Fourier Coefficients of Far Fields and the Convex Scattering Support

John Sylvester

University of Washington

Collaborators:

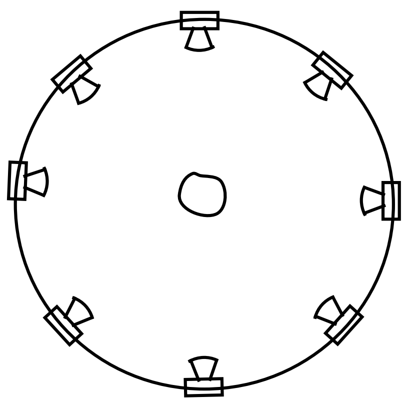
- Steve Kusiak
- Roland Potthast
- Jim Berryman

Gratefully acknowledge research support from ONR and NSF

Outline

- Fixed Frequency Scattering and Inverse Scattering
- Singular Values of Homogeneous Far Field Operator associated with a ball.
  - Source in a Ball iff Fourier Coefficients in a Box
  - Translation Formula for Fourier Coefficients of Far Fields
- Applications to Array Imaging – locating sources or scatterers
  - Algorithm to find a small scatterer
  - Find the convex hull of a big scatterer
  - Split the far fields of well separated scatterers
  - Analyze and enhance the focusing of TR eigenfunctions

Passive Remote Sensing



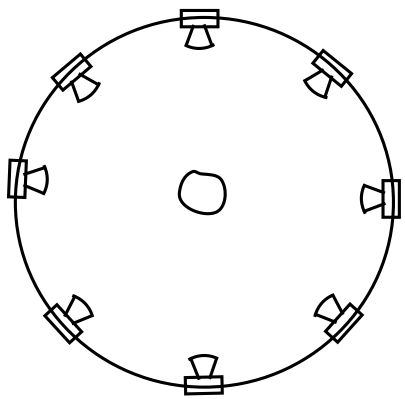
$$u_{tt} - \Delta u = F(x,t)$$

$$u(x,0) = 0$$

$$u_t(x,0) = 0$$

- The source radiates
- No Illumination – zero initial conditions
- Sensor Array is many wavelengths away
- Full (or partial) Aperture Observations
- Data measured by the sensor is called the Far Field
- Only one Far Field to measure

Active Remote Sensing



$$n^2(x)u_{tt} - \Delta u = 0$$

$$u_{tt} - \Delta u = (1 - n^2(x))u_{tt}$$

$$u(x, 0) = g(\theta)\delta(t - |x| - r_0)$$

$$u_t(x, 0) = g(\theta)\delta'(t - |x| - r_0)$$

- Incoming wave illuminates – incoming initial conditions
- Scatterer becomes an Induced Source
- Many Far Fields – one for each illuminating wave
- Nonlinear effects from strong scatterers
- Passive Location Algorithms apply directly

Time Harmonic Model

$$\begin{aligned} F(x,t) &= e^{i\omega t}F(x) \\ u(x,t) &= e^{i\omega t}u(x) \end{aligned}$$

Wave equation becomes Helmholtz equation ($k = n_0\omega$)

$$\left(\Delta + k^2\right) u = F(x,k)$$

- $G_0^+ F$ is the outgoing wave radiated by the source F.

Outgoing Waves and Far Fields

$$
\begin{aligned}
u(x,k) &= G_0^+ F \\
&\sim \frac{e^{ikr}}{r^{\frac{n-1}{2}}}\alpha(\Theta) \\
\mathcal{F}_0^+ F &:= \alpha(\Theta) = \widehat{F}(k\Theta)
\end{aligned}
$$

- $G_0^+ F$ is the outgoing wave radiated by the source F.
- $\mathcal{F}_0^+ F$ is the far field radiated by F.
- We can calculate $\mathcal{F}_0^+ F$, and it turns out to be the Fourier Transform of F, restricted to to the sphere of radius k.

The (Relative) Scattering Operator

$$\left(\Delta + k^2\right) u = qu \qquad\qquad (= k^2(n^2 - 1)u)$$

$$u_-^\infty = \beta(\Theta)$$

$$\mathcal{S}_q\beta := u_+^\infty - \beta(\Theta)$$

- u_-^∞ is the incoming far field and u_+^∞ is the outgoing far field.
- $\mathcal{S}_q$measures the difference between the outgoing far field you see and the one you would have seen if nothing were there.
- If the aperture is less than π, you don't see the $-\beta(-\Theta)$.
- ${\mathcal{F}_0^+}^* \mathcal{F}_0^+$ and $\mathcal{S}_q^*\mathcal{S}_q$ are far field versions of the time reversal operator.
- $\left(\mathcal{S}_q^*\mathcal{S}_q\right)^{\frac{1}{4}}$ is the Linear Sampling Operator

Singular Value Decomposition

The Far Field operator has a singular value decomposition

$$\mathcal{F}_q^+\Big|_{B_c(R)} = \sum \sigma_n \Phi_n \otimes \Psi_n$$

A source sits inside a ball <u>iff</u> the Fourier coefficients

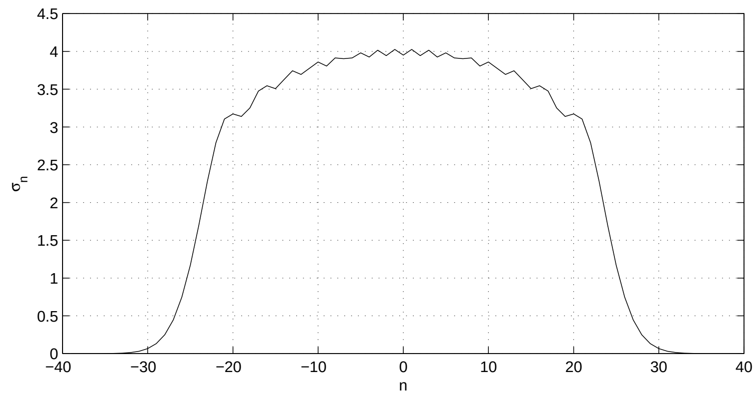
$$\left|\left(\mathcal{F}_q^+ F, \Phi_n\right)\right| \leq C_q \sigma_n(R)$$

For the homogeneous background, we can compute by hand

$$\begin{aligned}
\Phi_n &= e^{in\theta} \\
\Psi_n^c &= \chi_{kR_c} \frac{J_n(kr_c)e^{in\phi_c}}{\sigma_n} \\
\sigma_n^2(R) &= \int_0^R J_n^2(kr) r dr
\end{aligned}$$

A Box For the Fourier Coefficients

$\sigma_n(25)$



n

$$\sigma_n(R) = \left(\int_0^R J_n^2(kr)r\,dr\right)^{\frac{1}{2}}$$

$$\sim \begin{cases} ((kR)^2 - n^2)^{\frac{1}{4}} & |n| < kR \\ \frac{1}{\sqrt{n+\frac{1}{2}}}\left(\frac{ekR}{2(n+\frac{1}{2})}\right)^{(n+\frac{1}{2})} \sim 0 & |n| > kR \end{cases}$$

Rapid Transition to Evanescence

The homogeneous background exhibits a *rapid transition to evanescence.*

- $\sigma_n(R)$ uniformly big if $n < R$
- $\sigma_n(R)$ uniformly small if $n > R$
- Uniform contrast between big and small

Where is the Origin ?

All source points are effectively equidistant from the far field. Thus we can translate the origin c with a mathematical formula. You don't have to move the array.

$$\mathcal{F}_0^+ \left[F(x-c)\right] = e^{ik|c|\cos(\theta-\theta_c)}\alpha(\theta)$$

Fourier Coefficients of the Far Field

$$\mathcal{F}_0^+ F = \alpha(\theta) = \sum \alpha_n^0 e^{in\theta}$$

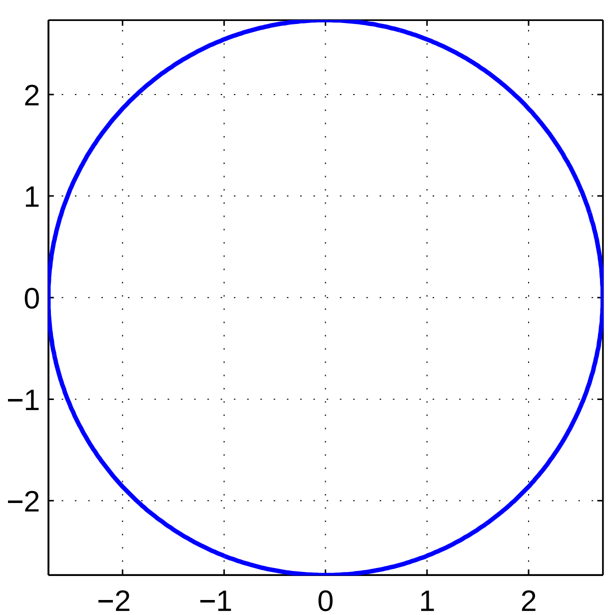
$$\mathcal{F}_0^+ \left[F(x-c)\right] = e^{ik|c|\cos(\theta-\theta_c)}\alpha(\theta) = \sum \alpha_n^c e^{in\theta}$$

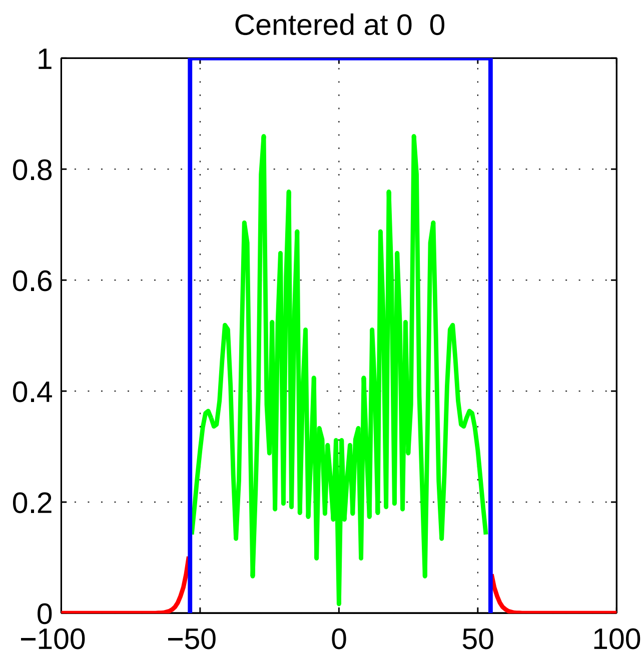
Rule for Locating A Source

- The α_n^0 are *zero* for $|n| > kR_0$ iff there is a source inside $B_{R_0}(0)$ that radiates α.
- The α_n^c are *zero* for $|n| > kR_c$ iff there is a source inside $B_{R_c}(c)$ that radiates α.
- If both conditions above are satisfied, there is a source inside the intersection of the two balls.

Finding a Source Triangle (k = 20)

A single layer source on a triangle (about 4 wavelengths on a side) is somewhere in a 3x3 box. We plot the modulus or the Fourier coefficients of Far Field it radiates on the right, measure the width, and draw the circle of radius W/k on the left.





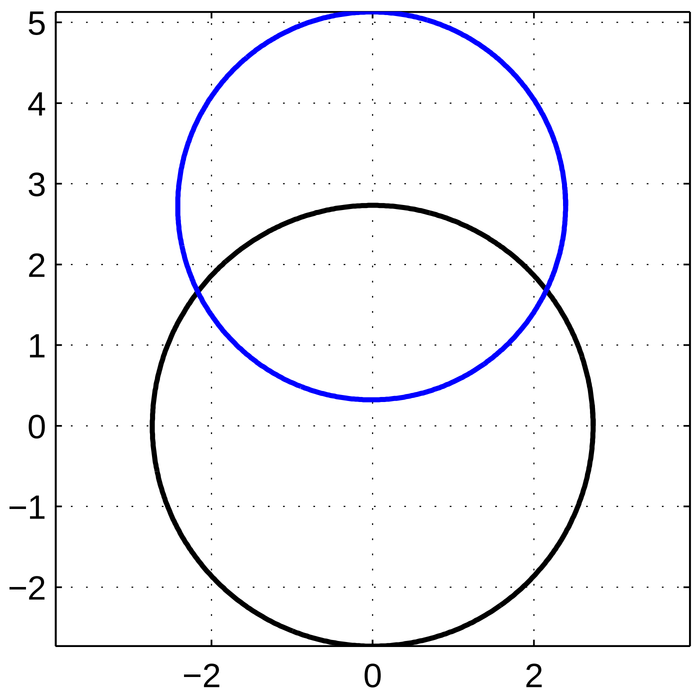
Ball of radius 55/20 about (0,0)

Modulus of α_n versus n

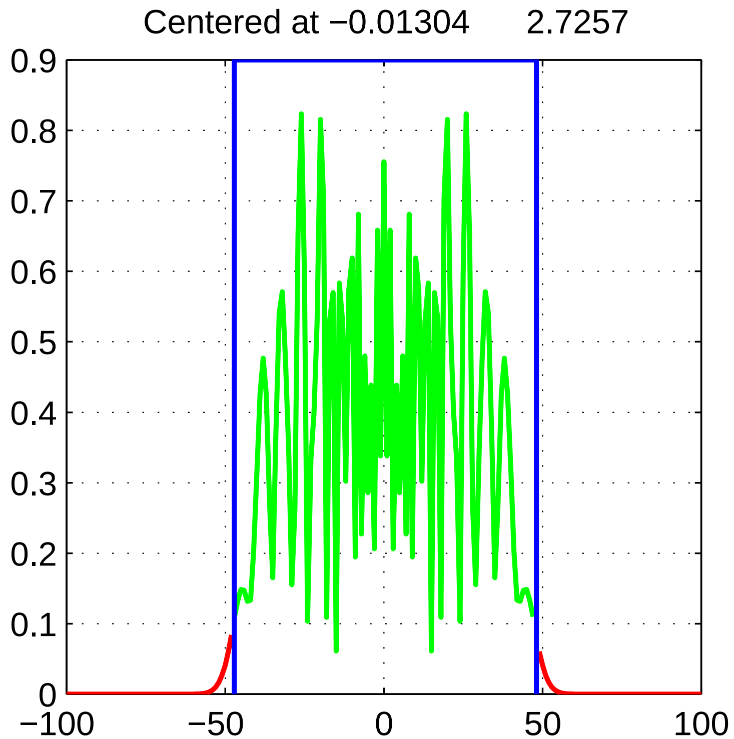
Choose a new center at the top of the old circle.

Plot the translated Fourier Coefficients.

Measure the width of the box and draw the new circle.



Ball of radius 49/20 about (0,2.7)

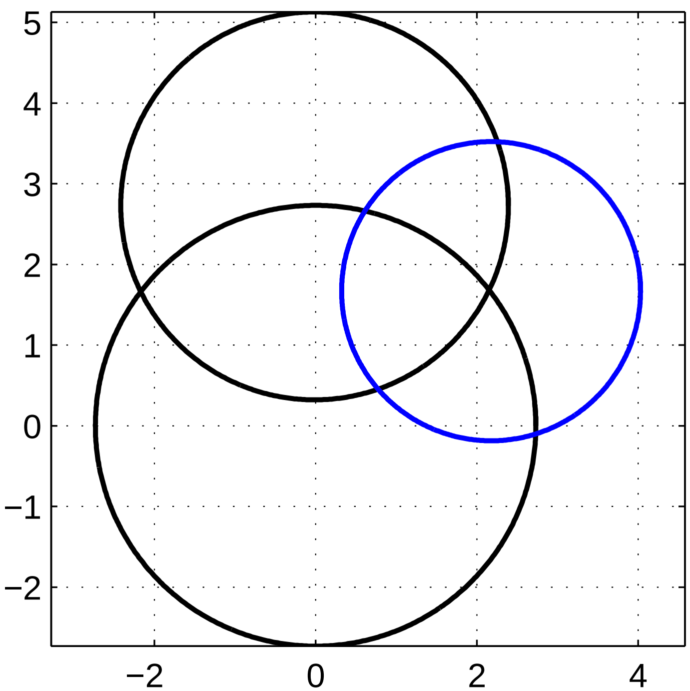


Modulus of α_n^c versus n

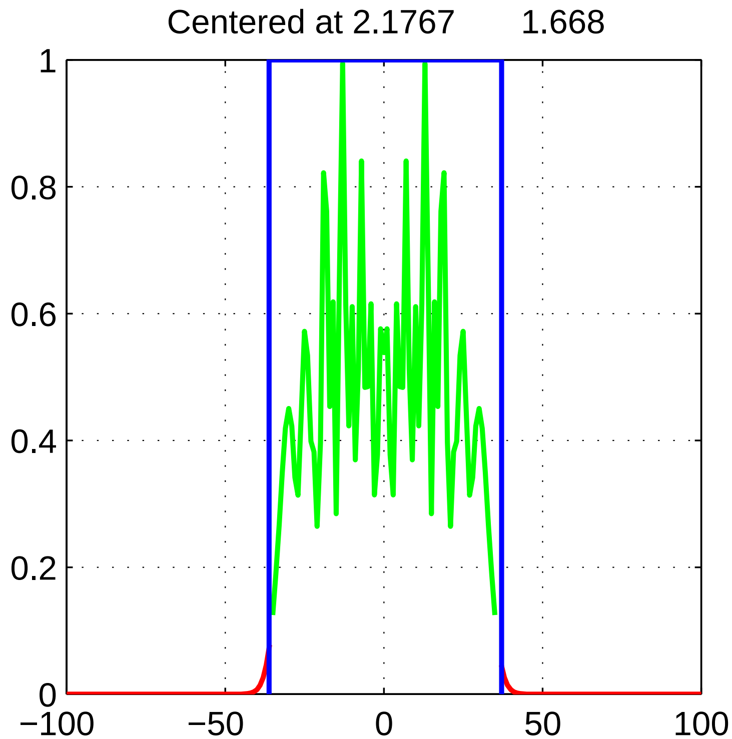
Choose a new center at the intesection of the old circles.

Plot the translated Fourier Coefficients.

Measure the width of the box and draw the new circle.

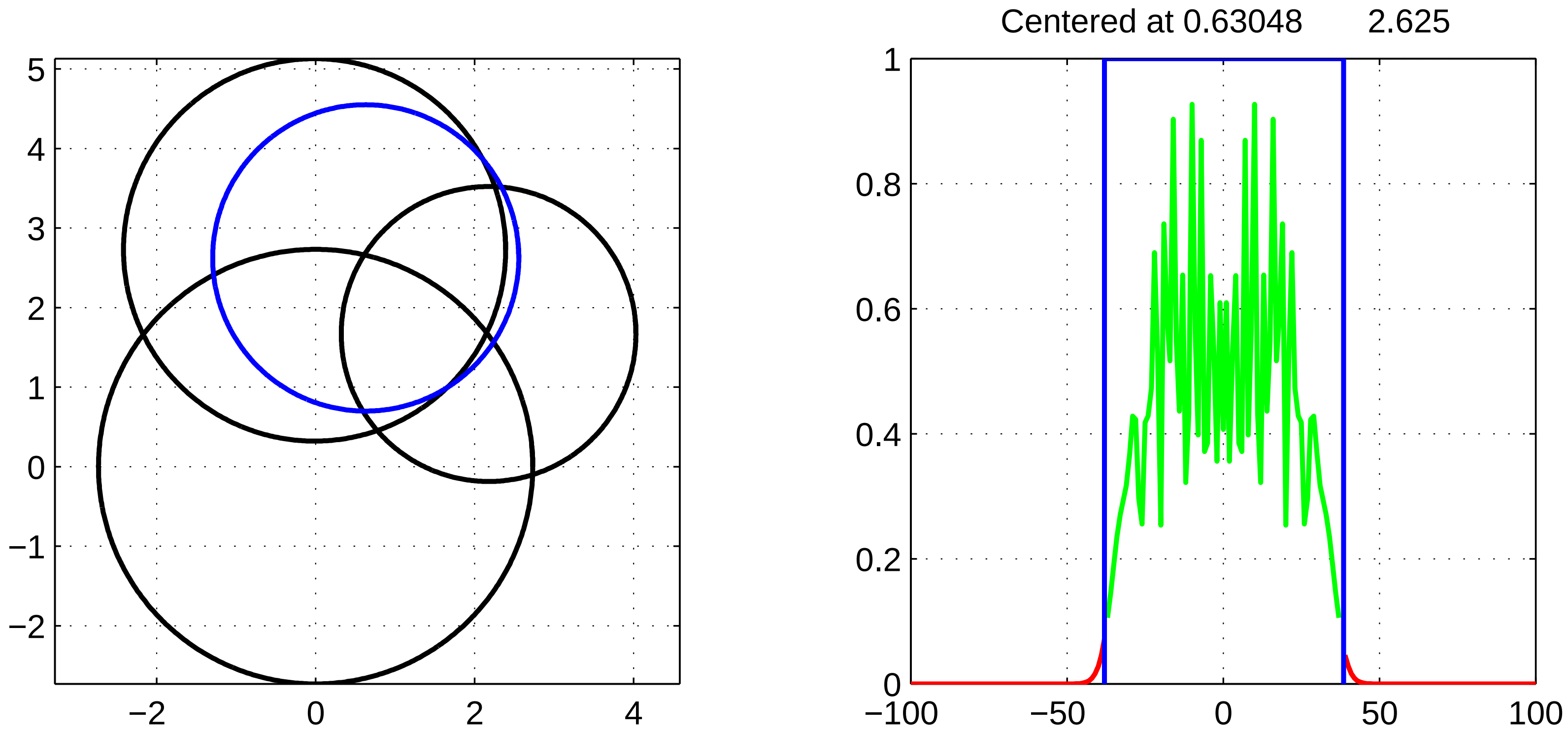


Ball of radius 36/20 about (2.2,1.6)



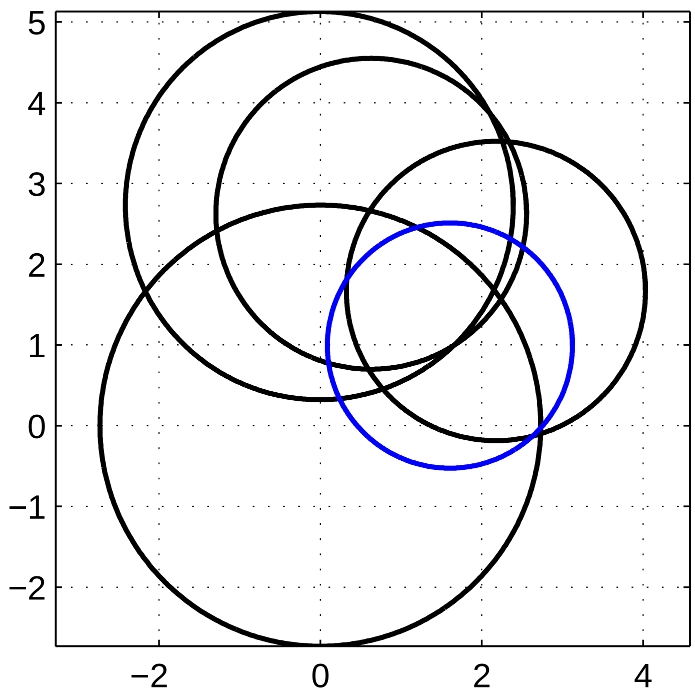
Modulus of α_n^c versus n

Translated Far Field (k = 20)

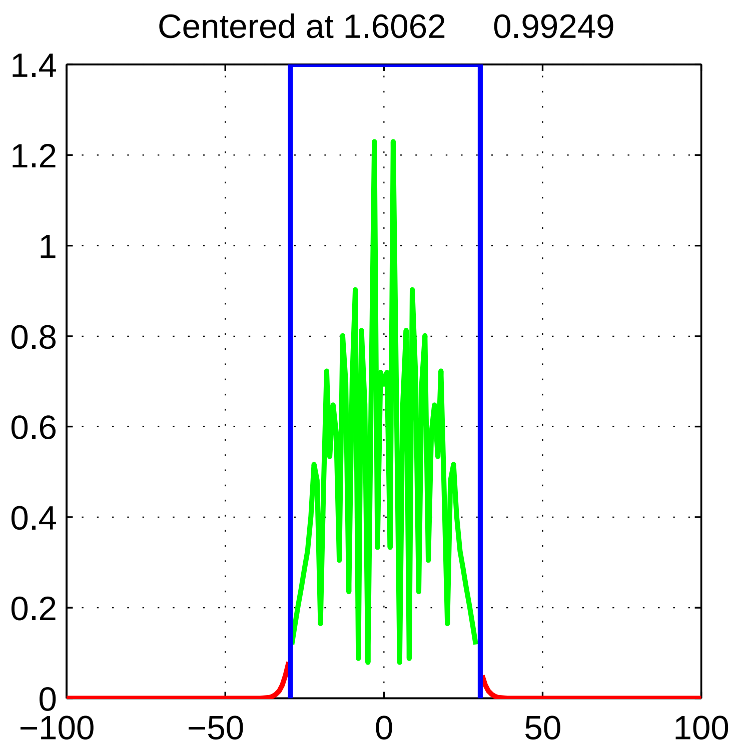


Ball of radius 40/20 about (0.6,2.6) Modulus of α_n^c versus n

Translated Far Field (k = 20)

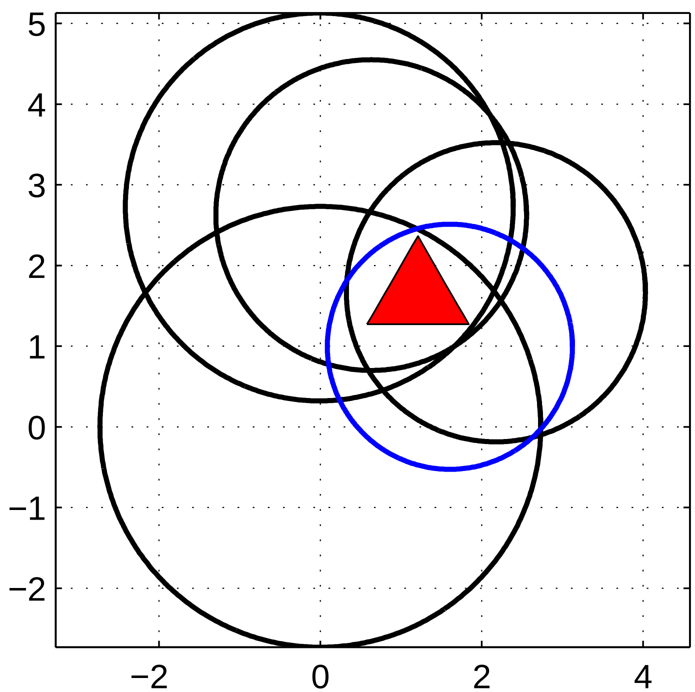


Ball of radius 30/20 about (1.6,1)

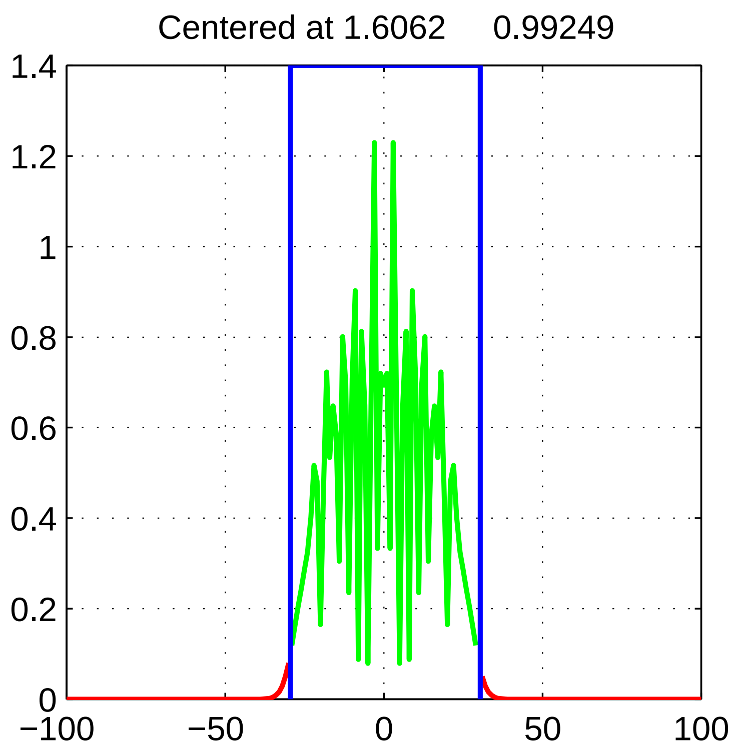


Modulus of α_n^c versus n

Translated Far Field (k = 20)



The source triangle



Modulus of α_n^c versus n

Confessions

- We can only find convex hulls by intersecting balls.
- We are actually finding the *Convex Scattering Support* of the far field, which can be smaller than the convex hull. Its impossible to do better if you measure only one far field.
  - **A point source and a spherically symmetric source produce the same far fields**.
  - **Its impossible to find an upper bound** using observations of a single (or finitely many) far fields.
  - **The convex hull of the true source must contain the** *Convex Scattering Support*.
  - **The** *Convex Scattering Support* **isn't too small.** There is always a source that fits inside that radiates the far field. Corners and components are always visible.

Narrow Boxes, High Amplitudes

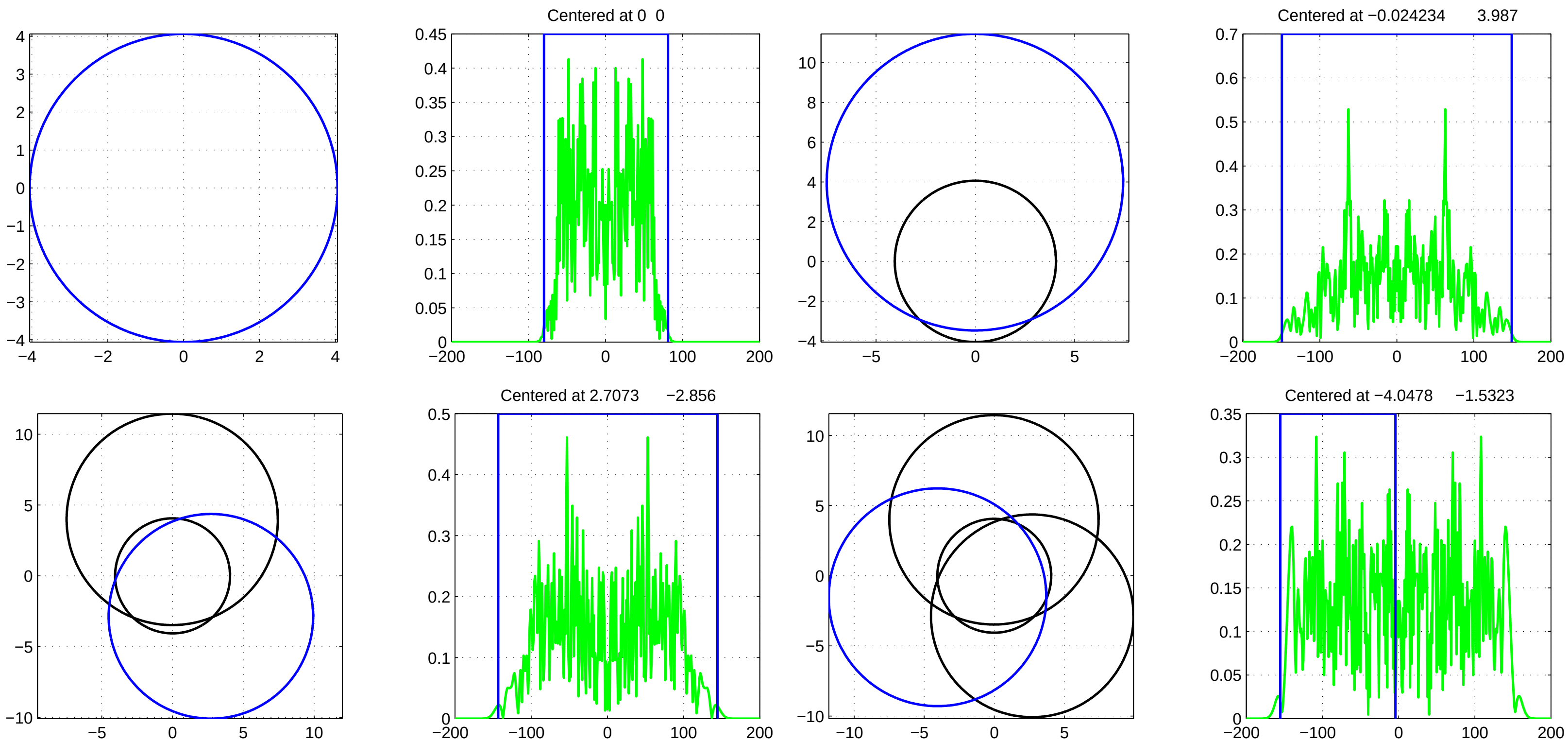
$$\mathcal{F}_0^+ F = \alpha(\theta) = \sum \alpha_n^0 e^{in\theta}$$

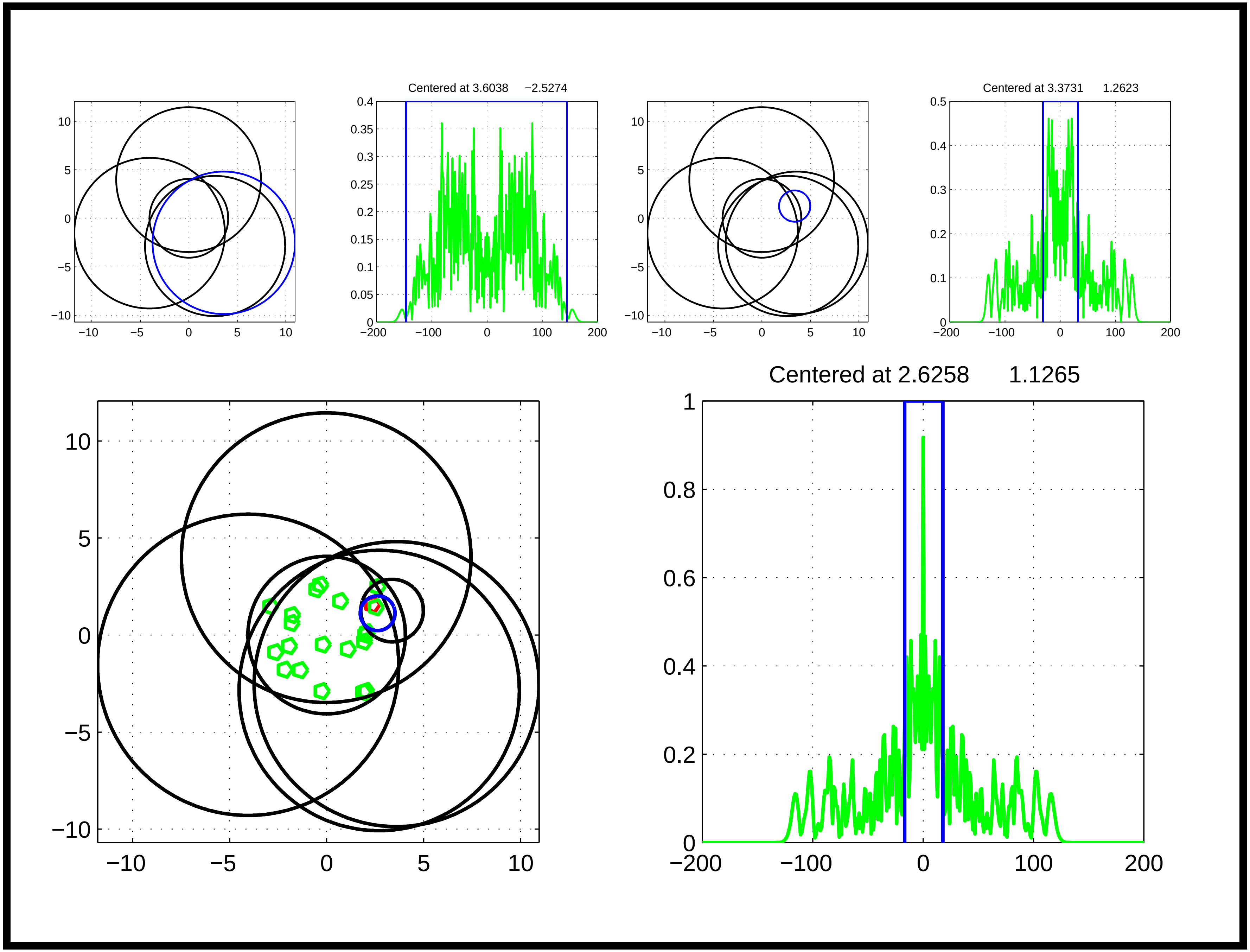
$$\mathcal{F}_0^+ \left[F(x-c)\right] = e^{ik|c|\cos(\theta-\theta_c)}\alpha(\theta) = \sum \alpha_n^c e^{in\theta}$$

- The translation operator is L^2 unitary
- As the center gets closer to the center of the source, the Fourier coefficients get squeezed into a narrow box.
- If you squeeze the Fourier coefficients into a narrow box, they get higher.
- High enough to stand out from a background.

A strong source among 20 weaker sources (20% strength)

21 pentagons (radius = 0.4) with point sources on the corners somewhere in 3x3 square.

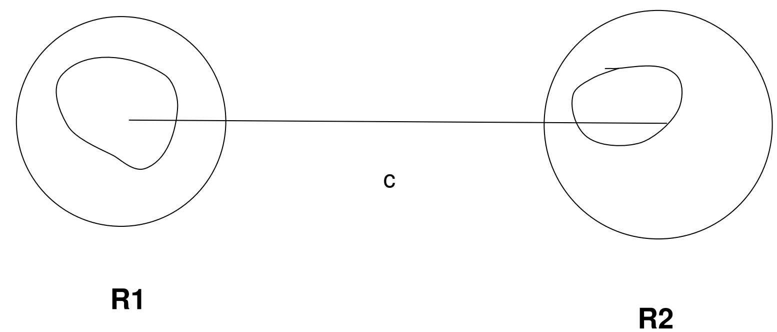



Centered at 3.6038 −2.5274
Centered at 3.3731 1.2623
Centered at 2.6258 1.1265

Other Applications of the Box

Splitting the Far Fields of two sources

$$\begin{aligned}\Phi &= \mathcal{F}_0^{+} F_1 + \mathcal{F}_0^{+} F_2 \\ &= \alpha_1 + \alpha_2\end{aligned}$$



Build a projection operator

- Translate to center c_1
- Cut out modes $n > kR_1$
- Translate back

$$P_1 = \quad T_{-c_1} \quad \chi_{n<kR_1} \quad T_{c_1}$$

Splitting the Far Fields of two sources

$$\alpha_1 + \alpha_2 = \Phi$$

$$\begin{aligned} \alpha_1 &= P_1\Phi - P_1\alpha_2 \\ \alpha_2 &= P_2\Phi - P_2\alpha_1 \end{aligned}$$

$$(I - P_1P_2)\,\alpha_1 = P_1\,(I - P_2)\,\Phi$$

We can represent P_1P_2 is a $2kR_1 \times 2kR_2$ toeplitz matrix whose entries are Bessel functions evaluated at $k|c| = k|c_1 - c_2|$. We can use this representation to compute and to estimate the stability of the inverse, at least when the scatterers are far apart.

$$||P_1P_2||^2 \leq k\frac{R_1R_2}{c}$$

Fourier Coefficients of $\mathcal{S}_q$ Fit in a Square

If q fits in a ball,

$$(\mathcal{S}_q)_n^m \leq C_q \sigma_n(R) \sigma_m(R)$$

- Conclusions persist in the presence of multiple scattering.
  Every reflected wave has a last bounce (and a first bounce).

Why its so ...

	Send it back	bounce it around	Send it in
$\mathcal{S}_q\beta =$	$\mathcal{F}_0^+$	$\left(I - qG_0^+\right)^{-1} q$	$\mathcal{H}\beta$
$\mathcal{S}_q\beta =$	$\mathcal{F}_0^+$	$q\left(I - G_0^+ q\right)^{-1}$	$\mathcal{H}\beta$
$\mathcal{S}_q\beta =$	$\left(\mathcal{F}_0^+ \chi_{\mathrm{supp}q}\right)$	$q\left(I - G_0^+ q\right)^{-1}$	$\left(\mathcal{F}_0^+ \chi_{\mathrm{supp}q}\right)^* \beta$

Eigenfunctions Fit in the Same Box

$$(\mathcal{S}_q)_n^m \leq C_q \sigma_n(R) \sigma_m(R)$$

$$e_n^\lambda \leq C_q \sigma_n(R)$$

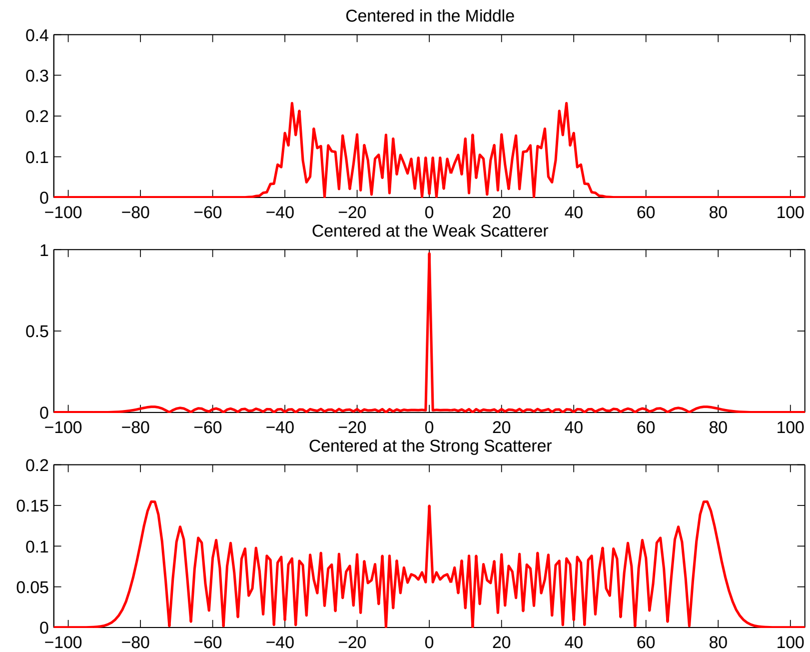
so location, splitting, and projection methods apply here as well. In particular, For separated scatterers,

$$|e_{q_1} \cdot e_{q_2}| \leq \sqrt{k} \frac{\min(R_1, R_2)}{\sqrt{c}}$$

Thus the eigenfunctions of the one scattering operator are nearly orthogonal to those of the another scattereirng operator if the scatterers are separated. Via perturbation theory, the eigenfunctions of the sum are nearly the union of the eigenvectors.

Translated Fourier Coefficients of an Eigenfunction

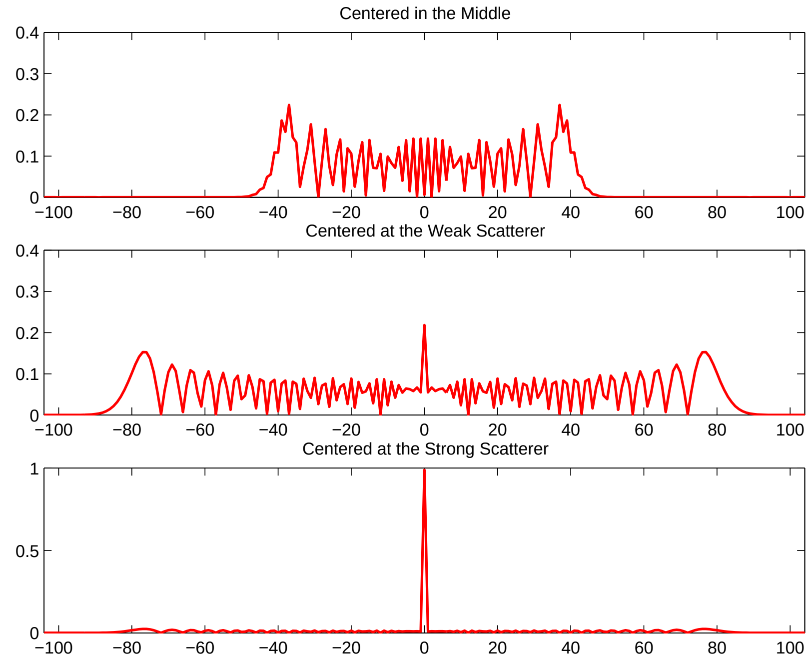
- Centered in the middle
- Centered at the strong scatterer
- Centered at the weak scatterer



- Born Approximation for two point scatterers ($k = 10$)
- One scatterer is 20% stronger
- Eigenfunction with largest eigenvalue

Translated Coefficients of Second Eigenfunction

- Centered in the middle
- Centered at the strong scatterer
- Centered at the weak scatterer



- Time Reversal Eigenfunctions localize in space
- Formulas analyze relation between individual scatterers and eigenfunctions in a simple way
- Plots of translated eigenfunctions show localization on the Fourier side.

Summary

- Narrow Box Principle gives a new way to probe for location and to analyze other aspects of localization on the other side of the Fourier Transform.
- No hypothesis about sources or scatterers.
- Algorithms contract down from the outside rather than sampling from the inside.
- Only convex hulls from single far field.
- Some indications we can work with **limited aperture**.
- Translation formula makes essential use of the Far Field.