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# A scattering support for broadband sparse far field measurements

**John Sylvester and James Kelly**

Department of Mathematics, University of Washington, Seattle, WA 98195, USA

E-mail: sylvest@math.washington.edu

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**Abstract**

The scattering support is an estimate of the support of a source or scatterer, based on a limited set of far field measurements. In this paper, we suppose that the far field is measured at all wavenumbers, but only at a few, say $N$, angles $\theta_i \in \Theta$. From these measurements, we produce a $\Theta$-convex polygon (a convex polygon with normals in the $\theta_i$ directions). We show that this polygon must be contained in the smallest $\Theta$-convex polygon which contains the source. We also show by explicit construction that, if (and only if) that polygon has *almost* $2N$ faces, there really is a source, supported in the $\Theta$-convex scattering support, which exactly produces these measurements.

## 1. Introduction

In this paper we are concerned with locating and estimating a general source (or scatterer) from broadband observations made by a few sensors (i.e. a sparse array) in the far field. We choose the far field because it provides a mathematical simplification and begin with the inverse source problem and the Born approximation to the inverse scattering problem because these problems are linear.

For the source problem (1), the far field is exactly the Fourier transform of source (4), restricted to a few lines through the origin (one line for each sensor located in our array) or, equivalently, the Radon transform of the source in a few directions (5) . This is a small set of data. As we will indicate in proposition 4 below, the corresponding far field operator has a large kernel, so it is impossible to uniquely reconstruct the source, or even to find an upper bound on its support. Nevertheless, any attempt to estimate (approximate) the source from this sort of data by a pseudo, regularized, or penalized inverse, must begin with some *a priori* decision as to where the source is supported . For example, we might choose to seek a source, $F$, supported on a set $K$, as the minimizer of the regularized functional

$$\|\widehat{F}(\tau\theta_i) - f_i(\tau)\|^2 + \|\nabla F\|^2$$

where $f_i(\tau)$ are the data measured from direction $\theta_i$. If we decide to minimize over the subspace $L^2(K)$; existence, uniqueness, well-posedness and the solution itself will clearly depend on our choice of $K$. Choosing that domain too large results in ill-posedness, while choosing it too small can make it impossible to fit the data. To the best of our knowledge, all such inverses tend to produce sources with supports as big as the *a priori* assumption allows them to be.

Our analysis below provides a convex set on which to produce such a regularized inverse, along with necessary and sufficient conditions on the data that guarantee the existence of a source, supported there, which fits that data. We will show that, although the data does not uniquely determine the source or any upper bound on its support, it does determine this unique *smallest* convex set. In the final section, we will provide a few examples to show that, while we generically find an approximation to the convex hull of the support of the source, in some cases this set may be much smaller than we would expect. Our main focus in this paper is theoretical, to show that even for such a small data set, we can make a meaningful statement about size and location. Nevertheless, we end the paper with a simple numerical computation to demonstrate that this meaningful set remains meaningful in the presence of noise.

Specifically, we find a lower bound on $K_f(\Theta)$, the $\Theta$-convex hull of the support (the smallest polyhedron, with normals in the directions of observations, which contains the support) of the source. That is, from the far field measured in (i.e. the Fourier or Radon transform restricted to) a collection of directions $\theta_i \in \Theta$, we compute another $\Theta$-convex polygon, $K_{\mathcal{R}f}(\Theta)$, which we prove must be a subset of the $\Theta$-convex hull of the support of any source $f$ which radiated that far field. We call this minimal polygon the $\Theta$-*convex scattering support* of the (restricted) far field. The development here parallels that used in [4] to define the convex scattering support of a far field measured at all angles but at only one wavenumber. See [7] for a method of computing that support using a regularized sampling method, and [5] for a different method that uses neither sampling nor regularization.

In contrast to the fixed wavenumber case, it may not be possible to construct a source supported in $K_{\mathcal{R}F}(\Theta)$ that reproduces the sparse broadband data we consider here. However, if the polyhedron $K_{\mathcal{R}F}(\Theta)$ has two faces for each $\theta_i \in \Theta$, as it should generically, then a source, supported in $K_{\mathcal{R}F}(\Theta)$, that reproduces the data always exists. This is an open condition, expressible as a finite number of strict inequalities. We will show that the closure of the this condition (i.e. allowing equality), (26), is both necessary and sufficient to guarantee the existence of a source, supported in $K_{\mathcal{R}F}(\Theta)$ which exactly reproduces the data. The condition depends only on the data, and if it is not satisfied we need to exclude one or more directions $\theta_i$ until it is satisfied.

## 2. Far fields and the Helmholtz equation

A scalar time harmonic wave radiated by a source in a homogeneous medium is modelled as a solution to the inhomogeneous Helmholtz equation:

$$(\Delta + k^2)u(x) = F(x), \qquad x \in \mathbb{R}^n. \tag{1}$$

Equation (1) has a unique *outgoing solution*, $u = G_0^+ F$, which satisfies the limiting absorption principle, or equivalently, the Sommerfeld radiation condition (see e.g. [8], p 147). It is given by the formula

$$u(x) = \int \frac{1}{4\mathrm{i}} \left( \frac{k}{|x-y|} \right)^{\frac{n-2}{2}} H_{\frac{n-2}{2}}{}^{+}(k|x-y|) F(y)\, \mathrm{d}y$$

$$\sim 2\mathrm{i}k^{\frac{n-1}{2}} \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|^{\frac{n-1}{2}}} \int \mathrm{e}^{\mathrm{i}k\Theta\cdot y} F(y)\,\mathrm{d}y \tag{2}$$

where $H_{\frac{n-2}{2}}{}^{+}$ denotes a Hankel function and $\sim$ means the large $|x|$ asymptotics. Thus the outgoing solution to the inhomogeneous Helmholtz equation has asymptotics of the form

$$G_0^+ F \sim 2\mathrm{i}k^{\frac{n-1}{2}} \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|^{\frac{n-1}{2}}} \alpha(\theta). \tag{3}$$

We call $\alpha = \mathcal{F}_k^+ F(\theta)$ the far field of the source $F$ at wavenumber $k$ and note that, according to (2), the far field of the source $F$ is just its Fourier transform

$$\mathcal{F}_k^+ F = \widehat{F}(k\theta). \tag{4}$$

We specify a far field sensor array by

$$\Theta = \{\theta_i \in S^{n-1}\}$$

where each unit vector $\theta_i$ denotes a sensor positioned at the point $r\theta_i$ (for some large $r > 0$) pointing at the origin . The data we observe are the $\{\widehat{F}(k\theta_i)|\theta_i \in \Theta\}$. Now,

$$\begin{aligned}
\widehat{F}(k\theta) &= \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}k\theta\cdot x} F(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}k\tau} \int_{\theta\cdot x=\tau} F(x)\,\mathrm{d}S \\
&= \int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}k\tau} [\mathcal{R}F](\tau,\theta)
\end{aligned} \tag{5}$$

where $\mathcal{R}F$ is the Radon transform of $F$. The calculation above, which shows that the one-dimensional Fourier transform of the Radon transform is the $n$-dimensional Fourier transform is called the *central slice theorem* [6, p 11]. Since we may easily compute the one-dimensional Fourier transform analytically or numerically, our data are equivalent to $\{[\mathcal{R}F](\tau,\theta_i)|\theta_i \in \Theta\}$.

The Born approximation for scattering from an inhomogeneous medium gives rise to the same sort of far field data, although the $\theta_i$ that arise incorporate both source and receiver directions. When we observe the far field of a wave scattered by an inhomogeneous medium, the mathematical model we use is the Helmholtz equation for an inhomogeneous medium:

$$(\Delta + k^2 n^2(x))u = 0.$$

We now have an array of sources and an array of receivers (usually the same) in the far field. If we excite the source in the far field in the direction $\phi \in S^{n-1}$, we express $u$ in the form

$$u = \mathrm{e}^{\mathrm{i}k\phi} + u_{sc}$$

where $\mathrm{e}^{\mathrm{i}k\phi}$ is called the incident wave and $u_{sc}$, the scattered wave, is the unique outgoing solution to

$$(\Delta + k^2)u_{sc} = k^2 q(x)(\mathrm{e}^{\mathrm{i}k\phi} + u_{sc}) \tag{6}$$

with $q(x) = 1 - n^2(x)$. The Born (or weak scattering) approximation is valid when $u_{sc}$ is small enough to be ignored on the right-hand side of (6). Thus the Born approximation, $u_B$, to $u_{sc}$ satisfies

$$(\Delta + k^2)u_B = k^2 q(x)\,\mathrm{e}^{\mathrm{i}k\phi}$$

which is just (1) with $F = k^2 q(x)\,\mathrm{e}^{\mathrm{i}k\phi}$. If we observe the far field at a sensor in the direction $\psi \in S^{n-1}$, the far field we see is the shifted Fourier transform of $q$, i.e.

$$\begin{aligned}
B(\phi,\psi) &= k^2(\widehat{q}(k(\phi-\psi))) \\
&= k^2(\widehat{q}(\tilde{k}(\theta)))
\end{aligned} \tag{7}$$

where

$$\tilde{k} = k\|\phi - \psi\|$$
$$= k\sqrt{2 - 2\phi \cdot \psi} \tag{8}$$

and

$$\theta = \frac{\phi - \psi}{\|\phi - \psi\|}. \tag{9}$$

Thus, both the inverse source problem and the Born approximation to the inverse scattering problem, with a sparse (i.e discrete) array in the far field, are equivalent to the mathematical problem of estimating the support of a function $F$ from its Fourier transform restricted to a set of lines through the origin. We set about this task in the following section.

## 3. The Θ-scattering support

In this section we introduce the notion of support that we will relate to our far field observations. Let $\Omega$ be a region in $\mathbb{R}^n$. The convex hull of region $\Omega$ is the intersection of the half planes

$$H_{s,\theta} = \{x | x \cdot \theta \leqslant s\}$$

which contain it, where $\theta$ varies on the unit sphere. That is,

$$\mathrm{ch}\,\Omega = \bigcap_{\theta \in S^{n-1}} \{x \cdot \theta \leqslant s_\Omega(\theta)\}$$

where

$$s_\Omega(\theta) = \sup_{x\in\Omega} x \cdot \theta$$

is the supporting function of $\Omega$. For convenience, the supporting function is extended to all $\xi \in \mathbb{R}^n$ by requiring it to be positively homogeneous of degree 1. For a source (i.e. a function or distribution defined on $\mathbb{R}^n$) we shall write

$$s_F(\xi) := s_{\mathrm{supp}\,F}(\xi).$$

Many properties of convex hulls are conveniently encoded in the supporting function. Every positively homogeneous convex function of degree 1, $s$, defines a convex set $K$ via

$$K_s = \bigcap_{\xi \in \mathbb{R}^n} \{x \cdot \xi \leqslant s(\xi)\}.$$

It is a simple consequence of the Hahn–Banach theorem that the support function of $K_s$ is equal to $s$ [2]. That is,

$$s_{K_s}(\phi) := \sup_{\substack{x\cdot\theta \leqslant s_\Omega(\theta) \\ \text{for all } \theta}} x \cdot \phi$$
$$= s_\Omega(\phi)$$

so there is a unique correspondence between convex sets and supporting functions (positively homogeneous convex functions of degree one).

In the previous section we noted that the far field of a source was just its restricted Fourier transform. The Paley–Wiener theorem [2] describes the way in which the supporting function is manifested in the Fourier transform of the source.

**Theorem 1** (Paley–Wiener theorem)**.** *If $F$ is a compactly supported distribution, $\widehat{F}(\xi)$ extends to a holomorphic function $\widehat{F}(\zeta)$ ($\zeta \in \mathbb{C}^n$) that satisfies*

$$|F(\zeta)| \leqslant C(1+|\zeta|)^m \, \mathrm{e}^{s_F(\mathrm{Im}(\zeta))}. \tag{10}$$

*Conversely, if $\widehat{F}(\zeta)$ is holomorphic and satisfies an estimate of the form (10), then $F$ has compact support and $s_F$ is the smallest supporting function for which (10) holds.*

In our scattering problem, because we have only a sparse (i.e. finite) array of sensors in the far field, we will have observations in a finite number, $N$, of directions (unit vectors). For such a collection of directions,

$$\Theta = \{\theta_i \in S^{n-1}\}$$

we will define the $\Theta$-convex hull of a region $\Omega$ by

$$K_{s_\Omega}(\Theta) = \bigcap_{\theta_i \in \Theta} \{x \cdot \theta_i \leqslant s_\Omega(\theta_i)\}.$$

The $\Theta$-convex hull is a polyhedron with faces whose normals belong to the collection of angles $\Theta$. It is the smallest such polyhedron that contains $\Omega$. $K_{s_\Omega}(\Theta)$ has its own supporting function

$$s_{K_{s_\Omega}(\Theta)}(\theta) = \sup_{\substack{x\cdot\theta_i \leqslant s_\Omega(\theta_i)\\ \theta_i\in\Theta}} x \cdot \theta. \tag{11}$$

Our far field data will provide us with a positively homogeneous function, $\sigma$, of degree one which will play a role similar to that of $s_\Omega$, but will not necessarily be convex. Nevertheless,

$$K_\sigma(\Theta) = \bigcap_{\theta_i \in \Theta} \{x \cdot \theta_i \leqslant \sigma(\theta_i)\}$$

still defines a convex polyhedron with supporting function

$$s_{K_\sigma(\Theta)}(\theta) = \sup_{\substack{x\cdot\theta_i \leqslant \sigma(\theta_i)\\ \theta_i\in\Theta}} x \cdot \theta. \tag{12}$$

The difference between (11) and (12) is that, in the first case, the two support functions agree on the $\theta_i$, i.e.

$$s_{K_{s_\Omega(\Theta)}}(\theta_i) = s_\Omega(\theta_i) \tag{13}$$

while, if $\sigma$ is not convex,

$$s_{K_\sigma(\Theta)}(\theta_i) \leqslant \sigma(\theta_i) \tag{14}$$

but equality may not hold. This distinction will play a role in (26) of theorem 7. Both (13) and (14) are consequences of the lemma below.

**Lemma 2.** *Let $\sigma$ be any homogeneous function of degree 1. Then $s_{K_\sigma(\Theta)}$ is the largest supporting function satisfying*

$$s(\theta_i) \leqslant \sigma(\Theta_i) \quad \textit{for all} \quad \theta_i \in \Theta. \tag{15}$$

**Proof.** Let $\beta$ be a supporting function that satisfies (15). According to (10)

$$\begin{aligned}\beta(\phi) &= s_{K_\beta}(\phi)\\ &= \sup_{\substack{x\cdot\theta\leqslant\beta(\theta)\\ \text{for all } \theta}} x\cdot\phi\\ &\leqslant \sup_{\substack{x\cdot\theta_i\leqslant\beta(\theta_i)\\ \theta_i\in\Theta}} x\cdot\phi\\ &\leqslant \sup_{\substack{x\cdot\theta_i\leqslant\sigma(\theta_i)\\ \theta_i\in\Theta}} x\cdot\phi\\ &= s_{K_\sigma(\Theta)}(\phi)\end{aligned}$$

□

Our $\sigma$ of interest arises from our far field observations (5) as follows. For each fixed $\theta$, $\sigma(\theta)$ will be the supporting function of the one-dimensional function $[\mathcal{R}F](\cdot,\theta)$. That is,

$$\sigma_{[\mathcal{R}F]}(\tau\theta) = \sup_{t\in\mathrm{supp}[\mathcal{R}F](\cdot,\theta)} t\tau.$$

$\sigma_{[\mathcal{R}F]}$ is always homogeneous of degree 1, but will not in general be convex. $K_{\sigma_{\mathcal{R}F}}(\Theta)$ provides a lower bound for the $\Theta$-convex hull of the support of $F$, i.e.

**Theorem 3.**

$$\sigma_{\mathcal{R}F}(\theta) \leqslant s_F(\theta) \tag{16}$$

*or, equivalently*

$$K_{\sigma_{\mathcal{R}F}}(\Theta) \subseteq K_{s_F}(\Theta).$$

**Proof.** The definition of the Radon transform

$$[\mathcal{R}F](\tau,\theta) = \int_{\theta\cdot x=\tau} F(x)\,\mathrm{d}S$$

shows clearly that

$$\tau \geqslant s_F(\theta) \quad\Longrightarrow\quad [\mathcal{R}F](\tau,\theta) = 0 \quad\Longrightarrow\quad \tau \geqslant \sigma_{\mathcal{R}F}(\theta)$$

i.e.

$$\sigma_{\mathcal{R}F}(\theta) \leqslant s_f(\theta)$$

for every $\theta$, which implies the desired conclusion on the corresponding convex sets they define.

□

One might hope to do better than merely a lower bound, but the following observation shows that, in the absence of ancillary information about $f$, no upper bound is possible.

**Proposition 4.** *Let $\Phi$ be any compactly supported smooth function and let $\theta_i^\perp$ denote some non-zero vector perpendicular to $\theta_i$ for each $i = 1,\ldots,N$. If $H$ is the Nth-order derivative of $\Phi$ given by*

$$H = \nabla_{\theta_1^\perp}\ldots\nabla_{\theta_N^\perp}\Phi \tag{17}$$

*then the restriction of $\widehat{H}(\xi)$ to all of the lines $\xi = \tau\theta_i$ vanishes identically.*

**Remark 5.** In the two-dimensional case, the kernel of the map from sources to restricted far fields is exactly the span of the functions of the form $H$. In dimensions 3 and higher, an exact description of the kernel is more complicated, depending in a detailed way on the geometry of the directions $\theta_i$ (how many are co-planar, e.g.).

**Proof.** Simply observe that

$$\widehat{H}(\xi) = \prod_i \left(\xi \cdot \theta_i^{\perp}\right)\widehat{\Phi}$$

has at least one linear factor which vanishes when $\xi = \tau\theta_i$. □

**Corollary 6.** *Given any compactly supported function $F$, any finite set of directions $\Theta$, and any compact set $\Omega \subset \mathbb{R}^n$, there exists a function $G$ such that*

$$\widehat{G}(\tau\theta_i) = \widehat{F}(\tau\theta_i) \qquad \Omega \subset \operatorname{ch\,supp} G. \tag{18}$$

**Proof.** Let $G = F + H$ where $H$ is of the form (17), with $\Phi$ chosen to be smooth and with support in $\Omega\backslash\operatorname{supp} F$. Note that because $\Phi$ is smooth and compactly supported, $\operatorname{ch\,supp} H = \operatorname{ch\,supp} \Phi$. □

Having established our lower bound, we might want to use it to find an approximate or regularized source. Our next order of business is to show that, under a condition that is generically satisfied and can be checked from the data, the $\Theta$-convex scattering support of $F$, $K_{\sigma_{\mathcal{R}F}}(\Theta)$, is in some sense the biggest set we can hope to identify. We will show that there exists an $\tilde{F}$ supported in $K_{\sigma_{\mathcal{R}F}}(\Theta)$ with

$$\widehat{\tilde{F}}(\tau\theta_i) = \widehat{F}(\tau\theta_i)$$

for all $\theta_i \in \Theta$, if and only if

$$\sigma_{\mathcal{R}F}(\theta_i) = s_{K_{\sigma_{\mathcal{R}F}}(\Theta)}(\theta_i). \tag{19}$$

We refer to condition (19) as the property that $K_\sigma(\Theta)$ has *almost* $2N$ *faces*[1]. To explain the relationship between (19) and the faces of $K_{\sigma_{\mathcal{R}F}}(\Theta)$, we point out that, according to (12), the boundary of $K_{\sigma_{\mathcal{R}F}}(\Theta)$ may consist only of faces which are subsets of the hyper-planes $\{x \cdot \theta_i = \sigma(\theta_i)\}$. If that intersection contains an open subset of that hyper-plane, it is genuinely a face of the polyhedron with outward pointing normal $\theta_i$. This means that $K_{\sigma_{\mathcal{R}F}}(\Theta)$ is strictly smaller than $K_{\sigma_{\mathcal{R}F}}(\Theta\backslash\{\theta_i\})$, or equivalently, that the support function of the former is strictly smaller than the support function of the latter. Combining this with (14), we see that

$$s_{K_\sigma(\Theta\backslash\{\theta_i\})}(\theta_i) > s_{K_\sigma(\Theta)}(\theta_i) \geqslant \sigma(\theta_i)$$

so that $K_{\sigma_{\mathcal{R}F}}(\Theta)$ has $2N$ nontrivial faces if and only if, for each $\theta_i \in \Theta$

$$s_{K_\sigma(\Theta\backslash\{\theta_i\})}(\theta_i) > \sigma(\theta_i). \tag{20}$$

Now, because it involves fewer conditions,

$$s_{K_\sigma(\Theta\backslash\{\theta_i\})}(\theta_i) \geqslant s_{K_\sigma(\Theta)}(\theta_i)$$

which combines with (14) to show that

$$s_{K_\sigma(\Theta\backslash\{\theta_i\})}(\theta_i) \geqslant \sigma(\theta_i) \tag{21}$$

follows from (19). Because (21) tells us that

$$K_\sigma(\Theta) = K_\sigma(\Theta\backslash\{\theta_i\}) \cap \{x \cdot \theta_i \leqslant \sigma(\theta_i\})\} \subset \{x \cdot \theta_i \leqslant \sigma(\theta_i\})\}$$

[1] Even though (19) is really a condition on the pair $(\sigma, \Theta)$, not the polygon $K_\sigma(\Theta)$.

which is just the geometric statement of (19), we see that (21) and (19) are equivalent. Thus any $(\sigma, \Theta)$ pair satisfying (19), satisfies (21), and is therefore a limit of pairs with $2N$ faces, i.e. has almost $2N$ faces.

**Theorem 7.** *Let $\{\theta_i\}$ be a set of N distinct unit vectors and let $f_i(\tau)$ be a corresponding set of compactly supported distributions defined on $\mathbb{R}$. Define*

$$\Theta = \{\pm\theta_i\}$$

*and*

$$\sigma(\pm\theta_i) = \sup_{\tau \in \operatorname{supp} f_i(\pm\tau)} \tau \tag{22}$$

*then there exists a compactly supported distribution F defined on $\mathbb{R}^n$ satisfying*

$$\widehat{F}(k\theta_i) = \widehat{f}_i(k) \tag{23}$$

$$\operatorname{supp} F \subset K_\sigma(\Theta) \tag{24}$$

*if and only if there is an Nth degree polynomial satisfying*

$$P^N(\tau\theta_i) = \widehat{f}_i(\tau) + O(\tau^{N+1}) \tag{25}$$

*and*

$$\sigma(\pm\theta_i) = s_{K_\sigma}(\pm\theta_i) \tag{26}$$

*i.e. $K_\sigma(\Theta)$ has almost 2N sides.*

**Remark 8.** Condition (25) is called the moment[2] condition for the Radon transform or the Helgason–Ludwig consistency conditions [6, p 36].

**Proof.** One direction of the proof is easy. If we start with (23) and (24), then we may satisfy (25) by choosing $P^N$ to be the $N$th-order Taylor polynomial of $\widehat{F}$, which is analytic because $F$ has compact support. To establish (26), note that (23) implies that, at each $\theta_i$,

$$\sigma(\theta_i) = \sigma_{\mathcal{R}F}(\theta_i) \leqslant s_F(\theta_i) \tag{27}$$

the final inequality following from (16). On the other hand, (24) guarantees that, for any $\theta$

$$s_f(\theta) \leqslant s_{K_\sigma(\Theta)}(\theta) \tag{28}$$

which, according to lemma 2

$$\leqslant \sigma(\theta). \tag{29}$$

Combining (27)–(29) shows that all the inequalities must be equalities at the $\theta_i$ and establishes (26).

The converse requires more work. We will construct $\widehat{F}$ from the $\widehat{f}_i$ with the help of the following lemma.

**Lemma 9.** *Let $f_i(\tau)$ be supported in $[\sigma(-\theta_i), \sigma(\theta_i)]$ and*

$$\widehat{f}_i(k) = O(k^N) \quad \text{at} \quad k = 0. \tag{30}$$

*Let $\Phi$ be a unit vector that is not perpendicular to $\theta_i$ and c a vector of arbitrary length that is perpendicular to $\theta_i$.*

[2] The Taylor coefficients of the Fourier transform at the origin are the moments, $\int t^n \mathcal{R}f(t, \theta)\,\mathrm{d}t$, of the Radon transform.

*Define*

$$\widehat{F}_i(\xi) = \prod_{j=1}^{N} \left[ \frac{\left(\theta_j^{\perp} \cdot \xi\right)(\theta_i \cdot \Phi)}{\left(\theta_j^{\perp} \cdot \theta_i\right)(\xi \cdot \Phi)} \right] \widehat{f}_i \left( \frac{\xi \cdot \Phi}{\theta_i \cdot \Phi} \right) \mathrm{e}^{\mathrm{i}\xi\cdot c} \tag{31}$$

*then $F_i(x)$ is a distribution supported on the line segment*

$$\gamma(t) = t\Phi + c$$

*between the planes*

$$x \cdot \theta_i = -\sigma(-\theta_i) \tag{32}$$

$$x \cdot \theta_i = \sigma(\theta_i) \tag{33}$$

*and*

$$\widehat{F}_i(k\theta_i) = \widehat{f}_i(k) \qquad \widehat{F}_i(k\theta_j) = 0 \quad \textit{for} \quad j \neq i. \tag{34}$$

**Proof of lemma.** Since $f_i(\tau)$ has compact support, its one-dimensional Fourier transform $\widehat{f}_i$ extends to be holomorphic in $\mathbb{C}$. Because of (30), $\frac{\widehat{f}_i(k)}{k^n}$ extends to be holomorphic as well. This implies that $\widehat{F}_i$, as defined in (31), extends to be holomorphic in $\mathbb{C}^n$.

The (one-dimensional) support function of $f_i$ is given by

$$s_{f_i}(t) = \begin{cases} \sigma(\theta_i)t & t > 0 \\ \sigma(-\theta_i)t & t \leqslant 0. \end{cases}$$

According to the Paley–Wiener estimate (10),

$$\widehat{f}_i(z) \leqslant C(1+|z|)^m \, \mathrm{e}^{s_{f_i}(\mathrm{Im}(z))}.$$

Therefore, according to formula (31)

$$\widehat{F}_i(\zeta) \leqslant C(1+|\zeta|)^M \, \mathrm{e}^{s_{f_i}(\mathrm{Im}(\zeta)\cdot\Phi)+\mathrm{Im}(\zeta)\cdot c}$$

so that

$$s_{F_i}(\eta) \leqslant \eta \cdot c + \begin{cases} \sigma(\theta_i)\eta \cdot \Phi & \eta \cdot \Phi > 0 \\ \sigma(-\theta_i)\eta \cdot \Phi & \eta \cdot \Phi \leqslant 0 \end{cases} \tag{35}$$

and the right-hand side of (35) is the support function of the line segment described in the lemma. □

We will prove the only if part of theorem 7 by finding

$$\widehat{F} = F_0 + \sum_{i=1}^{N} F_i. \tag{36}$$

The $F_i$ will come from the lemma. In order to use it, we must replace each $f_i$ with an $\tilde{f}_i$ that satisfies (30). To do this, let $d$ be a point in $K_\sigma$ and recall $P^N(\xi)$ from (25). We define

$$F_0 = Q^N(\xi)\, \mathrm{e}^{\mathrm{i}\xi\cdot d} \tag{37}$$

with $Q$ chosen so that

$$Q^N(\xi)\, \mathrm{e}^{\mathrm{i}\xi\cdot d} = P^N(\xi) + O(\xi^{N+1})$$

near $\xi = 0$. Now we apply the lemma to $\tilde{f}_i$ defined by

$$\widehat{\tilde{f}}_i = \widehat{f}_i - Q^N(k\theta_i)\, \mathrm{e}^{\mathrm{i}k\theta_i\cdot d}.$$

It follows from the lemma that $F$ as defined in (36) satisfies (23). Each $F_i$ can be constructed to have support on a line segment with one endpoint on each of the planes (32) and (33). Condition (26) guarantees that at least one point in each plane, and hence the line segment connecting them, belongs to the convex set $K_\sigma$, and thus completes the proof. □

If we replace the compactly supported distributions in theorem 7 by smooth functions $C_0^\infty$, the *only if* part of the theorem remains true as stated. However, we must modify the *if* part of the theorem slightly:

**Theorem 10.** *Let the $\theta_i$ and $f_i$ be as in theorem 7, and assume in addition that $f_i \in C_0^\infty(\mathbb{R})$, if*

*(1) There exists an Nth degree polynomial $P^N(\xi)$ satisfying*

$$P^N(\tau\theta_i) = \widehat{f}_i(\tau) + O(\tau^{N+1}).$$

*(2) $K_\sigma(\Theta)$ has almost 2N faces (i.e. $\sigma(\pm\theta_i) = s_{K_\sigma}(\pm\theta_i)$)*

*then, for any $\varepsilon > 0$, there exists an $F \in C_0^\infty(\mathbb{R}^n)$ satisfying*

*(1) $\widehat{F}(k\theta_i) = \widehat{f}_i(k)$*
*(2) $\operatorname{supp} F \subset N_\varepsilon(K_\sigma(\Theta))$*

*where $N_\varepsilon(K_\sigma)$ denotes an $\varepsilon$ neighbourhood of $K_\sigma$.*

**Proof.** We indicate the changes that need to be made to the construction of $\widehat{F}$ in the proof of theorem 10 to ensure that $F$ is smooth. Let $\phi_\varepsilon$ be an even smooth function of a single variable supported in $\{|\tau| \leqslant \varepsilon\}$ with enough vanishing moments that

$$\widehat{\phi}_\varepsilon(k) = 1 + O(k^{N+1})$$

near $k = 0$. We replace $\widehat{F}_0$ in (37) and each $\widehat{F}_i$ constructed in lemma 3 by

$$\widehat{F}_0 \mapsto \widehat{F}_0\widehat{\phi}_\varepsilon(|\xi|) \qquad \widehat{F}_i \mapsto \widehat{F}_i\widehat{\phi}_\varepsilon(|\xi - (\xi\cdot\theta_i)\theta_i|). \tag{38}$$

These changes ensure that $F$ is smooth, but do not change the values of the expansion of $\widehat{F}_0$ near 0 or the values of the $F_i(\tau\theta_j)$. They spread the support of $F_0$ and each $F_i$ to within an $\varepsilon$-neighbourhood of what they were before, which is enough to prove the theorem.

If we look a little more closely, we see that while the support of $F_0$ spreads in all directions to a ball about the point $d$, the support of each $F_i$ spreads away from the original line segment, but only in directions parallel to the planes $\{x\cdot\theta_i = \sigma(\theta_i)\}$. Thus, in the case that $K_{\sigma_{\mathcal{R}f}}(\Theta)$ has an interior point (which we may choose as $d$) and $2N$ faces (rather than almost $2N$ faces), we may take $\varepsilon = 0$ (i.e. we do not need the $\varepsilon$-neighbourhood). □

## 4. Examples

In the table below, we include a few examples of $K_{s_f}(\Theta)$ and $K_{\sigma_{\mathcal{R}f}}(\Theta)$ for a simple function. We take

$$f = \chi_r(x - p_1) - \chi_r(x - p_2) + \chi_r(x - p_3) \tag{39}$$

where $\chi_r$ is the indicator function on the ball of radius $r(= 0.2)$, and the $p_i$ are $(-1, 1)$, $(1, 1)$ and $(0, 0)$, respectively. In the plots, the light circles indicate the supports of the individual summands, the plus or minus signs indicate whether that summand is added or subtracted in (39).

In the first example, $\Theta$ includes the $x$-axis and the diagonal, in the second the $y$-axis and the anti-diagonal and in the third there are three directions, both coordinate axes and the diagonal. See figure 1.

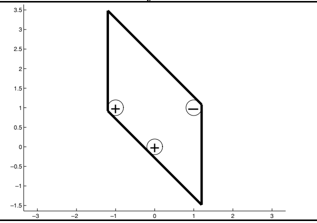
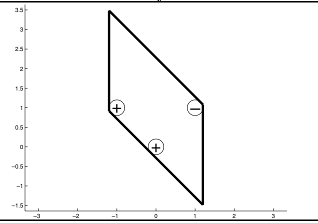
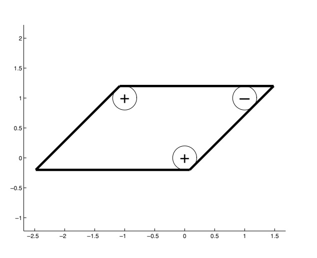
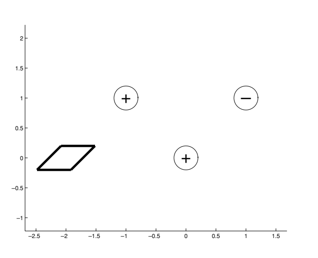
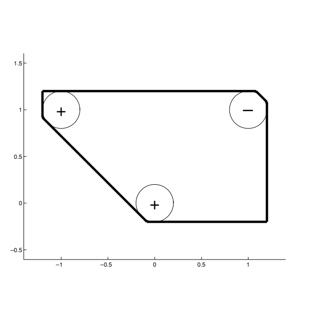
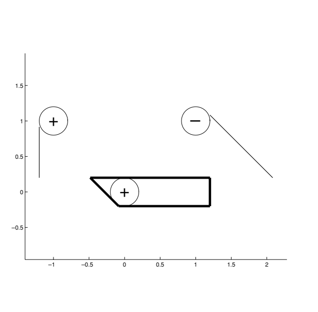
| $K_{s_f}(\Theta)$ | $K_{\sigma_{\mathcal{R}f}}(\Theta)$ | Comments |
|---|---|---|
|  |  | $K_{s_f}(\Theta) = K_{\sigma_{\mathcal{R}f}}(\Theta)$. This is the generic case. |
|  |  | $K_{\sigma_{\mathcal{R}f}}(\Theta) \subset K_{s_f}(\Theta)$. $K_{\sigma_{\mathcal{R}f}}(\Theta)$has $2N$ (4) faces, so we can find a source supported in $K_{\sigma_{\mathcal{R}f}}(\Theta)$ that interpolates $\widehat{f}$ in both directions. |
|  |  | $K_{\sigma_{\mathcal{R}f}}(\Theta) \subset K_{s_f}(\Theta)$. $K_{\sigma_{\mathcal{R}f}}(\Theta)$ does not have almost $2N$ faces. The thin lines represent the two *missing* faces. There can be no source supported in $K_{\sigma_{\mathcal{R}f}}(\Theta)$ that interpolates $\widehat{f}$ in all three directions. |

**Figure 1.** Three examples of the $\Theta$-convex scattering support.

In the generic example, $K_{s_f}(\Theta)$ and $K_{\sigma_{\mathcal{R}f}}(\Theta)$ are equal and approximate the convex hull of the support of $f$. The approximation gets better as we increase the number of $\theta$. The last two examples are non-generic because $f$ averages to zero along certain planes with normals in the direction of the *y*-axis and the anti-diagonal. We happen to have chosen one or more $\theta$ in precisely those directions.

Keep in mind when looking at these examples that if we have only the data in hand we see only $K_{\sigma_{\mathcal{R}f}}(\Theta)$, and neither $K_{s_f}(\Theta)$ nor the true supports (i.e. the three circles). In the last example, theorem 9 tells us that we must drop a direction before we can try to construct $\widehat{f}$. Any direction will do. If we choose to drop the (0, 1) direction, we wind up back at the first example.

A closer examination of the second example shows that the three sources at the corners of the parallelogram produce exactly the same far field data (in the two $\theta_i$ directions) as a single source at the only corner of the parallelogram which has no true source. In this case, $K_{\sigma_{\mathcal{R}f}}(\Theta)$ actually shows us the location of this *ghost source*, which is as far from the three true sources as it could be without contradicting theorem 3. Such a situation becomes more and more unlikely as the number of observation directions $\theta_i$ grows. Indeed, it is proved in [3] that for any fixed configuration of point sources, there is a large enough far field sensor array (i.e. enough directions $\theta_i$) to uniquely identify that configuration among all configurations of

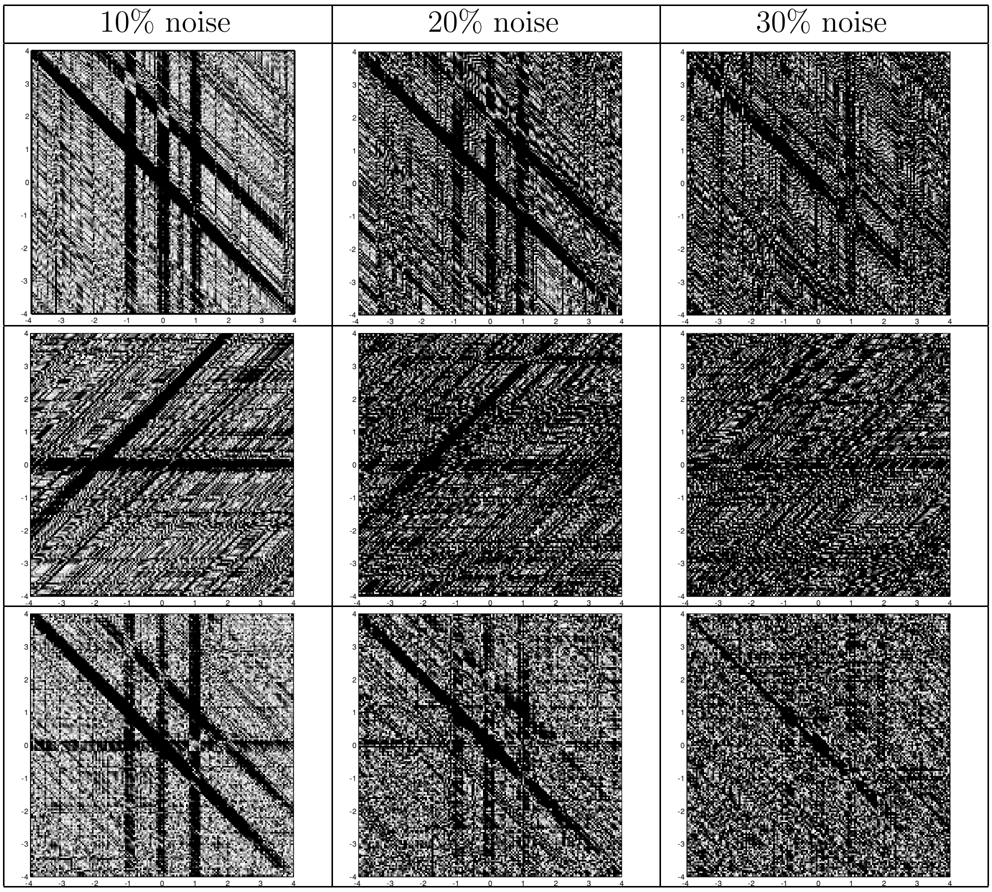



**Figure 2.** Numerical computations for the three examples in figure 1.

point sources. See also [1] for a description of the kernel of the map from point sources to a linear array of near field sensors.

Finally, we show a numerical computation for each of the three examples, illustrating one possible method for computing the polygon $K_{\sigma_{\mathcal{R}f}}(\Theta)$ and its performance in the presence of noise. While the support function served as a useful theoretical tool, we do not attempt to actually compute it. Instead we tensor the far field restricted to the line with direction $\theta_i$ with the dirac distribution, $\delta_{\theta_i}$, of that line[3] to produce a two-dimensional distribution. We then compute the inverse Fourier transform of each such distribution, which is a plane wave, and make an image plot to show the support of each function. $K_{\sigma_{\mathcal{R}f}}(\Theta)$ is the intersection of the convex hulls of these supports, each of which is a strip. We do not explicitly mark the intersection in the image plot.

To produce the figures in the table we evaluated the source $f$ on a $512 \times 512$ rectangular grid with spacing 0.05 and added normally distributed random noise with mean 0 and standard deviations 0.1, 0.2, or 0.3 for each column, respectively. A two-dimensional discrete Fourier transform (DFT) of the source produced the values of the far field on the dual grid. We formed the discrete analogues of the distributions discussed above by setting the far field to zero at all

[3] $\langle \delta_{\theta_i}, h \rangle = \int_{-\infty}^{\infty} h(t\theta_i)\, \mathrm{d}t$.

grid points that were more than 1.5 times the grid spacing (0.04 on the dual grid) from all of the lines in $\Theta$. Finally, we computed the inverse DFT, took absolute values and made image plots. We could have, but did not, filter the result. We expect that everyone would agree that the supports are clearly visible at the 10% noise level and no longer visible at the 30% noise level. See figure 2.

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## References

[1] Devaney A Super-resolution processing of multi-static data using time reversal and music *J. Acoust. Soc. Am.* at press

[2] Hörmander L 1983 *Analysis of Linear Partial Differential Equations I* (Berlin: Springer)

[3] Kirsch A 2002 The MUSIC algorithm and the factorization method in inverse scattering theory for inhomogeneous media *Inverse Problems* **18** 1025–40

[4] Kusiak S and Sylvester J 2003 The scattering support *Commun. Pure Appl. Math.* **56** 1525–48

[5] Kusiak S and Sylvester J 2005 The convex scattering support in a background medium *SIAM J. Math. Anal.* **36** 1142–8

[6] Natterer F 2001 The mathematics of computerized tomography *Classics in Applied Mathematics* vol 32 (Philadelphia, PA: SIAM) (Reprint of the 1986 original)

[7] Potthast J, Sylvester R and Kusiak S 2003 A range test for determining scatterers with unknown physical properties *Inverse Problems* **19** 533–47

[8] Taylor M E 1996 *Partial Differential Equations II: Qualitative Studies of Linear Equations (Applied Mathematical Sciences* 116*)* (Berlin: Springer)