INVERSE BOUNDARY VALUE PROBLEMS
AN OVERVIEW

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In this talk, we present an overview of some recent progress in inverse problems. We shall discuss several time independent problems. In general we consider a bounded region in $\mathbb{R}^n$, which represents a body, and a partial differential equation, or system of equations which represents the physics inside the body. The problem is to deduce the interior physical parameters—coefficients of the differential equation—from measurements made at the boundary of the region. We begin with the simplest example.

Schrödinger Equation in $\Omega \subset\subset \mathbb{R}^n$.

$$(\Delta + q)u = 0 \tag{1}$$

The physical parameter to be determined is the potential q. The boundary measurements are represented by the Cauchy Data, $\mathcal{C}_q$.

$$\mathcal{C}_q = \{(f,g) \mid f = u|_{\partial\Omega}\,,\, g = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}, u \text{ satisfies (1)}\} \tag{2}$$

The basic tool we use to relate the Cauchy Data to the coefficients is an orthogonality relation, which we derive below. Suppose that

$$\begin{aligned} (\Delta + q_1)u_1 &= 0 \qquad & (\Delta + q_2)u_2 &= 0 \\ u_1|_{\partial\Omega} &= f & u_2|_{\partial\Omega} &= g \end{aligned} \tag{3}$$

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then

$$0 = \int_\Omega u_2(\Delta + q_1)u_1$$

$$0 = -\int_\Omega \nabla u_2 \nabla u_1 + q u_1 u_2 + \int_{\partial\Omega} u_2 \frac{\partial u_1}{\partial \nu} \tag{4}$$

If we skew symmetrize (4), i.e., permute the indices 1 and 2 and then subtract the new equation from (4) we obtain

$$\begin{aligned} \int_\Omega (q_1 - q_2)u_1 u_2 &= \int_{\partial\Omega} u_2 \frac{\partial u_1}{\partial \nu} - u_1 \frac{\partial u_2}{\partial \nu} \\ &= \int_{\partial\Omega} g \frac{\partial u_1^f}{\partial \nu} - f \frac{\partial u_2^g}{\partial \nu} \end{aligned} \tag{5}$$

where we denote by u_i^f the solution to the Schrödinger equation with potential q_i and Dirichlet data f. Setting $q_1 = q_2 = q$ in (5) gives the symmetry relation

$$\int_{\partial\Omega} f \frac{\partial u^g}{\partial \nu} = \int_{\partial\Omega} g \frac{\partial u^f}{\partial \nu} \tag{6}$$

Making use of (6) we may rewrite (5) as

$$\int_{\partial\Omega} f \frac{\partial u_2^g}{\partial \nu} - f \frac{\partial u_1^g}{\partial \nu} = \int_\Omega (q_1 - q_2)u_1 u_2$$

If we assume that $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$, then

$$0 = \int_\Omega (q_1 - q_2)u_1 u_2 \tag{7}$$

The relation (7) is a variation of a formula first proposed in [C], the form above first appeared in [A].

The next step is to construct some special exponentially increasing solutions to (3). We notice that, for $\zeta \in \mathbb{C}^n$

$$\begin{aligned} \Delta e^{x\cdot\zeta} = 0 &\Longleftrightarrow \zeta \cdot \zeta = 0 \\ &\Longleftrightarrow \begin{cases} \mathrm{Re}\,\zeta \cdot \mathrm{Im}\,\zeta = 0 \\ |\mathrm{Re}\,\zeta| = |\mathrm{Im}\,\zeta| \end{cases} \end{aligned}$$

The following theorem is from [Sy-U]:

Theorem. *For $|\zeta|$ sufficiently large with $\zeta \cdot \zeta = 0$, there exist unique solutions to (1) of the form*

$$u = e^{x\cdot\zeta}\left(1 + O\left(\frac{1}{|\zeta|}\right)\right) \tag{8}$$

Here, $O\left(\frac{1}{|\zeta|}\right)$ stands for a smooth function of x and ζ which decays like $\frac{1}{|\zeta|}$ as $|\zeta|$ tends to infinity. We will choose u_1 and u_2 satisfying (3) to be of the form (8) with

$$\zeta_1 = l + i(k+m)$$
$$\zeta_2 = l - i(k-m)$$

where l, k, and m are real, mutually orthogonal and satisfy $|l|^2 = |k|^2 + |m|^2$. Note that this is only possible in dimension ≥ 3. Inserting these solutions into (7) gives

$$0 = \int_\Omega (q_1 - q_2) e^{2ix\cdot k}\left(1 + O\left(\frac{1}{|m|}\right)\right)$$

Letting $|m| \to \infty$ implies that the Fourier transform of q, and therefore q, is zero, which implies

Theorem.

$$\mathcal{C}_{q_1} = \mathcal{C}_{q_2} \Rightarrow q_1 = q_2$$

This theorem was first proved in [Sy-U] for $n \geq 3$. If q is a priori known to be positive, the same result in dimension 2 follows from a result of Nachman [N1].

2. Impedance Tomography.

$$d\gamma du = 0 \tag{10}$$

In the impedance tomography equation, u represents a voltage potntial, du, an electric field, and γdu the current flux (a 2-form). The conductivity, γ, is represented as a map from 1-forms to 2-forms. The Cauchy data is given by

$$\mathcal{C}_\gamma = \{(u, \gamma du)|_{\partial\Omega}\} = \left\{ \left(u, \left(\gamma^{ij}\nu_j \frac{\partial u}{\partial x_i}\right) dS\right)\Big|_{\partial\Omega} \right\} \tag{11}$$

where the last term on the right of (11) is just the expression for $\gamma du|_{\partial\Omega}$ in local coordinates. To derive the orthogonality relations we write

$$(12)\qquad \begin{aligned} 0 &= \int_\Omega u_2 d\gamma_1 du_1 \\ 0 &= -\int_\Omega du_2\gamma_1 du_1 + \int_{\partial\Omega} u_2\gamma_1 du_1 \end{aligned}$$

Skew symmetrizing (12), we find that

$$(13)\qquad \int_{\partial\Omega} u_1\gamma_2 du_2 - u_2\gamma_1 du_1 = \int_\Omega du_2(\gamma_1 - \gamma_2)du_1$$

so that, if $\mathcal{C}_{\gamma_1} = \mathcal{C}_{\gamma_2}$, we have

$$(14)\qquad 0 = \int_\Omega du_2(\gamma_1 - \gamma_2)du_1$$

The equations (10) can be shown to allow exponentially growing solutions just as the Schrödinger equation did. Inserting these solutions into (14) gives the following result.

Theorem. *Suppose that γ_1 and γ_2 are isotroppic, i.e. γ_1 and γ_2 can be written as scalar functions times the identity matrix. Then*

$$\mathcal{C}_{\gamma_1} = \mathcal{C}_{\gamma_2} \Longrightarrow \gamma_1 = \gamma_2$$

This theorem appeared in [Sy-U] for $n \geq 3$ and [N1] for $n = 2$.

If γ_1 and γ_2 are not isotropic, and Ψ is a diffeomorphism of Ω which is the identity on $\partial\Omega$, then

$$\Psi_*\gamma = \frac{D\Psi^T\gamma D\Psi}{\det D\Psi} \circ \Psi^{-1}$$

is a new conductivity which gives the same Cauchy data as γ. This suggests the following

Conjecture.

$$\mathcal{C}_{\gamma_1} = \mathcal{C}_{\gamma_2} \Longleftrightarrow \textit{there exists } \Psi \textit{ such that} \begin{cases} \Psi : \Omega \to \Omega \\ \Psi|_{\partial\Omega} = I \\ \Psi_*\gamma_1 = \gamma_2 \end{cases}$$

For $n = 2$, this conjecture is proved as a consequence of [N] and [Sy]. For $n \geq 3$, the conjecture is only known to be true if γ_1 and γ_2 are real analytic [L-U].

3. Time Harmonic Maxwell's Equations.

$$
\begin{cases} dE = i\omega\mu H \\ dH = -i\omega\epsilon E \quad ; \epsilon = \left(\widetilde{\epsilon} + \frac{i\sigma}{\omega}\right) \end{cases} \tag{15}
$$

In (15), we represent E and H as 1-forms, μ, $\widetilde{\epsilon}$, and σ are real symmetric ($\alpha_\wedge \mu\beta = \mu\alpha_\wedge\beta$) maps from 1-forms to $n-1$ forms. The Cauchy Data is just

$$
\mathcal{C}_{\mu,\epsilon} = \{\, (E,H)|_{\partial\Omega}\}
$$

The orthogonality relation is derived as follows:

$$
\begin{aligned}
\frac{1}{i\omega}\int_{\partial\Omega} E_1 \wedge H_2 &= \frac{1}{i\omega}\int_{\Omega} d(E_1 \wedge H_2) \\
\frac{1}{i\omega}\int_{\partial\Omega} E_1 \wedge H_2 &= \int_{\Omega} \mu_1 H_1 \wedge H_2 + E_1 \wedge \epsilon_2 E_2
\end{aligned} \tag{16}
$$

Skew symmetrizing (16) gives

$$
\frac{1}{i\omega}\int_{\partial\Omega} E_1 \wedge H_2 - E_2 \wedge H_1 = \int_{\Omega} H_1 \wedge (\mu_1 - \mu_2) H_2 + E_1 \wedge (\epsilon_1 - \epsilon_2) E_2 \tag{17}
$$

In the case where both μ and ϵ are isotropic, Ola, Parvarinta, and Sommersalo produced exponentially growing solutions to (15) in [O-P-S] and combined them with (17) to prove

Theorem. *Suppose that $\omega \neq 0$ and that $(\mu_i, \widetilde{\epsilon}_i, \sigma_i)$ are isotropic. Then*

$$
\mathcal{C}_1 = \mathcal{C}_2 \Longrightarrow (\mu_1, \widetilde{\epsilon}_1, \sigma_1) = (\mu_2, \widetilde{\epsilon}_2, \sigma_2)
$$

The anisotropic problem is completely open, although some linearized calculations may be found in [Sy3].

4. Linear Elasticity.

Let the displacement, u, be denoted by a vector field and its jacobian be a section of $\Lambda^1 \otimes V$, one forms tensor vector fields. Then the strain is defined by

$$
\epsilon(u) = Du + Du^*
$$

where $*$ denotes the adjoint—we use the euclidean metric. The stress is linearly related to the strain by

$$T(u) = C\epsilon(u) \tag{18}$$

In (18) the physical parameters are encoded in C which is a linear map from $\Lambda^1 \otimes \mathrm{Sym}(V)$ to $\Lambda^2 \otimes \mathrm{Sym}(V)$. With this notation, the equations of elasticity become

$$DT = 0 \tag{19}$$

The Cauchy data is just

$$\mathcal{C}_C = \{\, (u, T(u))|_{\partial\Omega}\}$$

The orthogonality relations can be obtained as in the previous example

$$\begin{aligned}\int_{\partial\Omega} u_1 T(u_2) &= \int_\Omega d(u_1 \cdot T(u_2)) \\ &= \int_\Omega Du_1 \cdot T(u_2) + u_1 \cdot DT(u_2) \\ \int_{\partial\Omega} u_1 T(u_2) &= \int_\Omega Du_1 \cdot T(u_2) \qquad (20)\end{aligned}$$

Skew symmetrizing (20) gives

$$\int_{\partial\Omega} u_1 T(u_2) - u_2 T(u_1) = \int_\Omega Du_1 T(u_2) - Du_2 T(u_1) \tag{21}$$

For isotropic linear elasticity, Nakamura and Uhlmann [N-U] have produced exponentially growing solutions to (19).

The isotropy hypothesis takes the form

$$T = \mu\epsilon + \lambda \mathrm{tr}(\epsilon I)$$

where μ and λ are the scalar Lamé parameters. In this case, if $\mathcal{C}_1 = \mathcal{C}_2$, (21) takes the form

$$0 = \int_\Omega (\mu_1 - \mu_2)\epsilon(u_1) \cdot \epsilon(u_2) + (\lambda_1 - \lambda_2)(\mathrm{d}ivu_1)(\mathrm{d}ivu_2)$$

The theorem proved in [N-U] is

Theorem.

$$\mathcal{C}_1 = \mathcal{C}_2 \Longrightarrow (\mu_1, \lambda_1) = (\mu_2, \lambda_2)$$

5. Schrödinger Equation with Magnetic Potential.

$$(d + iA) * (d + iA)u + qu = 0 \tag{22}$$

In (22), A is 1-form (vector potential) and $*$ represents the euclidean star operator. The Cauchy data is

$$\mathcal{C}_{A,q} = \{ (u, (d + iA)u)|_{\partial\Omega} \}$$

The orthogonality relations are obtained by

$$\begin{aligned} \int_{\partial\Omega} u_2 * (d + iA_1)u_1 &= \int_\Omega du_2 * (d + iA_1)u_1 + u_2 d * (d + iA_1)u_1 \\ \int_{\partial\Omega} u_2 * (d + iA_1)u_1 &= \int_\Omega du_{2\wedge} * du_1 \\ &\qquad + i * A_{1\wedge}(u_2 du_1 - u_1 du_2) + (A_{1\wedge} * A_1 - q)u_1 u_2 \end{aligned} \tag{23}$$

Skew symmetrizing (23) gives

$$\begin{aligned} &\int_{\partial\Omega} u_2 * (d + iA_1)u_1 - u_1(d + iA_2)u_2 \\ &\quad = \int_\Omega i * (A_1 - A_2)_\wedge (u_2 du_1 - u_1 du_2) \\ &\qquad + (A_{1\wedge} * A_1 - A_{2\wedge} * A_2 + q_2 - q_1)u_1 u_2 \end{aligned} \tag{24}$$

By contrasting exponentially growing solutions and using (24), it has been shown in [Su] and [N-Su-U] that

Theorem.

$$\mathcal{C}_1 = \mathcal{C}_2 \Longleftrightarrow dA_1 = dA_2 \ \ and \ q_1 = q_2$$

6. Nonlinear Schrödinger Equation.

$$\Delta u + q(x,u) = 0 \tag{25}$$

The Cauchy data is just

$$\mathcal{C}_q = \left\{ \left(u, \frac{\partial u}{\partial \nu} \right) \bigg|_{\partial\Omega} \right\}$$

The following theorem has been shown in [I-S].

Theorem. *Suppose that, for $i = 1, 2$, $q^i(x,0) = 0$, $|q^i(x,s)|$ and $\left|\frac{\partial q^i(x,s)}{\partial s}\right|$ are bounded on $\Omega \times \mathbb{R}$ and that $\frac{\partial q^i}{\partial s} \leq 0$, then*

$$\mathcal{C}_{q^1} = \mathcal{C}_{q^2} \Longrightarrow q^1 = q^2$$

The theorem is proved by differentiating the family of Dirichlet problems

$$\begin{aligned} \Delta u + q(x,u) &= 0 \\ u|_{\partial\Omega} &= \theta g \end{aligned} \tag{26}$$

with respect to the parameter θ to obtain (q_s denotes the derivative of q with respect to the second variable)

$$\begin{aligned} \Delta u_\theta + q_s(x,u)u_\theta &= 0 \\ u_\theta|_{\partial\Omega} &= g \end{aligned} \tag{27}$$

and showing that the equality of Cauchy data for the nonlinear problem (26) implies the equality of Cauchy data for the linear problem (27). From the uniqueness result for the linear problem we conclude that

$$q_s^1(x,u^1) = q_s^2(x,u^2)$$

and therefore that the unique solutions to (27) are equal, i.e.

$$u_\theta^1 = u_\theta^2$$

Integrating from $\theta = 0$, where $q^1(x,0) = q^2(x,0)$, to $\theta = 1$ gives

$$u^1 = u^2.$$

Once we know that all solutions to (27) are equal, it follows that

$$q_1(x,s) = q_2(x,s) \tag{28}$$

for every s which can be attained by a solution to (27) at x. As long as $\frac{\partial q}{\partial s}$ is bounded, it can be shown that (28) holds for all s.

7. Inverse Scattering at Fixed Energy.

Several authors ([C-K],[I],[N]) have observed that the results from Section 1 can be used to yield results about inverse scatting at fixed energy. We outline a new approach below. Consider the Schrödinger equation

$$(\Delta + q + \lambda)\psi = 0 \qquad \text{in } \mathbb{R}^n. \tag{29}$$

We assume that q has compact support and that $\lambda > 0$ is fixed. It is well known that for every $\omega \in S^{n-1}$, there exists a unique eigenfunction $\psi_q(x, \omega, \lambda)$ solving (29) which has the form

$$\psi_q(x, \omega, \lambda) = e^{i\sqrt{\lambda}x\cdot\omega} + \frac{e^{i\sqrt{\lambda}|x|}}{|x|^{\frac{n-1}{2}}} a_q\left(\frac{x}{|x|}, \omega, \lambda\right) + O\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right)$$

The function $a_q\left(\frac{x}{|x|}, \omega, \lambda\right)$ is called the scattering amplitude. The basic theorem is

Theorem. *Let q_1 and q_2 have support in the ball of radius R. If, for some $\lambda > 0$ and for all $\omega \in S^{n-1}$*

$$a_{q_1}\left(\frac{x}{|x|}, \omega, \lambda\right) = a_{q_2}\left(\frac{x}{|x|}, \omega, \lambda\right) \tag{30}$$

then

$$q_1 = q_2$$

The theorem is a consequence of the following proposition.

Proposition. *Suppose that $\operatorname{supp} q \subset \Omega \subset\subset \mathbb{R}^n$ and fix $\lambda > 0$. Then (f, g) is Cauchy data for (29) in Ω if and only if*

$$\int_{\partial\Omega} f\frac{\partial\psi}{\partial\nu} - g\psi = 0 \qquad \forall\ \omega \in S^{n-1} \tag{31}$$

According to the proposition, q_1 and q_2 have the same Cauchy data if their eigenfunctions agree outside some ball. But

$$(\Delta + \lambda)(\psi_{q_1} - \psi_{q_2}) = q_1\psi_1 - q_2\psi_2 \tag{32}$$

Since the right hand side of (32) has support inside the ball of radius R, the following variation of a theorem of Rellich asserts that (30) implies that $\psi_{q_1} - \psi_{q_2}$ is also supported inside that ball. Let $G_q(\lambda)f$ denote the unique solution to

$$(\Delta + q + \lambda)\psi = f \quad \text{in } \mathbb{R}^n$$

which satisfies the Sommerfeld radiation condition.

Lemma. *Suppose that $f \in H_0^s(B_R)$ with $-2 < s < 0$ and that $\lambda > 0$. Then the following are equivalent:*

$$
\begin{aligned}
&i) && G_q(\lambda)f \in H^s_\mu \quad \textit{for some} \mu > -1/2 \\
&ii) && (f, \psi_q(\cdot,\omega,\lambda)) = 0 \quad \textit{for all } \omega \in S^{n-1} \\
&iii) && supp(G_q(\lambda)f) \subset B_R
\end{aligned}
\tag{8}
$$

The hypothesis (30) implies that $(\psi_{q_1} - \psi_{q_2})$ satisfies i), which, according to the lemma, implies iii). Hence the Cauchy data $\mathcal{C}_{q_1+\lambda}(B_R)$ and $\mathcal{C}_{q_2+\lambda}(B_R)$ are equal. That this implies that $q_1 = q_2$ was shown in the first section of this report.

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