi_x)\,, \qquad \phi_x \in S^2\,,\; x = r\phi_x \in \mathbb{R}^3\,, \tag{3.2}$$

for any $Y_n \in \mathbb{Y}_n$ (see, e.g., [2, p. 32]). Here, j_n denotes the spherical Bessel function of degree n. If $\{Y_n^m \mid -n \leq m \leq n\}$ is any orthonormal basis of $\mathbb{Y}_n$, then

$$\mathcal{F}_{B_R(0)} f \,=\, \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \sigma_n^m \langle f, v_n^m \rangle_{L^2(S^2)} u_n^m \qquad \text{for any } f \in L^2(S^2) \tag{3.3}$$

i.e $(\sigma_n^m; u_n^m, v_n^m)$, $-n \leq m \leq n$, $n \in \mathbb{N}$, is a singular system for $\mathcal{F}_{B_R(0)}$, where σ_n^m is given by the squared singular values

$$(\sigma_n^m)^2 \,=\, 4\pi s_n^2(R) \,:=\, 4\pi \int_{B_R(0)} j_n^2(|y|)\,\mathrm{d}y\,. \tag{3.4a}$$

The left singular vectors are

$$u_n^m(\theta) \,=\, Y_n^m(\theta)\,, \qquad \theta \in S^2\,, \tag{3.4b}$$

and the right singular vectors are

$$v_n^m(y) \,=\, \frac{\sqrt{4\pi}\mathrm{i}^n j_n(|y|)}{s_n(R)} Y_n^m(\phi_y)\,, \qquad \phi_y \in S^2\,,\; y = |y|\phi_y \in B_R(0)\,. \tag{3.4c}$$

According to (3.1), any $\alpha \in L^2(S^2)$ can be uniquely represented as a sum of spherical harmonics,

$$\alpha \,=\, \sum_{n=0}^{\infty} \alpha_n\,, \qquad \alpha_n \in \mathbb{Y}_n\,, \tag{3.5}$$

which is called the *Fourier-Laplace series* of α. Here, the *n-spherical harmonic component* $\alpha_n \in \mathbb{Y}_n$ of α is given by

$$\alpha_n(\theta) \,=\, (\mathcal{P}_n\alpha)(\theta) \,:=\, \frac{2n+1}{4\pi} \int_{S^2} P_n(\theta\cdot\omega)\alpha(\omega)\,\mathrm{d}s(\omega)\,, \qquad \theta \in S^2\,, \tag{3.6}$$

where $\mathcal{P}_n : L^2(S^2) \to L^2(S^2)$ denotes the orthogonal projection onto $\mathbb{Y}_n$, and

$$P_n(t) \,:=\, \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n}(t^2-1)^n\,, \qquad t \in [-1,1]\,,$$

is the Legendre polynomial of degree n. As a consequence of (3.5), we have the Parseval identity

$$\|\alpha\|_{L^2(S^2)}^2 \,=\, \sum_{n=0}^{\infty} \|\alpha_n\|_{L^2(S^2)}^2\,, \qquad \alpha \in L^2(S^2)\,. \tag{3.7}$$

In particular, the sequence $\{\alpha_n\}$ of spherical harmonic components of α satisfies

$$\{\alpha_n\} \in l^2(L^2(S^2)) := \Big\{\{\beta_n\} \;\Big|\; \beta_n \in \mathbb{Y}_n \text{ for all } n \in \mathbb{N} \text{ and } \sum_{n=0}^{\infty} \|\beta_n\|^2_{L^2(S^2)} < \infty\Big\}\,.$$

It will sometimes be convenient to think of the projection kernel in (3.6) as a function of θ for each fixed $\omega \in S^2$. We will use the notation $P_n^\omega(\theta) := P_n(\omega \cdot \theta)$ to emphasize this point of view. For later use, we note that $P_n^\omega(\theta)$ is itself a spherical harmonic of degree n with

$$\|P_n^\omega\|_{L^2(S^2)} \;=\; \sqrt{\frac{4\pi}{2n+1}} \qquad \text{and} \qquad \|P_n^\omega\|_{L^\infty(S^2)} \;=\; 1\,, \tag{3.8}$$

independent of $\omega \in S^2$ (cf. [1, (2.39)–(2.40)]).

The rescaled squared singular values $\{s_n^2(R)\}$ of the restricted Fourier transform $\mathcal{F}_{B_R(0)}$ have a number of interesting properties with immediate consequences for the inverse source problem. Proofs of the results in the remainder of this section, which are (nontrivial) extensions of the corresponding results in [8], can be found in appendix A. The rescaled squared singular values satisfy

$$\sum_{n=0}^{\infty} (2n+1) s_n^2(R) \;=\; \frac{4\pi}{3} R^3\,,$$

and $s_n^2(R)$ decays rapidly as a function of n as soon as $n \geq R$,

$$s_n^2(R) \;\leq\; bR\Big(n+\frac{1}{2}\Big)^{\frac{2}{3}} \Big(\frac{R^2}{(n+1)^2} e^{1-\frac{R^2}{(n+1)^2}}\Big)^{n+1} \qquad \text{if } n \geq R\,,$$

where the constant $b \approx 4.791$ is independent of n and R. Moreover, the odd and even squared singular values, $s_n^2(R)$, are decreasing as functions of n, and asymptotically, as $R \to \infty$

$$s_n^2(R) \;\sim\; \begin{cases} 2\pi\sqrt{R^2 - \big(n+\frac{1}{2}\big)^2}\,, & n+\frac{1}{2} \leq R\,,\\ 0\,, & n+\frac{1}{2} > R\,, \end{cases}$$

(see also corollary A.10). This is illustrated in figure 3.1, where we include plots of $s_n^2(R)$ (solid line) together with plots of the asymptote $2\pi\sqrt{R^2 - (n+\frac{1}{2})^2}$ (dashed line) for $R = 10$ (left) and $R = 100$ (right).

The singular value decomposition (3.4) shows that the source $f_\alpha^* \in L^2(B_R(0))$ with smallest L^2-norm that is supported in $B_R(0)$ and radiates a given far field $\alpha \in L^2(S^2)$ has L^2-norm

$$\|f_\alpha^*\|^2_{L^2(B_R(0))} \;=\; \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{\|\alpha_n\|^2_{L^2(S^2)}}{s_n^2(R)}\,,$$

where $\{\alpha_n\}$ is again the sequence of spherical harmonic components of α. In the following we refer to f_α^* as the *minimal power source*, and to $\|\alpha\|^2_{L^2(S^2)}$ as the *radiated power* of the far field α. Since any physical source has *limited power*, which we denote by $P > 0$, and any receiver has a *power threshold*, which we denote by $p > 0$ (i.e., the receiver cannot detect a far field that has power less than p) not every source/farfield

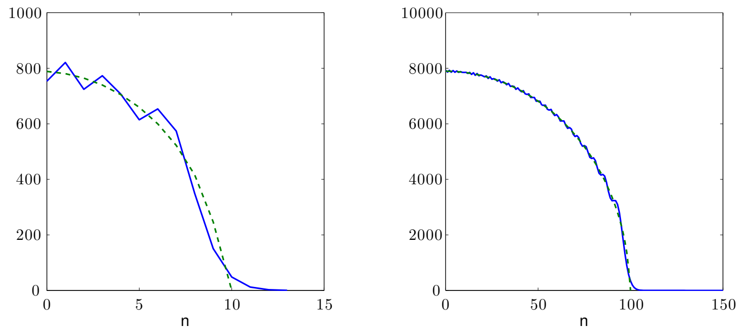

FIG. 3.1. *Rescaled squared singular values* $s_n^2(R)$ *(solid line) and asymptote* $2\pi\sqrt{R^2-(n+\frac{1}{2})^2}$ *(dashed line) for* $R=10$ *(left) and* $R=100$ *(right).*

combination is equally relevant in practice. Because the odd and even squared rescaled singular values, $s_n^2(R)$, are decreasing as functions of n, we may define

$$N(R,P,p) \,:=\, \sup_{4\pi s_n^2(R)\geq \frac{p}{P}} n\,. \tag{3.9}$$

So, if $\alpha \in L^2(S^2)$ is a far field with spherical harmonic components $\{\alpha_n\}$ radiated by a limited power source supported in $B_R(0)$ with $\|f_\alpha^*\|^2_{L^2(B_R(0))} \leq P$, then, for $N = N(R,P,p)$,

$$P \,\geq\, \frac{1}{4\pi}\sum_{n>N}\frac{\|\alpha_n\|^2_{L^2(S^2)}}{s_n^2(R)} \,\geq\, \frac{1}{4\pi}\frac{1}{s_{N+1}^2(R)}\sum_{n>N}\|\alpha_n\|^2_{L^2(S^2)} \,>\, \frac{P}{p}\sum_{n>N}\|\alpha_n\|^2_{L^2(S^2)}\,.$$

Therefore, $\sum_{n\geq N}\|\alpha_n\|^2_{L^2(S^2)} < p$ is below the power threshold, and accordingly the subspace of detectable far fields, that can be radiated by a power limited source supported in $B_R(0)$ is

$$\mathbb{Y}_{\leq N} \,:=\, \bigoplus_{n=0}^{N}\mathbb{Y}_n \,=\, \Big\{\alpha \in L^2(S^2) \;\Big|\; \alpha = \sum_{n=0}^{N}\alpha_n\,,\ \alpha_n \in \mathbb{Y}_n\Big\}\,.$$

We refer to $\mathbb{Y}_{\leq N}$ as the subspace of *nonevanescent far fields*, and to the orthogonal projection of a far field onto this subspace as the *nonevanescent part* of the far field. Since the rescaled squared singular values $s_n^2(R)$ decrease very rapidly for $|n| > R$, the number $N(R,P,p)$ is only a little larger than R for a wide range of P and p, if R is sufficiently large. Thus $\mathbb{Y}_{\leq N}$ is the subspace V_R^c referred to on page 2 of the introduction (with $k=1$ and $c=0$).

4. Uncertainty principles for far field translation. Because the far field is a restricted Fourier transform, the formula for the Fourier transform of the translation of a function,

$$\widehat{f(\cdot+c)}(\theta) \,=\, e^{\mathrm{i}c\cdot\theta}\widehat{f}(\theta)\,, \qquad \theta \in S^2\,,\ c \in \mathbb{R}^3\,,$$

plays an important role in the inverse source problem. We use $T_c : L^2(S^2) \to L^2(S^2)$ to denote the map given by

$$(T_c\alpha)(\theta) \,:=\, e^{\mathrm{i}c\cdot\theta}\alpha(\theta)\,, \qquad \theta \in S^2\,. \tag{4.1}$$

The operator T_c is unitary with $T_c^* = T_{-c}$. Combining the Jacobi-Anger expansion

$$e^{\mathrm{i}c\cdot\theta} = \sum_{l=0}^{\infty} \mathrm{i}^l(2l+1)j_l(|c|)P_l(\phi_c\cdot\theta)\,, \qquad c = |c|\phi_c \in \mathbb{R}^3\,,\ \theta,\phi_c \in S^2\,,$$

which is an immediate consequence of (3.2), with the Fourier-Laplace series (3.5) we find that

$$(T_c\alpha)(\theta) = \sum_{n=0}^{\infty}\alpha_n(\theta)\sum_{l=0}^{\infty}\mathrm{i}^l(2l+1)j_l(|c|)P_l(\phi_c\cdot\theta)\,, \qquad \theta\in S^2\,. \tag{4.2}$$

Substituting (4.2) into (3.6) shows that the spherical harmonic components $\{\alpha_m^c\}$ of $T_c\alpha$ satisfy, for $\theta \in S^2$,

$$\alpha_m^c(\theta) = \frac{2m+1}{4\pi}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\mathrm{i}^l(2l+1)j_l(|c|)\int_{S^2}\alpha_n(\omega)P_m(\theta\cdot\omega)P_l(\phi_c\cdot\omega)\,\mathrm{d}\omega\,. \tag{4.3}$$

Employing a slight abuse of notation, we also use T_c to denote the operator from $l^2(L^2(S^2))$ to itself that is given by

$$T_c(\{\alpha_n\}) = \{\alpha_m^c\}\,. \tag{4.4}$$

The following notation will be a useful shorthand. Let $\alpha \in L^2(S^2)$ with spherical harmonic expansion $\{\alpha_n\}$, then

$$\|\alpha\|_{L^p} := \Big(\int_{S^2}|\alpha(\theta)|^p\,\mathrm{d}\theta\Big)^{\frac{1}{p}}\,, \quad 1\le p<\infty\,, \quad \|\alpha\|_{L^\infty} := \operatorname*{ess\,sup}_{\theta\in S^2}|\alpha(\theta)|\,, \tag{4.5}$$

$$\|\alpha\|_{l^p} := \Big(\sum_{n=0}^{\infty}\|\alpha_n\|_{L^2(S^2)}^p\Big)^{\frac{1}{p}}\,, \quad 1\le p<\infty\,, \quad \|\alpha\|_{l^\infty} := \sup_{n\in\mathbb{N}}\|\alpha_n\|_{L^2(S^2)}\,. \tag{4.6}$$

The notation emphasizes that we treat the representation of the function α by its values, or by the sequence of its spherical harmonic components $\{\alpha_n\}$ as simply a way of inducing different norms. That is, both (4.5) and (4.6) describe different norms of the same function on S^2. Note that, because of the Plancherel equality (3.7), $\|\alpha\|_{L^2} = \|\alpha\|_{l^2}$, so we may just write $\|\alpha\|_2$, and we write $\langle\cdot,\cdot\rangle$ for the corresponding inner product. We will even extend this notation a little more and refer to the support of α in S^2 as its L^0-support and denote by $\|\alpha\|_{L^0}$ the measure of $\operatorname{supp}\alpha \subseteq S^2$. Similarly, we will call the indices of the nonzero spherical harmonic components in its Fourier-Laplace-expansion (3.5) the l^0-support of α.

We will prove our uncertainty principle, by showing that the far field translation operator is bounded on weighted l^p spaces. Given any sequence of nonnegative weights $\{w_n\} \subseteq [0,\infty)$ we define

$$l^p_w(L^2(S^2)) := \big\{\alpha \in L^2(S^2)\ \big|\ \|\alpha\|_{l^p_w} < \infty\big\}\,,$$

where

$$\|\alpha\|_{l^p_w} := \Big(\sum_{n=0}^{\infty}w_n\|\alpha_n\|_{L^2(S^2)}^p\Big)^{\frac{1}{p}}\,, \quad 1\le p<\infty\,, \qquad \|\alpha\|_{l^\infty_w} := \sup_{n\in\mathbb{N}}(w_n\|\alpha_n\|_{L^2(S^2)})\,.$$

The theorem below gives a lower bound on the angle between a nonevanescent far field α and the translate $T_c\beta$ of a nonevanescent far field β in terms of the l^0-support of α and β, and the distance c. We call this result an uncertainty principle for far field translation, and it will be the main ingredient of our analysis of the far field splitting problem.

THEOREM 4.1 (Uncertainty principle for far field translation). *Let $\alpha, \beta \in L^2(S^2)$ such that the corresponding spherical harmonic components $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy* $\operatorname{supp}\{\alpha_n\} \subseteq W_1$ *and* $\operatorname{supp}\{\beta_n\} \subseteq W_2$ *with $W_1, W_2 \subseteq \mathbb{N}$, and let $c \in \mathbb{R}^3$. Then,*

$$|\langle \alpha, T_c\beta\rangle|^2 \;\leq\; \frac{\sum_{n\in W_1}(2n+1)^2 \sum_{n\in W_2}(2n+1)^2}{|c|^{\frac{5}{3}}}\|\alpha\|_2^2\|\beta\|_2^2\,. \tag{4.7}$$

The proof of theorem 4.1 is a corollary of the following lemma.

LEMMA 4.2. *Let $c \in \mathbb{R}^3$ and let T_c be the operator introduced in (4.1) and (4.4). Then, the operator norm of $T_c : L^p(S^2) \longrightarrow L^p(S^2)$, $1 \leq p \leq \infty$, satisfies*

$$\|T_c\|_{L^p,L^p} \;=\; 1\,, \tag{4.8}$$

whereas $T_c : l^1_{2n+1} \longrightarrow l^\infty_{1/(2n+1)}$ fulfills

$$\|T_c\|_{l^1_{2n+1},\, l^\infty_{1/(2n+1)}} \;\leq\; \frac{1}{|c|^{\frac{5}{6}}}\,. \tag{4.9}$$

Proof. Let $\alpha, \beta \in L^2(S^2)$ with spherical harmonic components $\{\alpha_n\}, \{\beta_n\}$, and denote by $\{\alpha_n^c\}$ the spherical harmonic components of $T_c\alpha$. Recalling (4.3) and (3.6) we find that, for any $m \in \mathbb{N}$,

$$\begin{aligned}
&\langle \beta_m, \alpha_m^c\rangle \\
&= \frac{2m+1}{4\pi}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \mathrm{i}^l(2l+1)j_l(|c|)\int_{S^2} P_l(\phi_c\cdot\omega)\alpha_n(\omega)\int_{S^2} P_m(\theta\cdot\omega)\beta_m(\theta)\,\mathrm{d}\theta\,\mathrm{d}\omega\,,
\end{aligned}$$

and (C.1) tells us that the interior sum is finite, i.e.,

$$\langle \beta_m, \alpha_m^c\rangle \;=\; \sum_{n=0}^{\infty}\sum_{l=|m-n|}^{m+n} \mathrm{i}^l(2l+1)j_l(|c|)\int_{S^2} P_l(\phi_c\cdot\omega)\alpha_n(\omega)\beta_m(\omega)\,\mathrm{d}\omega\,.$$

We next use Hölder's inequality and then the bound on $\|P_l^{\phi_c}\|_{L^\infty}$ from (3.8),

$$\begin{aligned}
|\langle \beta_m, \alpha_m^c\rangle| \;&\leq\; \sum_{n=0}^{\infty}\|\alpha_n\|_2\Big(\sum_{l=|m-n|}^{m+n}(2l+1)|j_l(|c|)|\|P_l^{\phi_c}\|_{L^\infty}\Big)\|\beta_m\|_2 \\
&=\; \sum_{n=0}^{\infty}\|\alpha_n\|_2\Big(\sum_{l=|m-n|}^{m+n}(2l+1)|j_l(|c|)|\Big)\|\beta_m\|_2\,.
\end{aligned}$$

Setting $\beta = T_c\alpha$, we obtain, for any $m \in \mathbb{N}$,

$$\|\alpha_m^c\|_2 \;\leq\; \sum_{n=0}^{\infty}\|\alpha_n\|_2\Big(\sum_{l=|m-n|}^{m+n}(2l+1)|j_l(|c|)|\Big)\,. \tag{4.10}$$

To estimate the term in parentheses, we note that spherical Bessel functions satisfy

$$j_n(z) \;=\; \sqrt{\frac{\pi}{2z}}J_{n+\frac{1}{2}}(z)\,, \qquad z \in \mathbb{C}\,,\; n \in \mathbb{Z}\,, \tag{4.11}$$

where $J_{n+\frac{1}{2}}$ is a Bessel function, and employ the estimate of $J_{n+\frac{1}{2}}$ provided by the last inequality on page 199 of [15] to obtain

$$|j_n(x)| < \frac{c}{|x|^{\frac{5}{6}}} \qquad \text{with } c \approx 0.8459\,.$$

Combining this with the formula

$$\sum_{l=|m-n|}^{m+n} (2l+1) = (2m+1)(2n+1)$$

gives a further estimate for the right-hand side of (4.10),

$$\|\alpha_m^c\|_2 \le (2m+1)\frac{1}{|c|^{\frac{5}{6}}}\sum_{n=0}^{\infty}(2n+1)\|\alpha_n\|_2\,.$$

This shows that

$$\|T_c\alpha\|_{l^\infty_{1/(2n+1)}} = \sup_{m\in\mathbb{N}}\frac{\|\alpha_m^c\|_2}{2m+1} \le \frac{1}{|c|^{\frac{5}{6}}}\sum_{n=0}^{\infty}(2n+1)\|\alpha_n\|_2 = \frac{1}{|c|^{\frac{5}{6}}}\|\alpha\|_{l^1_{2n+1}}\,.$$

∎

Proof of theorem 4.1. Hölder's inequality and (4.9) imply that

$$\begin{aligned}|\langle\alpha,T_c\beta\rangle| &= \sum_{n=0}^{\infty}|\langle\alpha_n,\beta_n^c\rangle| \le \|\alpha\|_{l^1_{2n+1}}\|T_c\beta\|_{l^\infty_{1/(2n+1)}} \le \frac{1}{|c|^{\frac{5}{6}}}\|\alpha\|_{l^1_{2n+1}}\|\beta\|_{l^1_{2n+1}}\\ &\le \frac{1}{|c|^{\frac{5}{6}}}\Big(\sum_{n\in W_1}(2n+1)^2\Big)^{1/2}\|\alpha\|_{l^2}\Big(\sum_{n\in W_2}(2n+1)^2\Big)^{1/2}\|\beta\|_{l^2}\,.\end{aligned} \tag{4.12}$$

∎

We can improve the dependence on $|c|$ in (4.7) under hypotheses on α and β that are more restrictive, but well suited to the inverse source problem. Before we state the improved result, we recall that, for $N = N(R,P,p)$ as in (3.9), the space $\mathbb{Y}_{\le N}$ of spherical harmonics of degree less than or equal to N coincides with the subspace of nonevanescent far fields from section 3.

THEOREM 4.3. *Suppose that $\alpha\in\mathbb{Y}_{\le N}$, $\beta\in\mathbb{Y}_{\le M}$ with $M,N\ge 0$, and let $c\in\mathbb{R}^3$ such that $|c| > 2(M+N+\frac{3}{2})$. Then*

$$|\langle\alpha,T_c\beta\rangle|^2 \le b\,\frac{(N+\frac{1}{2})(N+1)(N+\frac{3}{2})(M+\frac{1}{2})(M+1)(M+\frac{3}{2})}{|c|^2}\|\alpha\|_2^2\|\beta\|_2^2 \tag{4.13}$$

with $b \approx 1.975$.

REMARK 4.4. *Setting $N\approx kR_1$ and $M\approx kR_2$, for some $R_1,R_2>0$, in (4.13) yields the estimate (1.2) from the introduction.*

Proof of theorem 4.3. Let $\alpha\in\mathbb{Y}_{\le N}$ with spherical harmonic components $\{\alpha_n\}$, and denote by $\{\alpha_n^c\}$ the spherical harmonic components of $T_c\alpha$. As in the proof of lemma 4.2 we find that

$$\|\alpha_m^c\|_2 \le (2m+1)\sup_{0\le l\le M+N}|j_l(|c|)|\sum_{n=0}^{N}(2n+1)\|\alpha_n\|_2\,.$$

Combining theorem 2 of [11] with (4.11) (see appendix B for details) we obtain

$$\sup_{0\leq l\leq M+N} |j_l(|c|)| \;\leq\; \frac{b}{|c|} \qquad \text{with } b\approx 1.111\,, \tag{4.14}$$

and hence

$$\|T_c\alpha\|_{l^\infty_{1/(2n+1)}} \;=\; \sup_{m\in\mathbb{N}} \frac{\|\alpha_m^c\|_2}{2m+1} \;\leq\; \frac{b}{|c|}\sum_{n=0}^{\infty}(2n+1)\|\alpha_n\|_2 \;=\; \frac{b}{|c|}\|\alpha\|_{l^1_{2n+1}}\,.$$

We now proceed as in (4.12), with the estimate for $\|T_c\beta\|_{l^\infty_{1/(2n+1)}}$ from (4.9) replaced by the estimate we have just established, and finally use the formula

$$\sum_{n=0}^{N}(2n+1)^2 \;=\; \frac{4}{3}\Big(N+\frac{1}{2}\Big)(N+1)\Big(N+\frac{3}{2}\Big)$$

to finish the proof. □

We will also make use of another uncertainty principle. A glance at (3.5)–(3.7) reveals that the operator $\mathcal{H}: L^2(S^2)\to l^2(L^2(S^2))$, $\mathcal{H}\alpha := \{\alpha_n\}$, which maps α to its spherical harmonic components $\{\alpha_n\}$, satisfies $\|\mathcal{H}\|_{2,2} \;=\; \|\mathcal{H}^{-1}\|_{2,2} = 1$. Furthermore, it follows from (3.6) and (3.8) that, for any $n\in\mathbb{N}$,

$$\|\alpha_n\|_{L^\infty} \;\leq\; \frac{2n+1}{4\pi}\sup_{\omega\in S^2}\|P_n^\omega\|_2\|\alpha\|_2 \;=\; \sqrt{\frac{2n+1}{4\pi}}\|\alpha\|_2\,.$$

Accordingly,

$$\|\alpha\|_{L^\infty} \;\leq\; \sum_{n=0}^{\infty}\|\alpha_n\|_{L^\infty} \;\leq\; \frac{1}{\sqrt{4\pi}}\sum_{n=0}^{\infty}\sqrt{2n+1}\|\alpha_n\|_2 \;=\; \frac{1}{\sqrt{4\pi}}\|\alpha\|_{l^1_{\sqrt{2n+1}}} \tag{4.15}$$

(with equality for $\alpha = P_n^\omega$ for any $\omega\in S^2$). Thus, $\mathcal{H}^{-1}: l^1_{\sqrt{2n+1}}(L^2(S^2)) \to L^\infty(S^2)$ is bounded with

$$\|\mathcal{H}^{-1}\|_{l^1_{\sqrt{2n+1}},\,L^\infty} \;=\; \frac{1}{\sqrt{4\pi}}\,.$$

On the other hand, for any $\alpha\in L^2(S^2)$ with spherical harmonic components $\{\alpha_n\}$ and any $\beta\in L^2(S^2)$, (3.6) implies that, for any $n\in\mathbb{N}$,

$$\langle\beta,\alpha_n\rangle \;=\; \frac{2n+1}{4\pi}\int_{S^2}\beta(\theta)\int_{S^2}P_n(\theta\cdot\omega)\alpha(\omega)\;\mathrm{d}s(\omega)\;\mathrm{d}s(\theta)\,.$$

Applying the Cauchy-Schwarz inequality and (3.8) yields

$$\begin{aligned}
&|\langle\beta,\alpha_n\rangle| \\
&\leq\; \frac{2n+1}{4\pi}\Big(\int_{S^2}\int_{S^2}|\beta(\theta)|^2|\alpha(\omega)|\,\mathrm{d}s(\omega)\,\mathrm{d}s(\theta)\Big)^{\frac{1}{2}}\Big(\int_{S^2}\int_{S^2}P_n^2(\theta\cdot\omega)|\alpha(\omega)|\,\mathrm{d}s(\omega)\,\mathrm{d}s(\theta)\Big)^{\frac{1}{2}} \\
&=\; \frac{2n+1}{4\pi}\|\beta\|_2\|\alpha\|_{L^1}^{\frac{1}{2}}\sqrt{\frac{4\pi}{2n+1}}\|\alpha\|_{L^1}^{\frac{1}{2}} \;=\; \sqrt{\frac{2n+1}{4\pi}}\|\beta\|_2\|\alpha\|_{L^1}\,,
\end{aligned}$$

and choosing $\beta = \alpha_n$ gives, for any $n\in\mathbb{N}$,

$$\|\alpha_n\|_2 \;\leq\; \sqrt{\frac{2n+1}{4\pi}}\|\alpha\|_{L^1}\,,$$

and therefore

$$\|\alpha\|_{l^\infty_{1/\sqrt{2n+1}}} = \sup_{n\in\mathbb{N}} \frac{\|\alpha_n\|_2}{\sqrt{2n+1}} \leq \frac{1}{\sqrt{4\pi}}\|\alpha\|_{L^1}\,.$$

Hence, the operator $\mathcal{H} : L^1(S^2) \to l^\infty_{1/\sqrt{2n+1}}$ is bounded with

$$\|\mathcal{H}\|_{L^1, l^\infty_{1/\sqrt{2n+1}}} \leq \frac{1}{\sqrt{4\pi}}\,.$$

The next theorem gives a lower bound on the angle between the translate $T_c\alpha$ of a nonevanescent far field α and a function β supported only on part of S^2 in terms of the l^0-support of α and the L^0-support of β. This result will be the main ingredient of our analysis of the data completion problem.

THEOREM 4.5. *Let $\alpha, \beta \in L^2(S^2)$ such that the corresponding spherical harmonic components $\{\alpha_n\}$ satisfy* $\operatorname{supp}\{\alpha_n\} \subseteq W$ *with $W \subseteq \mathbb{N}$, and let $c \in \mathbb{R}^3$. Then,*

$$|\langle T_c\alpha, \beta\rangle|^2 \leq \frac{\|\beta\|_{L^0}}{4\pi} \sum_{n\in W} (2n+1)\,\|\alpha\|_2^2\|\beta\|_2^2\,.$$

Proof. Combining Hölder's inequality with (4.8) and (4.15) we find that

$$\begin{aligned}|\langle T_c\alpha, \beta\rangle| &\leq \|T_c\alpha\|_{L^\infty}\|\beta\|_{L^1} \leq \|\alpha\|_{L^\infty}\|\beta\|_{L^1} \leq \Big(\frac{1}{\sqrt{4\pi}}\sum_{n\in W}\sqrt{2n+1}\|\alpha_n\|_2\Big)\|\beta\|_{L^1}\\ &\leq \frac{1}{\sqrt{4\pi}}\Big(\sum_{n\in W}(2n+1)\Big)^{\frac{1}{2}}\|\alpha\|_2\sqrt{\|\beta\|_{L^0}}\|\beta\|_2\,.\end{aligned}$$

□

For $\alpha \in \mathbb{Y}_{\leq N}$, we may use the formula $\sum_{n=0}^N (2n+1) = (N+1)^2$ to restate theorem 4.5 as follows.

COROLLARY 4.6. *Let $\alpha \in \mathbb{Y}_{\leq N}$, $\beta \in L^2(S^2)$, and $c \in \mathbb{R}^3$. Then*

$$|\langle T_c\alpha, \beta\rangle|^2 \leq \frac{\|\beta\|_{L^0}}{4\pi}(N+1)^2\|\alpha\|_2^2\|\beta\|_2^2\,. \tag{4.16}$$

REMARK 4.7. *Setting $N \approx kR$, for some $R > 0$, in (4.16) yields the estimate (1.4) from the introduction.*

5. Corollaries of the uncertainty principles. The uncertainty principles established in theorems 4.1, 4.3, and 4.5 have immediate consequences for data completion and far field splitting.

In [6, 7, 8] we recently developed two classes of reconstruction algorithms for these inverse problems in $\mathbb{R}^2$, one based on l^2 techniques (least squares) and the other using l^1 minimization (basis pursuit). We provided the corresponding stability analysis in [8], using uncertainty principles. Here we comment on modifications and extensions of these results that are needed for the three-dimensional setting. We will only do so for the least squares algorithms and note that similar modifications apply to the algorithms based on l^1 arguments. Since, apart from the new uncertainty principles, the proofs of the theorems below are relatively straightforward modifications of the corresponding results in the two-dimensional case, we omit them and refer the reader to [8].

The regularized Picard criterion from section 3 tells us that, up to measurement precision p, a far field radiated by a limited power source with power threshold P in $B_R(0)$ coincides with an α that belongs to the subspace of nonevanescent far fields $\mathbb{Y}_{\leq N}$, where $N = N(R,P,p)$ as in (3.9), is just a little bigger than R. Accordingly, the uncertainty principles from the previous section apply to this setting. The first result below gives conditions under which we can split the sum of two nonevanescent far fields radiated from well-separated localized sources into the original summands by solving a well-conditioned least squares problem and provides a stability estimate.

THEOREM 5.1. *Suppose that* $\gamma^0, \gamma^1 \in L^2(S^2)$, $c_1, c_2 \in \mathbb{R}^3$, *and* $N_1, N_2 \in \mathbb{N}$ *such that* $|c_1 - c_2| > 2(N_1 + N_2 + \frac{3}{2})$ *and*

$$C \;:=\; b\,\frac{(N_1+\frac12)(N_1+1)(N_1+\frac32)(N_2+\frac12)(N_2+1)(N_2+\frac32)}{|c_1-c_2|^2} \;<\; 1 \tag{5.1}$$

with $b \approx 1.975$, *and let*

$$\gamma^0 \overset{\mathrm{LS}}{=} T^*_{c_1}\alpha^0_1 + T^*_{c_2}\alpha^0_2\,, \qquad\qquad \alpha^0_i \in l^2(\mathbb{Y}_{\leq N_i})\,, \tag{5.2a}$$

$$\gamma^1 \overset{\mathrm{LS}}{=} T^*_{c_1}\alpha^1_1 + T^*_{c_2}\alpha^1_2\,, \qquad\qquad \alpha^1_i \in l^2(\mathbb{Y}_{\leq N_i})\,. \tag{5.2b}$$

Then, for $i = 1, 2$,

$$\|\alpha^1_i - \alpha^0_i\|_2^2 \;\leq\; (1-C)^{-1}\|\gamma^1 - \gamma^0\|_2^2\,. \tag{5.3}$$

The notation in (5.2) means that α^j_i are the least squares solutions to the equation $\gamma^j = T^*_{c_1}\alpha^j_1 + T^*_{c_2}\alpha^j_2$. Since a γ^j that is radiated by a limited power source supported in $B_{R_1}(c_1)\cup B_{R_2}(c_2)$ will typically not belong to the subspace $T^*_{c_1}l^2(\mathbb{Y}_{\leq N_1})\oplus T^*_{c_2}l^2(\mathbb{Y}_{\leq N_2})$ (although it is very close to this subspace if $N_i \gtrsim R_i$), α^j_1 and α^j_2 will usually not solve (5.2) exactly. However, the estimate (5.3) is always true and provides an explicit bound on the absolute condition number of the splitting operator, which maps γ^j to (α^j_1, α^j_2). The condition (5.1) essentially relates the radii R_i of the supports of the sources and the distance $|c_1 - c_2|$ between the sources.

Next we present a corresponding corollary of theorem 4.5, which is concerned with data completion. Assuming that a far field is radiated from a small ball $B_R(c)$ and measured on most of the sphere, then theorem 5.2 below says that its nonevanescent part can be recovered on the entire sphere. More precisely, we consider the case, where the farfield $\alpha = T^*_c\alpha^0$ cannot be measured on a subset $\Omega \subseteq S^2$, and instead we observe $\gamma = \alpha + \beta$, where $\beta = -\alpha|_\Omega$.

THEOREM 5.2. *Suppose that* $\gamma^0, \gamma^1 \in L^2(S^2)$, $c \in \mathbb{R}^3$, $N \in \mathbb{N}$, *and* $\Omega \subseteq S^2$ *such that*

$$C \;:=\; \frac{|\Omega|(N+1)^2}{4\pi} \;<\; 1\,, \tag{5.4}$$

and let

$$\gamma^0 \overset{\mathrm{LS}}{=} \beta^0 + T_c\alpha^0\,, \qquad\qquad \alpha^0 \in l^2(\mathbb{Y}_{\leq N}) \text{ and } \beta^0 \in L^2(\Omega)\,,$$

$$\gamma^1 \overset{\mathrm{LS}}{=} \beta^1 + T_c\alpha^1\,, \qquad\qquad \alpha^1 \in l^2(\mathbb{Y}_{\leq N}) \text{ and } \beta^1 \in L^2(\Omega)\,.$$

Then

$$\|\alpha^1 - \alpha^0\|_2^2 \;\leq\; (1-C)^{-1}\|\gamma^1 - \gamma^0\|_2^2 \tag{5.5a}$$

and

$$\|\beta^1 - \beta^0\|_2^2 \;\leq\; (1-C)^{-1}\|\gamma^1 - \gamma^0\|_2^2\,. \tag{5.5b}$$

The condition (5.4) essentially relates the radius R of the support of the source and the measure $|\Omega|$ of the missing data segment. The estimates (5.5) show that we can recover the nonevanescent part of the far field on Ω by solving a least squares problem and give explicit stability bounds for this algorithm.

Finally, we note that the results of theorem 5.1 can be extended to multiple well-separated components and that far field splitting and data completion can actually be combined.

THEOREM 5.3. *Suppose that $\gamma^0, \gamma^1 \in L^2(S^2)$, $c_i \in \mathbb{R}^3$, $N_i \in \mathbb{N}$, $i = 1, \dots, I$, and $\Omega \subseteq L^2(S^2)$ such that $|c_i - c_j| > 2(N_i + N_j + \frac{3}{2})$ for every $i \neq j$ and*

$$C_{\alpha,i} := \sqrt{\frac{|\Omega|}{4\pi}}(N_i + 1) + \sqrt{b}\sqrt{\Big(N_i + \frac{1}{2}\Big)(N_i + 1)\Big(N_i + \frac{3}{2}\Big)} \sum_{j\neq i} \frac{\sqrt{(N_j + \frac{1}{2})(N_j + 1)(N_j + \frac{3}{2})}}{|c_i - c_j|} < 1 \tag{5.6a}$$

for each i,

$$C_\beta := \sqrt{\frac{|\Omega|}{4\pi}} \sum_{i=1}^{I} (N_i + 1) < 1 \tag{5.6b}$$

with $b \approx 1.975$, and let

$$\gamma^0 \overset{LS}{=} \beta^0 + \sum_{i=1}^{I} T^*_{c_i}\alpha^0_i\,, \qquad \alpha^0_i \in l^2(\mathbb{Y}_{\leq N_i}) \text{ and } \beta^0 \in L^2(\Omega)\,, \tag{5.7a}$$

$$\gamma^1 \overset{LS}{=} \beta^1 + \sum_{i=1}^{I} T^*_{c_i}\alpha^1_i\,, \qquad \alpha^1_i \in l^2(\mathbb{Y}_{\leq N_i}) \text{ and } \beta^0 \in L^2(\Omega)\,. \tag{5.7b}$$

Then, for $i = 1, \dots, I$,

$$\|\alpha^1_i - \alpha^0_i\|_2^2 \leq (1 - C_{\alpha,i})^{-1}\|\gamma^1 - \gamma^0\|_2^2$$

and

$$\|\beta^1 - \beta^0\|_2^2 \leq (1 - C_\beta)^{-1}\|\gamma^1 - \gamma^0\|_2^2\,.$$

Because the sum of the diameters of the supports of well-separated localized source components may be much less than the diameter of a large ball that contains them all, combining data completion with splitting as in theorem 5.3 can improve the conditioning of the data completion problem.

6. An analytic example. Let f be a single-layer source supported on a horizontal two-dimensional disc of radius $W > 0$, and let g be the same source, translated vertically by a distance $d > 0$ (i.e., $c_1 = (0, 0, 0)$ and $c_2 = (0, 0, d)$). Specifically, with χ denoting the indicator function, and δ the Dirac mass we define

$$f := \frac{1}{2\pi W^2}\chi_{|(x_1,x_2)|<W}\delta_{x_3=0} \qquad \text{and} \qquad g := \frac{1}{2\pi W^2}\chi_{|(x_1,x_2)|<W}\delta_{x_3=d}\,. \tag{6.1}$$

Recalling that, for any $\xi \in \mathbb{R}^2$ (in two dimensions)

$$\int_{|\eta|<W} e^{-\mathrm{i}\eta\cdot\xi}\,\mathrm{d}\xi = \int_0^W \int_0^{2\pi} e^{-\mathrm{i}\rho|\xi|\cos\psi}\,\mathrm{d}\psi\rho\,\mathrm{d}\rho = 2\pi W\frac{J_1(W|\xi|)}{|\xi|}\,,$$

the far fields radiated by f and g satisfy

$$\alpha_f(\theta) \;=\; \frac{1}{2\pi W^2}\int_{|\xi|\leq W} e^{-\mathrm{i}(\theta_1,\theta_2)\cdot\xi}\;\mathrm{d}\xi \;=\; \frac{J_1(W\sin\vartheta)}{W\sin\vartheta}$$

and

$$\alpha_g(\theta) \;=\; e^{-\mathrm{i}d\cos\vartheta}\frac{J_1(W\sin\vartheta)}{W\sin\vartheta}$$

for $\theta=(\theta_1,\theta_2,\theta_3)=(\cos\varphi\sin\vartheta,\sin\varphi\sin\vartheta,\cos\vartheta)\in S^2$. Introducing

$$a(t) \;:=\; \frac{J_1(t)}{t} \;=\; \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k k!(k+1)!}t^{2k}\,, \qquad t\geq 0\,,$$

(see, e.g., [16, 10.2.2]) and $b(t):=(a(\sqrt{t}))^2$, $t\geq 0$, we note that both functions are analytic. This implies that a and b and their first and second derivatives are bounded on the compact interval $[0,1]$, and explicit calculations using the recurrence relations for Bessel functions immediately show that the same is also true on $(1,\infty)$. In particular, $|a(t)|\leq 1$ and the asymptotic behavior of Bessel functions for large argument (cf., e.g., [16, 10.17.3]) gives $|a(t)|\leq C_a t^{-3/2}$ for some $C_a>0$.

Accordingly,

$$\begin{aligned}\|\alpha_f\|_2^2 \;=\; \|\alpha_g\|_2^2 \;&=\; \int_0^{2\pi}\int_0^{\pi} a^2(W\sin\vartheta)\sin\vartheta\;\mathrm{d}\vartheta\;\mathrm{d}\varphi\\ &=\; 4\pi\int_0^1 a^2(Wt)\frac{t}{\sqrt{1-t^2}}\;\mathrm{d}t\\ &\leq\; 4\pi\int_0^{\frac{1}{W}}\frac{t}{\sqrt{1-t^2}}\;\mathrm{d}t + 4\pi C_a\int_{\frac{1}{W}}^1\frac{1}{(Wt)^3}\frac{t}{\sqrt{1-t^2}}\;\mathrm{d}t \;\leq\; CW^{-2}\end{aligned} \tag{6.2}$$

for some $C>0$. On the other hand,

$$\begin{aligned}\langle\alpha_f,\alpha_g\rangle \;&=\; \int_0^{2\pi}\int_0^{\pi} a^2(W\sin\theta)e^{\mathrm{i}d\cos\vartheta}\sin\vartheta\;\mathrm{d}\vartheta\;\mathrm{d}\varphi\\ &=\; 4\pi\,\mathrm{Re}\Big(\int_0^{\frac{\pi}{2}} a^2(W\sin\theta)e^{\mathrm{i}d\cos\vartheta}\sin\vartheta\;\mathrm{d}\vartheta\Big)\,.\end{aligned} \tag{6.3}$$

Integrating by parts twice and using the short hand notation $s(t):=1-t^2$, we find that

$$\begin{aligned}\int_0^{\frac{\pi}{2}} a^2(W\sin\theta)e^{\mathrm{i}d\cos\vartheta}\sin\vartheta\;\mathrm{d}\vartheta \;&=\; \int_0^1 a^2(W\sqrt{1-t^2})e^{\mathrm{i}dt}\;\mathrm{d}t \;=\; \int_0^1 b(W^2 s(t))e^{\mathrm{i}dt}\;\mathrm{d}t\\ &=\; b(W^2 s(t))\frac{e^{\mathrm{i}dt}}{\mathrm{i}d}\Big|_{t=0}^1 - \frac{W^2}{\mathrm{i}d}\int_0^1 b'(W^2 s(t))s'(t)e^{\mathrm{i}dt}\;\mathrm{d}t\\ &=\; \frac{1}{4}\frac{e^{\mathrm{i}d}}{\mathrm{i}d} - \frac{b(W^2)}{\mathrm{i}d} - \frac{W^2}{\mathrm{i}d}\bigg(b'(W^2 s(t))s'(t)\frac{e^{\mathrm{i}dt}}{\mathrm{i}d}\Big|_{t=0}^1\\ &\qquad - \int_0^1\Big(b''(W^2 s(t))W^2(s'(t))^2 + b'(W^2 s(t))s''(t)\Big)\frac{e^{\mathrm{i}dt}}{\mathrm{i}d}\;\mathrm{d}t\bigg)\,.\end{aligned}$$

The boundedness of b, b' and b'' on $[0,\infty)$ implies

$$\int_0^{\frac{\pi}{2}} a^2(W\sin\theta)e^{\mathrm{i}d\cos\vartheta}\sin\vartheta\;\mathrm{d}\vartheta \;=\; \frac{1}{4}\frac{e^{\mathrm{i}d}}{\mathrm{i}d} + \mathcal{O}(W^4 d^{-2})\,.$$

Substituting this into (6.3) yields

$$\langle \alpha_f, \alpha_g \rangle \;=\; \pi \frac{\sin d}{d} + \mathcal{O}\Big(\frac{W^4}{d^2}\Big) . \tag{6.4}$$

The combinination of (6.2) and (6.4) shows that

$$\frac{\langle \alpha_f, \alpha_g \rangle}{\|\alpha_f\|_2 \|\alpha_g\|_2} \;\gtrsim\; \frac{W^2}{d} \sin d + \mathcal{O}\Big(\frac{W^6}{d^2}\Big) ,$$

while the conclusion of theorem 4.3 with $d = |c|$ and $N = M \approx W$ is

$$\frac{\langle \alpha_f, \alpha_g \rangle}{\|\alpha_f\|_2 \|\alpha_g\|_2} \;\leq\; \frac{W^3}{d}$$

so that it may be possible to improve (4.13), but the dependence on the diameter W can be no better than W^2. Similar calculations in two dimensions, using a constant single-layer source supported on a line, yield $\frac{W}{\sqrt{d}}$ as both upper and lower bounds, which may be a reason to expect that theorem 4.3 is not sharp.

We can also use f as defined in (6.1) to check that the dependence on $\|\beta\|_{L^0}$ and N in corollary 4.6 is sharp. With $N \approx W$, β supported in Ω, and $c = 0$, inequality (4.16) becomes

$$\frac{\langle \alpha_f, \beta \rangle}{\|\alpha_f\|_2 \|\beta\|_2} \;\leq\; \sqrt{\frac{|\Omega| W^2}{4\pi}} . \tag{6.5}$$

If we choose Ω_ε, $\varepsilon > 0$, to be the neighborhood of the north pole

$$\Omega_\varepsilon \;:=\; \{(\cos\varphi \sin\vartheta, \sin\varphi \sin\vartheta, \cos\vartheta) \mid \vartheta < \varepsilon\} \subseteq S^2 ,$$

then

$$\lim_{\varepsilon \to 0} \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} \alpha_f^2(\theta) \, \mathrm{d}s(\theta) \;=\; \alpha_f^2(0) \;=\; \frac{1}{4} .$$

Now let β_ε be the restriction of α_f to Ω_ε and $\varepsilon = \frac{1}{W}$. Then

$$\frac{\langle \alpha_f, \beta_\varepsilon \rangle}{\|\alpha_f\|_2 \|\beta_\varepsilon\|_2} \;=\; \frac{1}{\|\alpha_f\|_2 \|\beta_\varepsilon\|_2} \int_{\Omega_\varepsilon} \alpha_f^2(\theta) \, \mathrm{d}s(\theta) \;=\; \frac{\|\beta_\varepsilon\|_2}{\|\alpha_f\|_2} \;\geq\; C^{-\frac{1}{2}} \|\beta_\varepsilon\|_2 W ,$$

where C is the constant in (6.2). But $\|\beta_\varepsilon\|_2 \to \frac{1}{2}\sqrt{|\Omega_\varepsilon|}$ as $\varepsilon \to 0$, so for small enough ε,

$$\frac{\langle \alpha_f, \beta_\varepsilon \rangle}{\|\alpha_f\|_2 \|\beta_\varepsilon\|_2} \;\geq\; \frac{C^{-\frac{1}{2}}}{4} \sqrt{|\Omega_\varepsilon|} W ,$$

which shows that the dependence on $|\Omega|$ and W in (6.5) is sharp.

7. A numerical example. We briefly discuss a numerical implementation of the least squares scheme for simultaneous data completion and splitting as considered in theorem 5.3. The algorithm is an extension of methods described in [6, 8] for the two-dimensional case.

Suppose that the far field $\alpha = \sum_{i=1}^{I} T_{c_i}^* \alpha_i$ is a superposition of far fields $T_{c_i}^* \alpha_i$ radiated by limited power sources supported in balls $B_{R_i}(c_i)$, for some $c_i \in \mathbb{R}^3$ and $R_i > 0$, and that we are unable to measure α on a subset $\Omega \subseteq S^2$. Given a priori information on the approximate location of the supports of the individual source components, i.e., $B_{R_i}(c_i)$, $i = 1, \dots, I$, we will recover an approximation of the nonevanescent part of $\alpha|_\Omega$ from $\alpha|_{S^1\setminus\Omega}$. Writing $\gamma := \alpha|_{S^1\setminus\Omega}$ for the observed far field data and $\beta := -\alpha|_\Omega$ implies that

$$\gamma \,=\, \beta + \sum_{i=1}^{I} T_{c_i}^* \alpha_i \,,$$

i.e., we are in the setting of theorem 5.3.

In the following we use the abbreviations $V_0 := L^2(\Omega)$ and $V_i := T_{c_i}^* l^2(\mathbb{Y}_{\leq N_i})$, $i = 1, \dots, I$. Denoting by $P_0, \dots, P_I$ the orthogonal projections onto $V_0, \dots, V_I$, respectively, the least squares problem (5.7) is equivalent to seeking approximations $a_i \in V_i$ of $T_{c_i}^* \alpha_i$, $i = 1, \dots, I$, and $b \in V_0$ of β satisfying the linear system

$$\begin{bmatrix} I & P_0P_1 & \cdots & P_0P_I \\ P_1P_0 & I & \cdots & P_1P_I \\ \vdots & \vdots & & \vdots \\ P_IP_0 & P_IP_1 & \cdots & I \end{bmatrix} \begin{bmatrix} b \\ a_1 \\ \vdots \\ a_I \end{bmatrix} = \begin{bmatrix} 0 \\ P_1\gamma \\ \vdots \\ P_I\gamma \end{bmatrix} . \tag{7.1}$$

In compliance with the regularized Picard criterion from section 3 we choose the numbers $N_1, \dots, N_I$ that determine the dimension of the individual subspaces $V_1, \dots, V_I$, such that $N_i \gtrsim kR_i$. The estimates from theorem 5.3 give bounds on the absolute condition number of the operator on the left-hand side of (7.1).

Assuming that the whole ensemble of sources is contained in a ball $B_R(0)$ of radius $R > 0$ around the origin, the nonevanescent part of α (and of each far field component $T_{c_1}^*\alpha_1, \dots, T_{c_I}^*\alpha_I$) is well approximated by its projection onto the subspace $\mathbb{Y}_{\leq N} \subseteq L^2(S^2)$ with $N \gtrsim kR$. In our numerical implementation we solve the restriction of (7.1) to this subspace.

A glance at the proofs of theorems 5.1 and 5.2 reveals that square roots of the left-hand sides of (5.1) and (5.4) are just upper bounds for the operator norms $\|P_iP_j\|_{2,2}$ of the entries of the block-operator on the left-hand side of (7.1). If (5.6) is satisfied, then this (self-adjoint) operator is strictly diagonally dominant, and thus positive definite. Accordingly, we apply the conjugate gradient method to solve the linear system (7.1) (restricted to $\mathbb{Y}_{\leq N}$), evaluating the projections P_iP_j using discrete spherical harmonic transforms.

In our numerical example below we use simulated far field data on an equiangular grid

$$\Theta \,:=\, \bigl\{\, \theta(\vartheta_m, \varphi_n) \bigm| \vartheta_m = m\pi/M,\ \varphi_n = 2n\pi/M,\ m,n = 0, \dots, M \,\bigr\} \tag{7.2}$$

on the unit sphere. Following, e.g., theorem 3 of [5], it is appropriate under our conditions to sample the far field α at M equidistant angles $\varphi \in [0, 2\pi)$ and the same number of equidistant angles $\vartheta \in [0, \pi]$, where M is given by

$$M \,\gtrsim\, 2R\,. \tag{7.3}$$

Example 7.1. We consider a scattering problem with two obstacles as shown in figure 7.1 (left), which are illuminated by an incident plane wave $u^i(x) = e^{\mathrm{i}kx\cdot d}$, $x \in \mathbb{R}^3$, with incident direction $d = (1,0,0)^T$ and wavenumber $k = 5$ (i.e., the

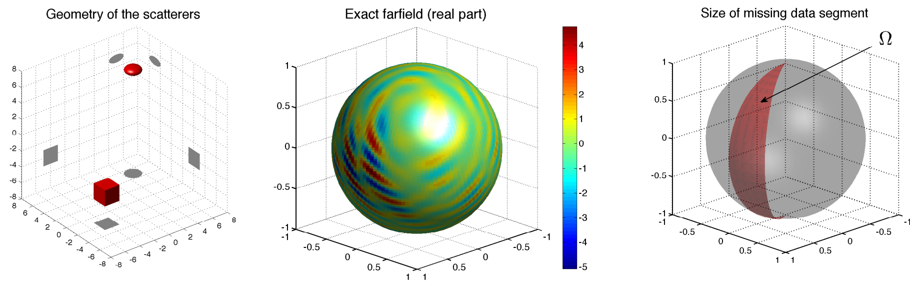

FIG. 7.1. *Left: Geometry of the scatterers. Center: Real part of the exact far field α. Right: Illustration of the missing data segment Ω.*

wavelength is $\lambda = 2\pi/5 \approx 1.26$). For better visualization this plot contains projections of the scatterers on the three coordinate axes. The diameters of the two obstacles (in conventional units) are 2.00 (ellipsoid) and 3.46 (cube), and both of them are contained in the ball $B_{10}(0)$ of radius $R = 10$ around the origin. Accordingly, we choose $N = \lceil \frac{e}{2} kR \rceil = 68$.

Assuming that the ellipsoid is sound soft whereas the cube is sound hard, the scattered field u^s satisfies the homogeneous Helmholtz equation outside the obstacles, the Sommerfeld radiation condition at infinity, and Dirichlet (ellipsoid) or Neumann boundary conditions (cube) on the boundaries of the obstacles. We simulate the far field pattern α of the scattered field on the equiangular grid $\Theta \subseteq S^2$ from (7.2) with $M = 128$ (i.e., $M \gtrsim 2kR = 100$ in compliance with (7.3)) using a boundary element method[5].

Figure 7.1 (center) shows a visualization of the real part of α over the sphere. Since the far field α can be written as a superposition of two far fields radiated by two individual smooth sources supported in arbitrarily small neighborhoods of the scattering obstacles (cf., e.g., [14, lemma 3.6]), this example fits into the framework of the previous sections.

We assume that the far field cannot be measured on the segment

$$\Omega = \{\theta = (\cos\varphi\cos\vartheta, \sin\varphi\cos\vartheta, \sin\vartheta) \in S^2 \mid -\pi/2 < \vartheta < \pi/2\,,\ 0 < \varphi < \pi/6\}\,,$$

which is illustrated illustrated in figure 7.1 (right); accordingly, $|\Omega| = \pi/3$. As a priori information on the location of the supports of the individual source components we use the balls $B_{R_i}(c_i)$, $i = 1, 2$, with $c_1 = (5, 5, 5)^T$, $c_2 = (-4, -2, -4)^T$ and $R_1 = 1.5$, $R_2 = 2.0$. We choose $N_1 = \lceil \frac{e}{2} kR_1 \rceil = 11$ and $N_2 = \lceil \frac{e}{2} kR_2 \rceil = 14$, and solve the linear system (7.1) (restricted to $\mathbb{Y}_{\leq N}$) using 500 conjugate gradient iterations.

Figure 7.2 shows plots of the real part of the observed data γ (left), the real part of the reconstruction of the missing data segment (center), and the real part of the difference between the exact far field and the reconstructed far field (right).

The relative approximation error of this reconstruction is

$$\frac{\|\alpha|_\Omega - b\|_{L^2(\Omega)}}{\|\alpha|_\Omega\|_{L^2(\Omega)}} \approx 9.1 \cdot 10^{-4} \qquad \text{and} \qquad \frac{\|\alpha - (a_1 + a_2)\|_2}{\|\alpha\|_2} \approx 4.3 \cdot 10^{-4}\,.$$

To put this into perspective, we note that the best approximation error of α in $\mathbb{Y}_{\leq N}$

[5]The data have been generated using the C++ boundary element library BEM++ (see [17]).

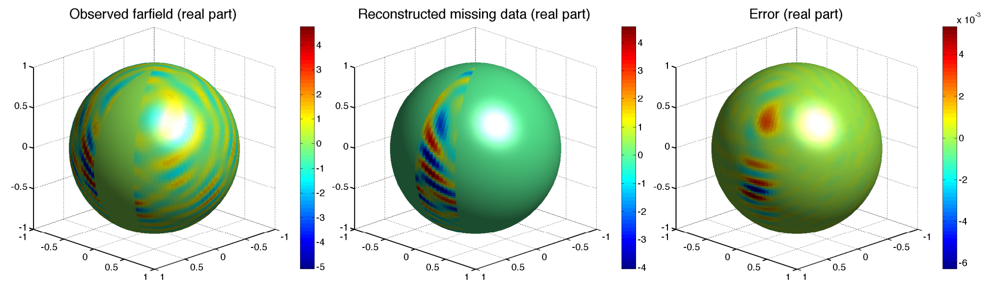

FIG. 7.2. *Left: Real part of the observed far field γ. Center: Real part of the reconstruction of the missing part $\alpha|_\Omega$. Right: Real part of the difference between exact far field and reconstructed far field on the whole sphere.*

satisfies

$$\frac{\|\alpha - \mathcal{P}_N\alpha\|_2}{\|\alpha\|_2} \approx 6.8 \cdot 10^{-9} \,.$$

Conclusions. We have derived uncertainty principles formulated as estimates for the cosine of an angle between subspaces, and combined them with a characterization of nonevanescent far fields to obtain explicit estimates of condition numbers for far field splitting and data completion for any number of well separated scatterers in $\mathbb{R}^3$. An important feature of these estimates is that they are unitless and have a simple direct physical interpretation in terms of wavelength, size, and distance between sources.

Several unanswered questions remain, the most significant being the sharpness of the dependence of the cosine estimate for the far field splitting operator. Does the estimate in (1.2) hold with $(kR_1)^{\frac{3}{2}}(kR_2)^{\frac{3}{2}}$ replaced by kR_1kR_2? In addition, while we believe the constants in theorem 5.2 are pretty close to optimal, the constants $C_{\alpha,j}$ and C_β in theorem 5.3 may well admit substantial improvement.

Acknowledgments. Part of this research was carried out while J.S. was visiting the University of Würzburg. This research stay was partially supported by the Simons Foundation and by the Mathematisches Forschungsinstitut Oberwolfach. J.S. was partially supported by the National Science Foundation's grant DMS-1309362.

Appendix A. Properties of the rescaled squared singular values.

Recalling (4.11) we obtain from [16, 10.22.5] that the rescaled singular values from (3.4a) satisfy

$$s_n^2(R) = 2\pi R^3 (j_n^2(R) - j_{n-1}(R) j_{n+1}(R)) \,, \qquad n \in \mathbb{N} \,.$$

As in [8, (SM1.2)] this implies that

$$\text{(A.1)} \qquad s_n^2(R) = \pi^2 \Big((R J'_{n+\frac{1}{2}}(R))^2 + \Big(R^2 - \Big(n + \frac{1}{2}\Big)^2 \Big) J^2_{n+\frac{1}{2}}(R) \Big) \,, \qquad n \in \mathbb{N} \,.$$

LEMMA A.1.

$$\sum_{n=0}^{\infty} (2n+1) s_n^2(R) = 4\pi \frac{R^3}{3} \,.$$

Proof. Applying formula 10.60.12 of [16], the definition (3.4a) yields

$$\sum_{n=0}^{\infty}(2n+1)s_n^2(R) \;=\; 4\pi\int_0^R\Big(\sum_{n=0}^{\infty}(2n+1)j_n^2(z)\Big)r^2\;\mathrm{d}r \;=\; \frac{4\pi}{3}R^3\,.$$

∎

The next lemma shows that odd and even rescaled squared singular values, $s_n^2(R)$, are monotonically decreasing as functions of n.

LEMMA A.2.

$$s_{n-1}^2(R) - s_{n+1}^2(R) \;\geq\; 0\,, \qquad n\geq 0\,.$$

Proof. Using recurrence relations for Bessel functions (cf. [16, 10.51.1, 10.51.2]) we find that

$$j_{n-1}^2(z) - j_{n+1}^2(z) \;=\; \frac{2n+1}{z}(j_n^2)'(z) + \frac{2n+1}{z^2}j_n^2(z)\,, \qquad z\in\mathbb{C}\,,\; n\in\mathbb{Z}\,.$$

Thus,

$$\begin{aligned} s_{n-1}^2(R) - s_{n+1}^2(R) \;&=\; 4\pi\Big(\int_0^R\Big((2n+1)r(j_n^2)'(r) + (2n+1)j_n^2(r)\Big)\;\mathrm{d}r\Big) \\ &=\; 4\pi(2n+1)Rj_n^2(R) \;\geq\; 0\,, \end{aligned}$$

where in the last step we integrated by parts. ∎

Integrating sharp estimates for $J_n(r)$ from [12], we obtain upper bounds for $s_n^2(R)$ when $n\geq R>0$.

LEMMA A.3. *Suppose that $n\geq R>0$. Then*

$$s_n^2(R) \;\leq\; bR\Big(n+\frac{1}{2}\Big)^{\frac{2}{3}}\Big(\frac{R^2}{(n+1)^2}e^{1-\frac{R^2}{(n+1)^2}}\Big)^{n+1}\,, \tag{A.2}$$

where the constant $b\approx 4.791$ is independent of n and R.

Proof. From theorem 2 of [12] we obtain for any $\nu>0$ satisfying $0<r\leq\nu+\frac{1}{2}$ that

$$J_\nu^2(r) \;\leq\; \frac{2^{\frac{2}{3}}}{3^{\frac{4}{3}}\big(\Gamma(\frac{2}{3})\big)^2}\frac{r^{2\nu}}{\nu^{2\nu+\frac{2}{3}}}e^{\frac{\nu^2-r^2}{\nu+\frac{1}{2}}}\,,$$

i.e., setting $\nu=n+\frac{1}{2}$ (and accordingly $0<r\leq n$) and using (4.11),

$$j_n^2(r) \;\leq\; \frac{\pi}{2r}\frac{2^{\frac{2}{3}}}{3^{\frac{4}{3}}\big(\Gamma(\frac{2}{3})\big)^2}\frac{r^{2n+1}}{(n+\frac{1}{2})^{2n+\frac{5}{3}}}e^{\frac{(n+\frac{1}{2})^2-r^2}{n+1}}\,.$$

Substituting this into (3.4a) yields

$$\begin{aligned} s_n^2(R) \;&\leq\; 4\pi\frac{\pi}{2^{\frac{1}{3}}}\frac{1}{3^{\frac{4}{3}}\big(\Gamma(\frac{2}{3})\big)^2}\frac{e^{\frac{(n+\frac{1}{2})^2}{n+1}}}{(n+\frac{1}{2})^{2n+\frac{5}{3}}}\int_0^R r^{2n}e^{-\frac{r^2}{n+1}}r^2\;\mathrm{d}r \\ &=\; \frac{4\pi^2}{2^{\frac{1}{3}}3^{\frac{4}{3}}\big(\Gamma(\frac{2}{3})\big)^2}\frac{e^{\frac{(n+\frac{1}{2})^2}{n+1}}}{(n+\frac{1}{2})^{2n+\frac{5}{3}}}\frac{(n+1)^{n+\frac{3}{2}}}{2}\int_0^{\frac{R^2}{n+1}}t^{n+\frac{1}{2}}e^{-t}\;\mathrm{d}t\,. \end{aligned}$$

Since $t^n e^{-t}$ is monotonically increasing for $0 < t < \frac{R^2}{n+1} \le n + \frac{1}{2}$, it follows that

$$s_n^2(R) \le \frac{4\pi^2}{6^{\frac{4}{3}}\big(\Gamma(\frac{2}{3})\big)^2\sqrt{e}}\Big(\frac{n+1}{n+\frac{1}{2}}\Big)^{2n+2} R\Big(n+\frac{1}{2}\Big)^{\frac{2}{3}}\Big(\frac{R^2 e^{1-\frac{R^2}{(n+1)^2}}}{(n+1)^2}\Big)^{n+1}.$$

Estimating $((n+1)/(n+\frac{1}{2}))^{2n+2} \le 4$ (the maximum is at $n = 0$) yields (A.2). ∎

On the other hand, the rescaled squared singular values $s_n^2(R)$ are not small for $n < R$. To see this, we estimate the right-hand side of (A.1). While complete asymptotics for the Bessel and Hankel functions are well known, we have been unable to find explicit error estimates in the literature, so we include them here.

THEOREM A.4. *Suppose that $r > \nu \ge 0$, define $\mu \in (0, \pi)$ by $\cos\mu := \frac{\nu}{r}$, and assume that $\sin\mu \ge \delta > 0$. There exists $C = C(\delta)$ such that*

$$\Big|H_\nu^{(1)}(r) - \sqrt{\frac{2}{\mathrm{i}\pi r \sin\mu}} e^{\mathrm{i}r(\sin\mu - \mu\cos\mu)}\Big| \le \frac{C}{r}\,, \tag{A.3a}$$

$$\Big|(H_\nu^{(1)})'(r) - \sqrt{\frac{2}{\mathrm{i}\pi r \sin\mu}} (\mathrm{i}\sin\mu) e^{\mathrm{i}r(\sin\mu - \mu\cos\mu)}\Big| \le \frac{C}{r}\,, \tag{A.3b}$$

where the constant C is independent of ν and r.

Proof. Following [3], p. 468, Hankel functions may be represented as a contour integral[6]

$$H_\nu^{(1)}(r) = \int_\gamma e^{\mathrm{i}r\phi_\mu(z)}\,\mathrm{d}z\,, \tag{A.4}$$

where $\phi_\mu(z) := \sin z - z\cos\mu$ and γ is the contour

$$\gamma(t) := \begin{cases} -\mathrm{i}t\,, & t < 0\,,\\ t\,, & 0 \le t \le \pi\,,\\ \pi - \mathrm{i}(t-\pi)\,, & t > \pi\,. \end{cases}$$

Denoting the three parts of γ by γ_1, γ_2, and γ_3, we have the following estimates.

LEMMA A.5.

$$\Big|\int_{\gamma_1} e^{\mathrm{i}r\phi_\mu(z)}\,\mathrm{d}z\Big| \le \frac{2}{(1-\cos\mu)r} \qquad \textit{and} \qquad \Big|\int_{\gamma_3} e^{\mathrm{i}r\phi_\mu(z)}\,\mathrm{d}z\Big| \le \frac{2}{(1+\cos\mu)r}\,. \tag{A.5}$$

Proof. With $\psi_\mu(t) := \sinh t - t\cos\mu$, integration by parts gives

$$\begin{aligned}
\Big|\int_{\gamma_1} e^{\mathrm{i}r\phi_\mu(z)}\,\mathrm{d}z\Big| &= \Big|\int_0^\infty e^{-r\psi_\mu(t)}\,\mathrm{d}t\Big| = \Big|\int_0^\infty (e^{-r\psi_\mu(t)} r\psi_\mu'(t))\frac{\mathrm{d}t}{r\psi_\mu'(t)}\Big|\\
&= \frac{1}{r}\Big|\frac{1}{1-\cos\mu} + \int_0^\infty e^{-r\psi_\mu(t)}\frac{\sinh t}{(\cosh t - \cos\mu)^2}\,\mathrm{d}t\Big|\\
&\le \frac{1}{r}\Big|\frac{1}{1-\cos\mu} + \int_0^\infty \frac{\sinh t}{(\cosh t - \cos\mu)^2}\,\mathrm{d}t\Big|\\
&= \frac{1}{r}\Big(\frac{1}{1-\cos\mu} + \frac{1}{1-\cos\mu}\Big).
\end{aligned}$$

The second estimate in (A.5) follows similarly. ∎

[6] This is actually the contour in [3] combined with the change of variables $z \mapsto -z$. The representation holds for $\nu \in \mathbb{C}$ and $\operatorname{Re} r > 0$.

Thus the asymptotic behavior as $r \to \infty$ depends only on the second part of the contour, which we will estimate using stationary phase. To prepare, note that

$$\int_{\gamma_2} e^{\mathrm{i}r\phi_\mu(z)}\,\mathrm{d}z = \int_0^\pi e^{\mathrm{i}r\phi_\mu(t)}\,\mathrm{d}t$$

and let $a_\varepsilon(t) := a\big(\frac{t}{\varepsilon}\big)$, $t \in \mathbb{R}$, be a C^∞ cutoff function satisfying

$$a_\varepsilon(t) = \begin{cases} 1 & \text{if } |t| > 2\varepsilon \\ 0 & \text{if } |t| < \varepsilon \end{cases} \qquad \text{and} \qquad |a_\varepsilon^{(j)}(t)| \le \frac{C_j}{\varepsilon^j} \tag{A.6}$$

with the $C_j > 0$ independent of $\varepsilon > 0$. We define $A_\varepsilon(t) := a_\varepsilon(\phi_\mu'(t))$, $t \in (0,\pi)$, then

$$A_\varepsilon(t) = \begin{cases} 1 & \text{if } |\phi_\mu'(t)| > 2\varepsilon\,, \\ 0 & \text{if } |\phi_\mu'(t)| < \varepsilon\,, \end{cases}$$

and let $B_\varepsilon := 1 - A_\varepsilon$. Since $\phi_\mu'(t) = \cos t - \cos\mu$ and $\phi_\mu''(t) = -\sin t$, the phase function ϕ_μ has a stationary point at μ, and ϕ_μ'' vanishes at 0 and π. Only the part of the integral near the stationary point $t = \mu \in \operatorname{supp} B_\varepsilon$ contributes to the asymptotics.

LEMMA A.6.

$$\Big|\int_0^\pi e^{\mathrm{i}r\phi_\mu(t)} A_\varepsilon(t)\,\mathrm{d}t\Big| \le \frac{2}{r\varepsilon} + \frac{\pi(2C_1 + \varepsilon)}{r\varepsilon^3}$$

Proof. We again use integration by parts,

$$\begin{aligned}\Big|\int_0^\pi \Big(e^{\mathrm{i}r\phi_\mu(t)}\mathrm{i}r\phi_\mu'(t)\Big)\frac{A_\varepsilon(t)}{\mathrm{i}r\phi_\mu'(t)}\,\mathrm{d}t\Big| &= \Big|\Big[\frac{e^{\mathrm{i}r\phi_\mu(t)}A_\varepsilon(t)}{\mathrm{i}r\phi_\mu'(t)}\Big]_{t=0}^{\pi}\Big| - \int_0^\pi e^{\mathrm{i}r\phi_\mu(t)}\Big(\frac{A_\varepsilon(t)}{\mathrm{i}r\phi_\mu'(t)}\Big)'\,\mathrm{d}t\Big| \\ &\le \frac{2}{r\varepsilon} + \frac{\pi}{r}\sup_{0\le t\le\pi}\Big|\Big(\frac{A_\varepsilon(t)}{\mathrm{i}r\phi_\mu'(t)}\Big)'\Big| \le \frac{2}{r\varepsilon} + \frac{\pi}{r}\Big(\frac{2C_1}{\varepsilon^3} + \frac{1}{\varepsilon^2}\Big)\,,\end{aligned}$$

where we have used (A.6), the fact that $|\phi_\mu'| > \varepsilon$ on $\operatorname{supp} A_\varepsilon$, and that all derivatives of ϕ_μ' are bounded by one to estimate the supremum in the last line. ∎

Finally, we quote a special case of the stationary phase lemma, theorem 7.7.5 on page 225 of [10].

LEMMA A.7. *Suppose that B and ϕ are smooth functions satisfying*

(i) μ is the unique stationary point of ϕ in $\operatorname{supp} B$,
(ii) $\operatorname{supp} B$ is a compact subset of $(0,\pi)$,
(iii) $|\phi''| \ge \delta_1 > 0$ on $\operatorname{supp} B$,

then

$$\Big|\int_0^\pi e^{\mathrm{i}r\phi(t)}B(t)\,\mathrm{d}t - e^{\mathrm{i}r\phi(\mu)}\sqrt{\frac{2\pi\mathrm{i}}{r\phi''(\mu)}}\Big| \le \frac{C}{r}\|B\|_{C^2}\,, \tag{A.7}$$

where C depends only on δ_1 and $\|\phi'\|_{C^3(\operatorname{supp} B)}$.

We need only verify the hypotheses of lemma A.7 to complete the proof of theorem A.4. We have already seen that $t = \mu$ is the unique stationary point of ϕ_μ in $(0,\pi)$. We will use the lemma below to show that, if we choose $\varepsilon < \frac{\sin^2\mu}{8}$, then the remaining two hypotheses of the stationary phase lemma A.7 are satisfied with $\delta = \frac{\sin\mu}{2}$.

LEMMA A.8. *For $\mu, t \in [0,\pi]$ and $\varepsilon_1 > 0$ we have that*

(i) if $|\cos t - \cos\mu| \le \varepsilon_1$, then $|\sin t - \sin\mu| \le \frac{2}{\sin\mu}\varepsilon_1$.

(ii) if $|\sin t - \sin\mu| > \frac{2}{\sin\mu}\varepsilon_1$, then $|\cos t - \cos\mu| > \varepsilon_1$.

Proof. The second statement is the converse of the first, and the first has been proved in [8]. ∎

If we choose $\varepsilon_1 = 2\varepsilon$, and set $t = 0$, the second item in lemma A.8 tells us that $\frac{\sin^2\mu}{4} < \varepsilon$ implies that $|1-\cos\mu| > 2\varepsilon$, i.e. $|\phi_\mu'(0)| > 2\varepsilon$, which means that $0 \notin \operatorname{supp} B_\varepsilon$. Choosing $t = \pi$ similarly implies that, with the same choice of ε, $\pi \notin \operatorname{supp} B_\varepsilon$. Thus the second hypothesis of lemma A.7 is satisfied. Finally, on $\operatorname{supp} B_\varepsilon$, $|\cos t - \cos\mu| \le 2\varepsilon$, so the first item in the lemma implies that

$$|\phi_\mu(t)''| \,=\, \sin t \,\ge\, \sin\mu - \frac{4\varepsilon}{\sin\mu} \,\ge\, \frac{\sin\mu}{2}$$

as long as $\varepsilon \le \frac{\sin^2\mu}{8}$. Thus the third hypothesis is verified with $\delta_1 = \frac{\sin^2\mu}{8}$. Just as is true for A_ε, the j'th derivative of B_ε is bounded by $\frac{C_j}{\varepsilon^j}$, so the constant C in theorem A.4 is uniform as long as $\sin\mu \ge \delta$.

The calculation for (A.3b) is analogous with (A.4) replaced by

$$(H_\nu^{(1)})'(r) \,=\, \int_\gamma \mathrm{i}\phi_\mu'(z)e^{\mathrm{i}r\phi_\mu(z)}\,\mathrm{d}z\,,$$

which has the same phase and hence the same stationary points. The only difference is that an additional factor $\mathrm{i}\sin\mu$ appears in the second term on the left-hand side of (A.7). ∎

An immediate corollary of theorem A.4 is the analogous estimate for the asymptotics of the Bessel function.

COROLLARY A.9. *Suppose that $r > \nu \ge 0$, define $\mu \in (0,\pi)$ as $\cos\mu := \frac{\nu}{r}$, and suppose that $\sin\mu \ge \delta > 0$. There exists $C = C(\delta)$ such that*

$$\left| J_\nu(r) - \sqrt{\frac{2}{\pi r\sin\mu}}\cos\Big(r(\sin\mu - \mu\cos\mu) - \frac{\pi}{4}\Big)\right| \,\le\, \frac{C}{r}\,, \tag{A.8}$$

$$\left| J_\nu'(r) + \sqrt{\frac{2}{\pi r\sin\mu}}\sin\mu\sin\Big(r(\sin\mu - \mu\cos\mu) - \frac{\pi}{4}\Big)\right| \,\le\, \frac{C}{r}\,, \tag{A.9}$$

where the constant C is independent of ν and r.

We insert (A.8) and (A.9) into the equality (A.1) to obtain, for $n \in \mathbb{N}$, $\nu = n + \frac{1}{2}$, and $\cos\mu = \frac{\nu}{r} < 1 - \delta$,

$$\left| s_n^2(R) - 2\pi\sqrt{R^2 - \Big(n + \frac{1}{2}\Big)^2}\right| \,\le\, C(\delta)\sqrt{R}\,,$$

and state the conclusion as a corollary:

COROLLARY A.10. *Let $s_{\nu R}^2(R)$ be defined for any $\nu, R \ge 0$ by (3.4a). Then*

$$\lim_{R\to\infty} \frac{s_{\nu R}^2(R)}{2\pi R} \,=\, \begin{cases} \sqrt{1-\nu^2}\,, & \nu \le 1\,,\\ 0\,, & \nu > 1\,. \end{cases}$$

Appendix B. An estimate for spherical Bessel functions. Here we give a proof of estimate (4.14). Let $\nu \in \mathbb{Z}$, $\nu > -\frac{1}{2}$ and $\mu := (2\nu+1)(2\nu+3) > 0$. Then theorem 2 of [11] establishes that for $r > \sqrt{\mu + \mu^{2/3}}/2$,

$$J_\nu^2(r) \le \frac{4(4r^2 - (2\nu+1)(2\nu+5))}{\pi((4r^2-\mu)^{3/2} - \mu)} . \tag{B.1}$$

The following lemma shows that, under the assumptions of theorem 4.3, the estimate (B.1) implies the inequality (4.14).

LEMMA B.1. *Let $M, N \ge 0$ and $r > 2(M + N + \frac{3}{2})$, then*

$$\sup_{0\le n\le M+N} j_n(r) \le \frac{b}{r} \qquad \textit{with } b \approx 1.111 .$$

Proof. Let $0 \le n \le M + N$, $\nu := n + \frac{1}{2}$, and $\eta := \sqrt{(\nu + \frac{1}{2})(\nu + \frac{3}{2})}$. Then $\frac{\mu}{4} = \eta^2 = \nu^2 + 2\nu + \frac{3}{4}$, i.e.

$$\frac{3}{4} \le \eta^2 \le (\nu+1)^2 , \tag{B.2}$$

and therefore our assumption $r > 2(M + N + \frac{3}{2})$ implies that

$$r > 2(M + N + \frac{3}{2}) \ge 2(\nu + 1) \ge 2\eta . \tag{B.3}$$

Accordingly,

$$\frac{1}{2}\sqrt{\mu + \mu^{2/3}} = \eta\sqrt{1 + \frac{1}{(4\eta^2)^{1/3}}} \le \eta\sqrt{1 + \frac{1}{3^{1/3}}} \le \sqrt{2}\eta \le \frac{r}{\sqrt{2}} \le r .$$

This shows that the assumptions of theorem 2 of [11] are satisfied.

Next we consider (B.1) and further estimate its right-hand side,

$$\begin{aligned} J_\nu^2(r) &\le \frac{4(4r^2 - (2\nu+1)(2\nu+5))}{\pi((4r^2-\mu)^{3/2} - \mu)} \le \frac{4(4r^2 - 4\eta^2)}{\pi(8(r^2-\eta^2)^{3/2} - 4\eta^2)} \\ &= \frac{2}{\pi}\frac{1}{(r^2-\eta^2)^{1/2}\Big(1 - \frac{1}{2}\frac{\eta^2}{(r^2-\eta^2)^{3/2}}\Big)} = \frac{2}{\pi}\frac{1}{r}\frac{1}{\big(1 - \big(\frac{\eta}{r}\big)^2\big)^{1/2}}\frac{1}{1 - \frac{1}{2}\frac{\eta^2}{(r^2-\eta^2)^{3/2}}} . \end{aligned}$$

Since $r > 2(M + N + \frac{3}{2}) \ge 3$, applying (B.2) and (B.3) yields

$$\frac{\eta^2}{(r^2-\eta^2)^{3/2}} = \frac{1}{r}\frac{\big(\frac{\eta}{r}\big)^2}{\big(1 - \big(\frac{\eta}{r}\big)^2\big)^{3/2}} \le \frac{1}{r}\frac{\frac{1}{4}}{\big(\frac{3}{4}\big)^{3/2}} = \frac{2}{3\sqrt{27}} ,$$

whence

$$J_\nu^2(r) \le \frac{2}{\pi}\frac{1}{r}\Big(\frac{4}{3}\Big)^{\frac{1}{2}}\frac{1}{1 - \frac{1}{3\sqrt{27}}} .$$

Recalling (4.11) this shows that

$$j_n(r) = \frac{\pi}{2r}J_\nu(r) \le \frac{b}{r} \qquad \text{with } b \approx 1.111 .$$

$\square$

Appendix C. Triple products of spherical harmonics. Finally, we prove an "orthogonality property" of triple products products of spherical harmonics that is well known to experts (and closely related to so-called Wigner 3-j symbols or Clebsh-Gordan coefficients) but that we haven't been able to find in the literature.

LEMMA C.1. *Let $l, m, n \in \mathbb{N}$, $\alpha_l \in \mathbb{Y}_l$, $\beta_m \in \mathbb{Y}_m$, and $\gamma_n \in \mathbb{Y}_n$. If $l > m+n$ or $l < |m-n|$, then*

$$\int_{S^2} \alpha_l(\theta)\beta_m(\theta)\gamma_n(\theta)\,\mathrm{d}\theta \;=\; 0\,. \tag{C.1}$$

Proof. In the following we denote for any $j \in \mathbb{N}$ by $\mathbb{H}_j$ the space of homogeneous polynomials of degree j in 3 dimensions. By definition the spherical harmonics $\beta_m \in \mathbb{Y}_m$ and $\gamma_n \in \mathbb{Y}_n$ extend to homogeneous harmonic polynomials $B_m \in \mathbb{Y}_m(\mathbb{R}^3)$ and $G_n \in \mathbb{Y}_n(\mathbb{R}^3)$, respectively. Accordingly, the product $B_m G_n$ is a homogeneous polynomial of degree $m+n$, i.e., $B_m G_n \in \mathbb{H}_{m+n}$. Since

$$\Bigl(\sum_{j=0}^{m+n} \mathbb{H}_j\Bigr)\Big|_{S^2} \;=\; \bigoplus_{j=0}^{m+n} \mathbb{Y}_j$$

(cf. [1, corollary 2.19]), we obtain (C.1) for $l > m+n$, because $\mathbb{Y}_l \perp \bigoplus_{j=0}^{m+n} \mathbb{Y}_j$. Finally, permuting the roles of l, m and n yields (C.1) for $l < |m-n|$. ∎

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