A 'range test' for determining scatterers with unknown physical properties.

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Abstract

We describe a new scheme for determining the *convex scattering support* of an unknown scatterer when the physical properties of the scatterers are not known. The convex scattering support is a subset of the scatterer and provides information about its location and estimates for its shape. For convex polygonal scatterers the scattering support coincides with the scatterer and we obtain full shape reconstructions. The method will be formulated for the reconstruction of the scatterers from the far field pattern for *one* or a few incident waves. The method is *non-iterative* in nature and belongs to recent type of generalized sampling schemes like the 'No response test' of Luke-Potthast.

The *range test* operates by testing whether it is possible to analytically continue a far field to the exterior of any test domain Ω_{test}. By intersecting the convex hulls of various test domains we can produce a minimal convex set , the *convex scattering support* which must be contained in the convex hull of the support of any scatterer which produces that far field.

The convex scattering support is calculated by testing the range of special integral operators for a sampling set of test domains. The numerical results can be used as an approximation for the support of the unknown scatterer. We prove convergence and regularity of the scheme and show numerical examples for sound-soft, sound-hard and medium scatterers.

We can apply the *range test* to non-convex scatterers as well. We can conclude that an Ω_{test} which passes the range test has a non-empty intersection with the infinity-support (the complement of the unbounded component of the complement of the support) of the true scatterer, but cannot find a minimal set which must be contained therein.

1 Introduction

We consider the scattering of time-harmonic acoustic waves by some possibly multiply connected scatterer Ω in $\mathbb{R}^m$ for $m = 2, 3$. An incident wave u^i is scattered into the scattered field u^s with far field pattern u^∞. The inverse problem is to reconstruct the location, shape and properties of the unknown scatterer.

The main purpose of the paper is to introduce a new scheme, which we call *range test*, for determining information about the location and estimates for the shape of unknown

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Method for *shape reconstruction*	Year	data needed
Colton-Kirsch / Linear Sampling Method see [2]	1995/96	all waves
Kirsch / Factorization Method see [10]	1998	all waves
Ikehata / Probe Method see [8]	1998	all waves
Potthast / Singular Sources Method see [19]	1999/2000	all waves
Luke-Potthast / No-Response Test see [14]	2002	*one* wave
Potthast-Sylvester-Kusiak / Range Test	2002	*one* wave

Table 1: Methods for reconstructing the shape of an unknown scatterer when the physical properties of the scatterer are not known.

scatterers when the physical properties of the scatterers are not known. Due to the lack of knowledge about the boundary condition, in this situation most conventional schemes like Newton's Method, the Kirsch-Kress potential fit or the point-source method of Potthast (see [20] and [3]) cannot be applied. The scattering support is a subset of the unknown scatterer and thus provides some general information about its location. For convex polygonal we show that the scattering support coincides with the scatterer and, thus, the range test provides full reconstructions of convex polygonal scatterers. In general, there might be special situations where the range test delivers only one single point in the interior of the unknown scatterer. However, our numerical experiments confirm that for most practical situation it provides a rather good (however still rough) estimate for the shape and size of the scatterer from the far field pattern of *one* scattered time-harmonic plane wave.

The range test uses an approach different from former schemes like the 'no response test' of Luke-Potthast [14], the singular sources method of Potthast [19], the 'probe method' of Ikehata [9], the 'factorization method' of Kirsch [10] or the 'linear sampling method' of Colton-Kirsch [2]. The singular sources method, the probe method and the linear sampling method need to know the far field pattern for *all* plane waves for reconstructions. The range test, however, needs only one or a few far field patterns to provide reasonable results. Table 1 gives a survey about the different reconstruction methods which do not use the physical optics or the Born approximation. The range test is particularly suited for inverse scattering in the resonance region, since it does not perform any high- or low-frequency approximations for the scattering problem like recent methods of Bucci, Capozzoli and Elia [1].

The basic idea of the *range test* is to determine the *convex scattering support*, the minimal convex set, such that the scattered field may be analytically extended to its complement. This set will be a subset of the convex hull of the unknown scatterer Ω. The method will not deliver full reconstructions of the shape of scatterers, but reconstruct a subset of the convex hull of the obstacle, based on one or more far field patterns scattered from one or more incident waves. The determination of the convex scattering support will be efficiently carried out using integral equations of the first kind on a number of test domains, where the integral equation

$$\int_{\partial G} e^{i\kappa \hat{x}\cdot d} g(d) = u^{\infty}(\hat{x}), \quad \hat{x} \in \mathbb{S}^{m-1}, \tag{1.1}$$

has a solution in $L^2(\partial G)$ if and only if the scattered field can be analytically extended up to the boundary ∂G of the test domain G. Since this tests the range of the integral operator the method is called *range test*.

A range test for the extension of u^{∞} into u^s could be performed by expansion of the scattered field and far field pattern with respect to spherical harmonics (see Theorem 2.16 of [3]). Here, we prefer the integral equation approach. Also, implicitly the 'factorization method' of Kirsch [10] carries out a range test. However, the method of Kirsch needs to know the far field pattern for all plane waves, where for our range test algorithm it is sufficient to know u^{∞} for scattering of one plane wave.

The plan for this paper is as follows. In Section 2 we provide some basic material about the scattering problems under consideration. In Section 3 we define the convex scattering support $\mathrm{cS}_{\kappa}\mathrm{supp}\, u^{\infty}$ of a far field pattern u^{∞}. We will also define the $\mathcal{D}$-convex scattering support, $\mathrm{S}_{\mathcal{D}}\mathrm{supp}\Omega$, of a scatterer Ω illuminated by a collection of plane waves $\mathcal{D}$, to be the union of the convex scattering supports of the individual scattered waves. We prove some basic properties of solutions of integral equations of the first kind and use these properties to derive a range characterization which can be used to decide whether $\mathrm{cS}_{\kappa}\mathrm{supp}\, u^{\infty}$ is a subset of some test domain G. In Section 4 the range test is described and we prove convergence and regularity of the scheme to reconstruct $\mathrm{S}_{\kappa}\mathrm{supp}\, u^{\infty}$ from the far field pattern u^{∞} for scattering of one incident wave. In Section 5 we describe some efficient implementation of the range test and show numerical examples for the reconstruction of sound-soft obstacles, sound-hard obstacles or medium scatterers when the boundary condition and physical properties of the scatterer are not known.

Finally, we would like to provide some remarks about the measurement domain. The far field is an analytic function, so that its restriction to an arbitrary open subset of $\mathbb{S}^{m-1}$ is theoretically enough to determine the entire far field. Here, we will assume that the far field pattern is known on the whole unit sphere $\mathbb{S}^{m-1}$. For a pilot paper about a new method this is a simple and well-known setting in which the method can be compared to a number of other schemes (see [3]). We expect to be able to apply the method in many related settings, for example, when the measurements are taken on a line or a plane (or even an open subset of that line or plane) Γ on one side of the scatterer. In this case, the scattered field $u^s|_{\Gamma}$ uniquely determines the field u^s in $\mathbb{R}^m \setminus \Omega$ and thus the scattering support. In this case the far field operator $S^{\infty} : L^2(\partial G) \to L^2(\mathbb{S}^{m-1})$ can be replaced by the operator single-layer operator $S : L^2(\partial G) \to L^2(\Gamma)$ and the whole theory will work analogously.

2 Obstacle and medium scattering problems.

This section serves to briefly review the key elements of scattering by bounded objects or media, and to provide some tools for the inversion scheme described in section 4.

Let v^i be an incident field that satisfies the Helmholtz equation,

$$\triangle v + \kappa^2 v = 0, \tag{2.1}$$

equation

with wave number $\kappa > 0$ on $\mathbb{R}^m$. The incident field produces a scattered field v^s that solves the Helmholtz equation on the exterior of the scatterer Ω and is radiating, i.e. it satisfies the *Sommerfeld radiation condition*

$$r^{\frac{m-1}{2}}\Big(\frac{\partial}{\partial r} - i\kappa\Big)v(x) \to 0, \quad r = |x| \to \infty$$

uniformly in all directions. For impenetrable scatterers we consider cases where the scatterer is either sound-soft (a perfect conductor), sound-hard (a perfect reflector) or some mixture of these. Each of these types of scatterers is modeled by a total field,

$$v = v^i + v^s,$$

that satisfies either *Dirichlet, Neumann* or *impedance* boundary conditions. These boundary conditions are given respectively as

$$v|_{\partial\Omega} = 0, \qquad \frac{\partial v}{\partial \nu}|_{\partial\Omega} = 0, \qquad \frac{\partial v}{\partial \nu}|_{\partial\Omega} + \lambda v|_{\partial\Omega} \;=\; 0,$$

with the impedance function $\lambda \in C(\partial\Omega)$. We also treat penetrable scatterers, where the inhomogeneity is modeled by a refractive index $n \in L^\infty(\Omega)$ $(\Omega \subset \mathbb{R}^m)$ and where $n(x) := 1$ for $x \in \mathbb{R}^m \setminus \Omega$. Then the total field $v \in H^2_{loc}(\mathbb{R}^m)$ solves the Helmholtz equation for an inhomogeneous medium,

$$\triangle v + \kappa^2 n v = 0,$$

in $\mathbb{R}^m$, and $v^s = v - v^i$ satisfies the Sommerfeld radiation condition.

For the solution of the Dirichlet problem we represent the scattered field as a combined single- and double layer potential

$$v^s(x) = \int_{\partial\Omega} \left\{ \frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\Phi(x,y) \right\} \varphi(y) ds(y), \quad x \in \mathbb{R}^m \setminus \partial\Omega,$$

where $\Phi(x,y)$ denotes the free-space fundamental solution to the Helmholtz equation in two or three dimensions, respectively. For this representation of the scattered field and the boundary condition, the density φ must satisfy the integral equation

$$\varphi + K\varphi - iS\varphi = -2v^i, \tag{2.2}$$

where S is the single-layer operator,

$$(S\varphi)(x) := 2\int_{\partial\Omega} \Phi(x,y)\varphi(y)ds(y), \quad x \in \partial\Omega \tag{2.3}$$

and K is the double-layer operator,

$$(K\varphi)(x) := 2\int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi(y)ds(y), \quad x \in \partial\Omega.$$

The equation (2.2) has a unique solution that depends continuously on the right-hand side in $C(\partial\Omega)$.

For the Neumann problem we use the modified approach due to Panich [15]

$$v^s(x) = \int_{\partial\Omega} \left\{ \Phi(x,y)\varphi(y) + i\frac{\partial\Phi(x,y)}{\partial\nu(y)}(S_0^2\varphi)(y) \right\} ds(y), \quad x \in \mathbb{R}^m \setminus \partial\Omega, \tag{2.4}$$

where S_0 denotes the single layer operator in the limit as $\kappa \to 0$. For this representation of the scattered field, the density φ can be shown to satisfy the boundary integral equation

$$\varphi - K'\varphi - iTS_0^2\varphi = 2\frac{\partial v^i}{\partial\nu} \tag{2.5}$$

where

$$(K'\varphi)(x) := 2\int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial\nu(x)}\varphi(y)ds(y), \quad x \in \partial\Omega,$$

and

$$(T\varphi)(x) := 2\frac{\partial}{\partial\nu(x)}\int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi(y)ds(y), \quad x \in \partial\Omega.$$

Both Eq.(2.2) and Eq.(2.5) have unique solutions that depend continuously on the incident field in $C(\partial\Omega)$.

For the impedance boundary value problem we follow the same approach using the representation (2.4). An application of the jump relations leads to the equation

$$\Big[I - K' - iTS_0^2 - \lambda S - i\lambda(I+K)S_0^2\Big]\varphi = 2\frac{\partial v^i}{\partial\nu} + 2\lambda v^i.$$

Under suitable assumptions on the impedance λ (basically ensuring uniqueness of the impedance scattering problem) the integral equation Eq.(2.6) has a unique solution which depends continuously on the incident field in $C(\partial\Omega)$.

For the penetrable inhomogeneous medium we use the volume potential approach

$$v(x) = \int_{\Omega} \Phi(x,y)(1-n)(y)\varphi(y)dy, \quad x \in \mathbb{R}^m.$$

Then the scattering problem can be reduced to the *Lippmann-Schwinger* equation

$$\Big(I + \kappa^2 V(1-n)\Big)v \;=\; v^i \tag{2.6}$$

in $C(\Omega)$ with the volume potential operator

$$(V\psi)(x) \ := \int_{\Omega} \Phi(x,y)\psi(y)dy, \quad x \in \Omega.$$

The Lippmann-Schwinger equation has a unique solution in $C(\Omega)$ that depends continuously on the incident field v^i.

3 The scattering support

Let u^∞ be the far field pattern of the scattered (radiating) field u^s, defined in the exterior of some ball B_R with $R > 0$. Because it satisfies (2.1), the scattered field u^s is a real analytic function on $\mathbb{R}^m \setminus B_R$, and therefore may be analytically continued (e.g. by translating its power series expansion) to (possibly) larger unbounded connected open sets. Because this continuation is real analytic, the extension of u^s continues to solve (2.1) wherever it is defined.

We say that a test domain Ω supports u^∞ if u^∞ can be continued to solve the free Helmholtz equation in $\mathbb{R}^n \setminus \Omega$.

Lemma 3.1 *If Ω_1 and Ω_2 are convex sets which support the same far field u^∞. Then $\Omega_1 \bigcap \Omega_2$ supports u^∞.*

Proof. Suppose that we have two continuations, one which defines a continuation u_1 outside Ω_1 and another which defines u_2 outside Ω_2. Rellich's lemma and the unique continuation theorem for the homogeneous Helmholtz equation ([3]) guarantee that the two continuations agree on the unbounded open component of $\mathbb{R}^n \setminus (\Omega_1 \bigcup \Omega_2)$. If Ω_1 and Ω_2 are convex, then this set has just one component and therefore

$$u_{12} = \begin{cases} u_1 & \text{on } \mathbb{R}^n \setminus \Omega_1 \\ u_2 & \text{on } \mathbb{R}^n \setminus \Omega_2 \end{cases}$$

is well defined and satisfies the free Helmholtz equation outside $\Omega_1 \bigcap \Omega_2$. Thus $\Omega_1 \bigcap \Omega_2$ supports u^∞. □

DEFINITION 3.2 *We call the intersection of all convex Ω which support u^∞ the* convex scattering support *of the far field u^∞ or* $\mathrm{cS}_\kappa\mathrm{supp}\ u^\infty$.

We describe some simple properties of the convex scattering support in the lemma below (see [12] for a more elaborate discussion).

Lemma 3.3 *The convex scattering support has the following properties:*

1. If $u^\infty \neq 0$, then $\mathrm{cS}_\kappa\mathrm{supp}\ u^\infty$ is not empty.

2. Suppose u^∞ is the far field pattern produced by a scatterer with convex hull Ω when it is illuminated by an incident field u^i. Then $\mathrm{cS}_\kappa\mathrm{supp}\ u^\infty \subset \overline{\Omega}$.

3. The convex scattering support $cS_\kappa \text{supp } u^\infty$ *contains all the singularities of the scattered field* u^s *in the closure of the unbounded component of the complement of the convex hull of the support of the scatterer. In particular, this includes all corners of sound-soft or sound-hard scatterers which are on the boundary of the convex hull of the support of the scatterer.*

Proof. If $cS_\kappa \text{supp } u^\infty$ is empty, then there exist a finite number of convex Ω's with empty intersection. According to lemma 3.1 u^s extends to a solution to the free Helmholtz equation (2.1) in all of $\mathbb{R}^m$, but the only radiating solution to the free Helmholtz equation is identically zero. This proves 1.

Item 2. is a direct consequence of the definition.

The scattered field, u^s has a unique real analytic continuation to the complement of $cS_\kappa \text{supp } u^\infty$, so is *a fortiori* bounded in a neighborhood of any x in that open set. The hypothesis of item 3 implies that x is in the closure of that set, and that the unique real analytic continuation is unbounded in any neighborhood of x. □

So far, we have defined the convex scattering support of a single far field pattern u^∞. Suppose now that it is possible to illuminate some scatterer Ω with several incident plane waves, $u^i(x,d) = e^{i\kappa x \cdot d}$ with wavenumbers κ and direction d, or superpositions thereof. To simplify notation, we let $D = \kappa d$ and let $\mathcal{D}$ denote the collection of D's that parameterize these incident waves. Also we denote the corresponding scattered fields by $u^s_\kappa(\cdot, d)$ and their far field patterns by $u^\infty_\kappa(\cdot, d)$.

Definition 3.4 *The* convex scattering support *of a scatterer* Ω*, illuminated by a collection of plane waves* $\mathcal{D}$*, is defined to be the union of the convex scattering supports of the scattered waves, that is,*

$$cS_{\mathcal{D}}\text{supp}\Omega := \bigcup_{D \in \mathcal{D}} cS_\kappa \text{supp } u^\infty_\kappa(\cdot, d), \tag{3.1}$$

In the following section, the determination of the convex scattering support of a far field will be based on the following property of the Tikhonov regularization for integral equations of the first kind. We will formulate the result in a general form.

Theorem 3.5 *Let* G *be a bounded domain with boundary of class* C^2*. We consider an injective linear integral operator with continuous kernel and dense range*

$$(A\varphi)(x) := \int_{\partial G} k(x,y)\varphi(y)ds(y), \quad x \in \mathbb{S}^{m-1}, \tag{3.2}$$

from the Hilbert space $X = L^2(\partial G)$ *into the Hilbert space* $Y = L^2(\mathbb{S}^{m-1})$*. Then, for the Tikhonov solution*

$$\varphi_\alpha := (\alpha I + A^* A)^{-1} A^* f \tag{3.3}$$

with regularization parameter α of the equation

$$A\varphi = f \tag{3.4}$$

we obtain the behaviour

$$\lim_{\alpha\to 0} ||\varphi_\alpha|| = \begin{cases} \infty, & \text{if } f \notin A(X), \\ ||\varphi^*|| < \infty, & \text{if } A\varphi^* = f. \end{cases} \tag{3.5}$$

Proof. First, assume that $f \in A(X)$, i.e. there is $\varphi^* \in X$ such that $A\varphi^* = f$. Then, according to Theorem 15.23 and (15.5) of [11], for the Tikhonov equation we have

$$\begin{aligned} \lim_{\alpha\to 0} \varphi_\alpha &= \lim_{\alpha\to 0} (\alpha I + A^*A)^{-1}A^*f \\ &= \lim_{\alpha\to 0} (\alpha I + A^*A)^{-1}A^*A\varphi^* \\ &= \varphi^*, \end{aligned} \tag{3.6}$$

which proves the second line of (3.5).

Second, assume that $f \notin A(X)$. Assume that there is a constant C such that φ_α is bounded for sufficiently small $\alpha > 0$. Then there is a sequence $\alpha_j \to 0$ for $j \to \infty$ such that the weak convergence $\varphi_j \rightharpoonup \tilde{\varphi}$, $j \to \infty$ holds, where we set $\varphi_j := \varphi_{\alpha_j}$. The linear integral operator A maps the weakly convergent sequence into a strongly convergent sequence, i.e. we obtain

$$A\varphi_j \to A\tilde{\varphi} =: \tilde{f} \tag{3.7}$$

with some $\tilde{f} \in A(X) \subset Y$. Passing to the limit $j \to \infty$ in

$$\Big(\alpha_j I + A^*A\Big)\varphi_j = A^*f \tag{3.8}$$

we obtain

$$A^*A\tilde{\varphi} = A^*\tilde{f} = A^*f. \tag{3.9}$$

The density of the range of A is equivalent to the injectivity of A^*, which implies that $\tilde{f} = f$. But this yields the contradiction $f \in A(X)$ and the proof is complete. □

Now, consider the operator

$$(S^\infty\varphi)(\hat{x}) := \int_{\partial G} \gamma_m e^{-i\kappa\hat{x}\cdot y}\varphi(y)ds(y), \;\; \hat{x} \in \mathbb{S}^{m-1}, \tag{3.10}$$

with

$$\gamma_m := \begin{cases} \frac{e^{i\pi/4}}{\sqrt{8\pi\kappa}}, & m = 2, \\ \frac{1}{4\pi}, & m = 3, \end{cases} \tag{3.11}$$

which maps some density $\varphi \in L^2(\partial G)$ onto the far field pattern of the single layer potential

$$v(x) = \int_{\partial G} \Phi(x,y)\varphi(y)ds(y), \;\; x \in \mathbb{R}^m \setminus \overline{G}, \tag{3.12}$$

in the exterior of G.

Lemma 3.6 *Let G be be an open C^2 domain such that the interior homogeneous Dirichlet problem for the Helmholtz equation has only the trivial solution. If G supports u^∞, then u^∞ is in the range of the operator S^∞. Conversely, if u^∞ is in the range of S^∞, then $\overline{G}$ supports u^∞.*

Proof. If G supports u^∞, then the field u^s with far field pattern u^∞ can be analytically extended into the open exterior of the domain G and into a neighborhood of ∂G. We now solve the equation

$$S\varphi = u^s \qquad \text{on } \partial G, \tag{3.13}$$

which has a unique solution since S maps $L^2(\partial G)$ bijectively into $H^1(\partial G)$. On ∂G, the single layer potential v now coincides with u^s and by the solution of the exterior Dirichlet problem for the domain G it coincides with u^s in $\mathbb{R}^m \setminus G$. Thus, the far field pattern $S^\infty\varphi$ of v and u^∞ coincide as well and we have proven that u^∞ is in the range of S^∞.

Now, assume that u^∞ is in the range of S^∞. Then there is a function $\varphi \in L^2(\partial G)$ such that $S^\infty\varphi = u^\infty$. We define the single-layer potential v with density φ and obtain an analytic extension of u^s into $\mathbb{R}^m \setminus \overline{G}$. Thus $\overline{G}$ supports u^∞. □

As a consequence of the preceding result we derive the following test.

Theorem 3.7 **(Range characterization.)** *If G supports u^∞, then*

$$\lim_{\alpha\to 0} \left|\left|(\alpha I + S^{\infty,*}S^\infty)^{-1}S^{\infty,*}u^\infty\right|\right| \; < \; \infty. \tag{3.14}$$

If (3.14) is true for the test domain G, where the homogeneous interior Dirichlet problem for G has only the trivial solution, then $\overline{G}$ supports u_∞.

In particular, if G is convex, then (3.14) implies $\mathrm{cS}_\kappa\mathrm{supp}\, u_\infty \subset \overline{G}$ *and* $\mathrm{cS}_\kappa\mathrm{supp}\, u_\infty \subset G$ *implies (3.14).*

Proof. We combine Theorem 3.5 and Lemma 3.6 to obtain the statement. □

4 The range test, its convergence and regularity

The main goal of this section is to formulate the range test and show convergence and regularity for the reconstruction of the convex scattering support of some far field pattern u^∞. Later, we will use the algorithm of the range test to determine the support of unknown scatterers given the far field pattern for scattering of one incident wave.

We set up the range test as follows.

Algorithm 4.1 **(Range Test.)** *The range test calculates an estimate for the convex scattering support* $\mathrm{cS}_\kappa\mathrm{supp}\, u^\infty$ *of some far field pattern u^∞ in the following steps.*

1. *Choose a set of convex test domains $\mathcal{N} := \{G_j : j \in \mathcal{J}\}$ with boundary of class C^2 such that the homogeneous interior Dirichlet problem for G_j, $j \in \mathcal{J}$, does have only the trivial solution. Here, $\mathcal{J}$ denotes some index set, which for numerical purposes needs to be finite.*

2. *Choose regularization parameters $\alpha, c > 0$.*

3. *For each $G_j \in \mathcal{N}$, $j \in \mathcal{J}$, calculate*

$$\mu(\alpha, G_j) := \left|\left|(\alpha I + S^{\infty,*}S^{\infty})^{-1}S^{\infty,*}u^{\infty}\right|\right|_{L^2(\partial G_j)}. \tag{4.1}$$

4. *Calculate an estimate $M_{\alpha,c}$ for $\mathrm{cS}_\kappa \mathrm{supp}\ u^\infty$ by*

$$M_{\alpha,c} := \bigcap_{\mu(\alpha,G_j)<c} \overline{G_j}. \tag{4.2}$$

The calculation of the functional μ can be performed by standard means of integral equations, see for example [11]. We will describe some efficient way to deal with sets of domains in the following section. Here, we first study convergence for exact data and the behavior of the range test for data with error $\delta > 0$.

Theorem 4.2 **(Convergence of the range test.)** *Given some far field pattern u^∞ we have the convergence for the range test described by Definition 4.1 in the sense that for each test domain G we can decide whether $\mathrm{cS}_\kappa \mathrm{supp}\ u^\infty \subset \overline{G}$ or $\mathrm{cS}_\kappa \mathrm{supp}\ u^\infty \not\subset \overline{G}$. Further, using appropriate increasing sets of sampling domains $\mathcal{N}$, there exists a decreasing sequence M_k, $k \in \mathbb{N}$, of domains M_k such that $\mathrm{cS}_\kappa \mathrm{supp}\ u^\infty \subset \overline{M_k}$ and*

$$\text{for each domain } M \text{ with } \mathrm{cS}_\kappa \mathrm{supp}\ u^\infty \subset \overline{M} \text{ we have } M_k \subset M \tag{4.3}$$

for all sufficiently large indices $k \in \mathbb{N}$.

Remark. The above theorem states the existence of suitable sampling domains such that the unknown scattering support is approximated up to a given precision. For practical construction of these domains here it seems to be necessary to construct a very large set of test domains to be able to come close to the true scattering support. In section 5 we describe a procedure to reduce the number of test domains by methods from image algebra. Another simple method to reduce the number of test domains, called 'set-handling approach', can be found in [21].

Proof. Given some test domain G, the convergence of the decision about the statement $\mathrm{cS}_\kappa \mathrm{supp}\ u^\infty \subset \overline{G}$ is obtained from Theorem 3.7 and equation (3.5) as follows. Assume that $\mathrm{cS}_\kappa \mathrm{supp}\ u^\infty \subset \overline{G}$. If c is chosen such that for the true solution φ^* of the equation $S^\infty \varphi = u^\infty$ we have $c > ||\varphi^*||$, then the range test will deliver the right answer $\mathrm{cS}_\kappa \mathrm{supp}\ u^\infty \subset \overline{G}$ for sufficiently small $\alpha > 0$. Also, if $\mathrm{cS}_\kappa \mathrm{supp}\ u^\infty \not\subset \overline{G}$, then the range test will always deliver the right answer for α sufficiently small.

To construct the sequence M_k, $k \in \mathbb{N}$, we need to define increasing sets of sampling domains $\mathcal{N}_l$, $l \in \mathbb{N}$, such that they contain a decreasing sequence of sets M_k which contain $\mathrm{cS}_\kappa \mathrm{supp}\ u^\infty$ and have the property (4.3) stated in the theorem. Using the standard Hausdorff distance in $\mathbb{R}^m$ it is clear that these sampling sets exist, for example using

$$\tilde{M}_k := \left\{ y \in \mathbb{R}^m : \ d(y, \mathrm{cS}_\kappa \mathrm{supp}\ u^\infty) \le \tfrac{1}{k} \right\} \tag{4.4}$$

and using appropriate sets $\tilde{M}_k \subset M_k \subset \tilde{M}_{k-1}$ to satisfy the condition on the homogeneous interior Dirichlet problem and to obtain boundaries of class C^2. Thus, the second part of the theorem is proven. □

For some inverse problem $A(\varphi) = f_\delta$ with data f_δ, regularity studies the convergence of regularized solutions $\varphi^{(\delta)} = R_{\alpha(\delta)} f_\delta$ towards the exact solution φ for $\delta \to 0$. Here, we assume that the data error $||f_\delta - f||$ is bounded by

$$||f_\delta - f|| \le \delta, \tag{4.5}$$

and f is the exact data corresponding to φ.

Definition 4.3 *A reconstruction scheme R_α with regularization parameters α depending on δ is called* regular, *if for $\varphi^{(\delta)} = R_{\alpha(\delta)} f_\delta$ we have*

$$\varphi^{(\delta)} \to \varphi, \quad \delta \to 0. \tag{4.6}$$

We will now show in what sense there is a regular choice of the regularization parameters for the range test to obtain the convergence of the type (4.6).

Theorem 4.4 **(Regularity of the range test, part 1.)** *Consider the measured far field pattern u_δ^∞ of the scattered field u^s with far field pattern u^∞, where we assume*

$$||u_\delta^\infty - u^\infty||_{L^2(\mathbb{S}^{m-1})} \le \delta. \tag{4.7}$$

For a finite set of sampling domains $\mathcal{N}$ there is a constant $c = c(\delta, u^\infty)$, $c > 0$, and a parameter $\alpha = \alpha(\delta, u^\infty)$, $\alpha > 0$, such that for sufficiently small $\delta > 0$ the range test delivers the right answer to the question whether

$$\mathrm{cS}_\kappa \mathrm{supp}\ u^\infty \subset \overline{G} \qquad or \qquad \mathrm{cS}_\kappa \mathrm{supp}\ u^\infty \not\subset \overline{G} \tag{4.8}$$

for all $G \in \mathcal{N}$.

Proof. We use the boundedness of

$$(\alpha I + S^{\infty,*} S^\infty)^{-1} S^{\infty,*} \tag{4.9}$$

for fixed $\alpha > 0$ as follows. First, for a finite set of sampling domains $\mathcal{N}$ there is a subset $\mathcal{N}' \subset \mathcal{N}$ of domains such that $\mu(\alpha, G)$ remains bounded for $\alpha \to 0$ for all $G \in \mathcal{N}'$ and

$\mu(\alpha, G) \to \infty$ for $G \in \mathcal{N} \backslash \mathcal{N}'$. Then, there are constants $c, \alpha_0 > 0$, such that $\mu(\alpha, G) < c/2$, $\alpha \in (0, \alpha_0]$, for all $G \in \mathcal{N}'$ and $\mu(\alpha, G) > 2c$, $\alpha \in (0, \alpha_0]$, for all $G \in \mathcal{N} \backslash \mathcal{N}'$. We abbreviate

$$\begin{aligned} \tilde{R}_\alpha &:= (\alpha I + S^{\infty,*} S^{\infty})^{-1} S^{\infty,*} \\ \tilde{c} &:= \|\tilde{R}_{\alpha_0}\| \end{aligned} \tag{4.10}$$

and choose $\alpha = \alpha_0$. With the operator $\tilde{R}_{\alpha_0}$ we calculate for $G \in \mathcal{N}'$ the estimate

$$\begin{aligned} \mu(\alpha_0, G) &= \|\tilde{R}_{\alpha_0} u_\delta^\infty\| \\ &\leq \|\tilde{R}_{\alpha_0} u^\infty\| + \|\tilde{R}_{\alpha_0}(u_\delta^\infty - u^\infty)\| \\ &\leq c/2 + \tilde{c} \cdot \delta \\ &\leq c \end{aligned} \tag{4.11}$$

for $0 \leq \delta \leq c/2\tilde{c}$. Analogously, we estimate the functional $\mu(\alpha_0, G)$ for $G \in \mathcal{N} \setminus \mathcal{N}'$.

$$\begin{aligned} \mu(\alpha_0, G) &= \|\tilde{R}_{\alpha_0} u_\delta^\infty\| \\ &\geq \|\tilde{R}_{\alpha_0} u^\infty\| - \|\tilde{R}_{\alpha_0}(u_\delta^\infty - u^\infty)\| \\ &\geq 2c - \tilde{c} \cdot \delta \\ &\geq 3c/2 \end{aligned} \tag{4.12}$$

for $0 \leq \delta < c/2\tilde{c}$. In these cases the range test delivers the right decomposition of $\mathcal{N}$ into $\mathcal{N}'$ and $\mathcal{N} \setminus \mathcal{N}'$ and the proof is complete. □

According to the preceding regularity theorem, for each finite set of domains $\mathcal{N}$ we obtain some $\delta > 0$ such that for data with error less than δ the range test finds exactly those domains which contain $\mathrm{cS}_\kappa \mathrm{supp}\ u^\infty$. There exist arbitrarily fine finite coverings $\mathcal{N}$ of the set of all domains to obtain results up to some prescribed error ϵ in the Hausdorff metric.

We expect the dependence of the parameter δ on ϵ to be such that $\delta \to 0$ as $\epsilon \to 0$, since the norm of $\tilde{R}_\alpha$ tends to infinity as $\alpha \to 0$. Finally, we obtain the following important result.

Theorem 4.5 **(Regularity of the range test, part 2.)** *For each $\epsilon > 0$ let $\mathcal{N}(\epsilon)$ be a finite set of domains such that*

$$\min_{G \in \mathcal{N}(\epsilon)} d(G, \mathrm{cS}_\kappa \mathrm{supp}\ u^\infty) \leq \epsilon. \tag{4.13}$$

We use the range test to determine the smallest set $M(\epsilon)$ in $\mathcal{N}(\epsilon)$ which contains $\mathrm{cS}_\kappa \mathrm{supp}\ u^\infty$ from measured data u_δ^∞ for δ sufficiently small. Then, we can choose ϵ to depend on the data error δ such that we obtain

$$d(M(\epsilon), \mathrm{cS}_\kappa \mathrm{supp}\ u^\infty) \leq \epsilon(\delta) \to 0, \quad \delta \to 0 \tag{4.14}$$

for the Hausdorff distance d between $M(\epsilon)$ and $\mathrm{cS}_\kappa \mathrm{supp}\ u^\infty$.

Proof. We consider the function $\delta(\epsilon)$ as constructed above. It can be chosen as a monotonicly increasing function with $\delta(\epsilon) \to 0$ for $\epsilon \to 0$. We invert this function to obtain a function $\epsilon(\delta)$ such that for all data u_δ^∞ with error less than δ the range test for $\mathcal{N}(\epsilon)$ delivers the right decision about the question (4.8). Choosing the smallest set $M(\epsilon)$ in $\mathcal{N}(\epsilon)$ which contains $\mathrm{cS}_\kappa \mathrm{supp}\ u^\infty$ we obtain (4.14) and the proof is complete. □

5 Numerical examples

In this last section we show numerical examples using the range test to determine the support of obstacles or medium scatterers. We describe an efficient implementation of the method and a streamlined treatment of the test domains. We describe how we choose the cut-off parameter c for reconstructions. Finally, we will suggest modifications of the range test to enhance the contrast of the reconstructions.

We show examples of sound-soft obstacles, sound-hard obstacles and medium scatterers, with reconstructions calculated from the far field pattern of *one* wave. We do not utilize any *a priori* information about the boundary condition or physical nature of the scatterer under consideration.

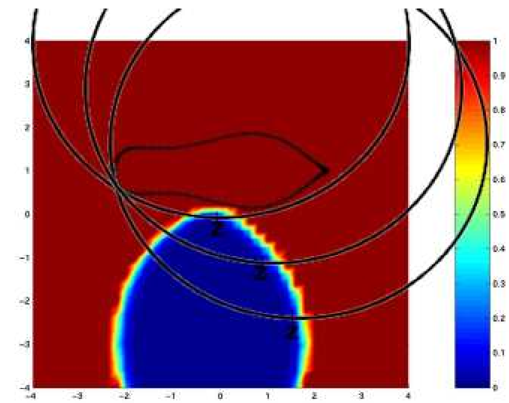
The following algorithm is a realization of the method described in the convergence and regularity theorems due to the following two arguments. First, we remark that there is no need to test the intersection $G := G_1 \cap G_2$ of two test domains, if the test for G_1 and G_2 has been performed and has been positive (i.e. $\mu(G_1) < c$ and $\mu(G_2) < c$). If u^s can be analytically extended into $\mathbb{R}^m \setminus \overline{G_1}$ and into $\mathbb{R}^m \setminus \overline{G_2}$, then it can be analytically extended into $\mathbb{R}^m \setminus (\overline{G_1} \cap \overline{G_2})$. Thus the test for $G_1 \cap G_2$ will be positive as well. Second, if we know some sets which are in the exterior of the scattering support, then the union of these sets will be in the exterior of the scattering support. If the complement M of this union is an isolated domain in $\mathbb{R}^m$, then the field u^s can be analytically extended into M^c and the range test for this set M should give a positive result, i.e. we can use M as an upper bound for the scattering support. Thus, by taking intersections of domains $G(z)$ or (as carried out here) unions of subsets of the complement of the scattering support we construct special sets $M = M_{c,\alpha}$ for which we derived existence and convergence statements in section 4.

First, we describe details of our special implementation for the range test which was introduced by Definition 4.1.

1. We choose some rectangular sampling grid $\mathcal{G}$ which covers the unknown scatterer. For each point $z \in \mathcal{G}$ we construct a test domain from some reference domain G_0 by

$$G_z := G_0 + z, \quad z \in \mathcal{G}, \tag{5.1}$$

where G_0 is chosen with the conditions stated in Definition 4.1, part 1, and such that $0 \in \partial G_0$. Choosing test domains which are translates of one fixed domain gives us a very quick way to calculate the solution of the corresponding integral equations. To this end



(a)

Figure 1: Demonstration of an image $\mathcal{I}_l$ for one rotation as described in 3., where the test domain G_0 is a circle of radius $r = 4$ and center $(0, 4)$. The images shows the result for a sound-soft scatterer Ω indicated by the black line. The points z such that $\Omega \subset G_z$ can clearly be found by the blue area. The blue area is part of the exterior of the unknown scatterer, where the special form of the blue area is a consequence of the form of the approximation domain $G(z)$. We show three different translated versions of $G(z)$. For the rotation shown, z is the lowest point on the circle which is the boundary of $G(z)$. As long as $\Omega \subset G(z)$ we obtain blue (=small) values, otherwise we observe red (=large) values.

we remark that

$$\begin{aligned}(S^\infty[G_z]\varphi)(\hat{x}) &= \int_{\partial G_z} e^{-i\kappa\hat{x}\cdot y}\varphi(y)ds(y),\\ &= e^{-i\kappa\hat{x}\cdot z}\int_{\partial G_0} e^{-i\kappa\hat{x}\cdot y}\tilde{\varphi}(y)ds(y)\\ &= e^{-i\kappa\hat{x}\cdot z}(S^\infty[G_0]\varphi)(\hat{x}) \qquad (5.2)\end{aligned}$$

where $\tilde{\varphi}(y) := \varphi(y+z)$ for $y \in \partial G_0$. Using (5.2) we can calculate

$$\begin{aligned}&(\alpha I + S_z^{\infty,*}S_z^\infty)^{-1}S_z^{\infty,*}u^\infty\\ &= (\alpha I + S_0^{\infty,*}S_0^\infty)^{-1}S_0^{\infty,*}\Big(e^{i\kappa\hat{x}\cdot z}u^\infty(\hat{x})\Big), \qquad (5.3)\end{aligned}$$

i.e. for each reference domain G_0 we need to solve only one equation with different right-hand sides given by

$$e^{i\kappa\hat{x}\cdot z}u^\infty(\hat{x}),\ \ \hat{x} \in \mathbb{S}^{m-1}. \qquad (5.4)$$

2. We choose the regularization parameter α by trial and error. For our examples, we worked with $\alpha = 10e-9$. The constant c is determined using the far field pattern u_0^∞ for some known reference domain Ω_0 and determining

$$c := ||\tilde{R}_\alpha u_0^\infty||_{L^2(\partial G_{\tilde{z}})} \qquad (5.5)$$

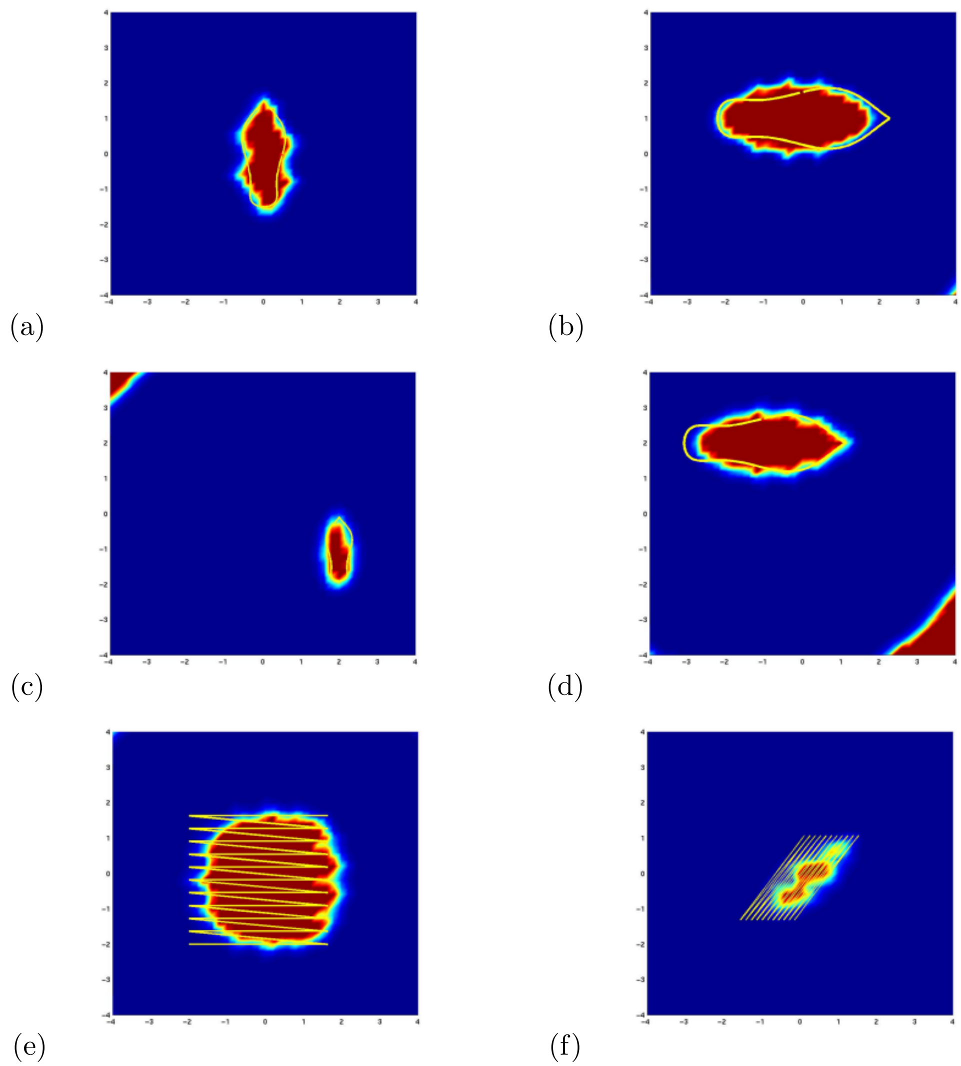


Figure 2: Reconstructions of sound-soft obstacles (a) and (b), of sound-hard obstacles (c) and (d) and medium scatterers (e) and (f) (the jagged lines indicate the support of the scatterer) with $p = 3$. Here we used the wave number $\kappa = 2$, aperture opening $\theta = 1.8\pi;$, regularization parameter $\alpha = 10^{-9}$ for an incident wave with direction $d = (-1, 0)$. The far field pattern contains 1-2% errors. For these images the same algorithm with identical parameters is applied to different data sets.

with $\tilde{R}_\alpha$ defined by (4.10) for the domain $G_{\tilde{z}}$ where $\tilde{z}$ is a point on $\partial\Omega_0$. All images have been normalized by multiplication with $1/c$.

3. For each domain G_z we calculate $\mu(\alpha, G_z)$ by (4.1) as discussed in Definition 4.1. For our simple choice of domains , the images are produced as follows. We choose G_0 to be a circle or ellipse, respectively. These domains are only suitable to reconstruct the convex hull of scatterers, but are sufficient to demonstrate the feasibility of the method. The values $\mu(\alpha, G_z)$ are mapped onto a the grid $\mathcal{G}$ by $z \mapsto \mu(\alpha, G_z)$ (as demonstrated in Figure 1). In principle, this image could be used to find the scatterer by taking the intersection of all domains G_z where z is in the blue area. This approach is worked out in [21].

Instead, we repeat the calculations above with several new rotations $G_0^{(l)}$ for $l = 1, ..., L$ of the domain G_0 around the origin. For each rotation an image $\mathcal{I}_l$ is produced by

$$\mathcal{I}_l \ := \{\min(\mu(\alpha, G_z), c) : \ \ z \in \mathcal{G}\}, \tag{5.6}$$

see for example Figure 1, where the test domain G_0 is located in the upper half space. We will use a simple trick to obtain this intersection using the other images $\mathcal{I}_l$, $l = 1, ..., L$. The main idea of the trick is as follows: Instead of taking the intersection of those domains $G(z)$ where $\mu(G(z)) < C$, we can take the union of its complements. The blue area in Figure 1 is a subset of this complement. Thus we obtain a lower estimate for the complement of the unknown scatterer by taking the union of the blue areas which arise when rotating the domain of approximation. This corresponds to taking the minimum over the images under consideration. This is formalized as follows.

4. We perform the operation (4.2) by taking the minimum over all images $\mathcal{I}_l$, $l = 1, ..., L$, where the rotation angles β_l are chosen as

$$\beta_l \ := \ \frac{2\pi(l-1)}{L}, \ \ l = 1, ..., L. \tag{5.7}$$

A number of examples for

$$\mathcal{I} \ := \ \min\{\mathcal{I}_l : \ \ l = 1, ..., L\} \tag{5.8}$$

is shown in the figures below.

Finally, we would like to discuss some slight modifications of the range test to obtain better contrast. Instead of the functional $\mu(\alpha, G)$ defined by (4.1) we define

$$\mu_2(\alpha, G) := \left|\left|(\alpha I + S^{\infty,*} S^\infty)^{-p} S^{\infty,*} u^\infty\right|\right|_{L^2(\partial G_j)} \tag{5.9}$$

for $p \in \mathbb{N}$. For $p = 1$ this coincides with the functional $\mu(\alpha, G)$. For $p = 2$ it calculates the norm of the derivative

$$\varphi'_\alpha \ := \ \frac{\partial \varphi_\alpha}{\partial \alpha} \tag{5.10}$$

of

$$\varphi_\alpha := (\alpha I + S^{\infty,*}S^{\infty})^{-1}S^{\infty,*}u^{\infty} \tag{5.11}$$

and for higher p it is the norm of the $(p-1)$st derivative modulo some constants. For all pictures in this paper we used $p = 3$. We remark that $||\varphi_\alpha|| \to \infty$ yields $||\varphi_\alpha^{(p-1)}|| \to \infty$, $p \in \mathbb{N}$. In general $||\varphi_\alpha|| \leq c$ does not yield the boundedness of the derivatives, but we expect the rate of divergence of these derivatives for $\alpha \to 0$ to be smaller if $\mathrm{cS}_\kappa \mathrm{supp}\ u^\infty \subset \overline{G}$ than for $\mathrm{cS}_\kappa \mathrm{supp}\ u^\infty \not\subset \overline{G}$, which is confirmed by the numerical results.

Figure 2 shows the reconstruction of several scatterers of different size, location and physical nature. For all these reconstructions we used the same algorithm with fixed parameters. For all images we chose $\kappa = 2$ and considered one wave with direction of incidence $d = (-1, 0)$. We measured the far field pattern at 121 points.

The influence of the size of the sampling domains and the value of the cut-off parameter c is shown in Figure 3. The size of the sampling domain does *not* significantly change the reconstructions. However, the influence of the cut-off parameter is large. Strategies to choose the this cut-off parameter in dependence of the data and the reconstructions need to be part of future research.

References

[1] Bucci, O.M., Capozzoli, A. and D'Elia, G.: Determination of the convex hull of radiating or scattering systems: a new, simple and effective approach. *Inverse Problems* **18**, 1621-1638 (2002).

[2] Colton, D. and Kirsch, A.: "A simple method for solving inverse scattering problems in the resonance region." *Inverse problems* **12**(4), 383-93 (1996).

[3] Colton, D. and Kress, R.: *Inverse Acoustic and Electromagnetic Scattering Theory.* 2nd Ed., Springer-Verlag (1998).

[4] Colton, D. and Monk, P.: "A novel method for solving the inverse scattering problem for time-harmonic waves in the resonance region." *SIAM J. Appl. Math.* **45**, 1039-1053 (1985).

[5] Colton, D. and Monk, P.: "A novel method for solving the inverse scattering problem for time-harmonic waves in the resonance region II." *SIAM J. Appl. Math.* **46**, 506-523 (1986).

[6] Engl, H., Hanke, M. and Neubauer, A.: *Regularization of Inverse Problems.* Kluwer 1996.

[7] Gilbarg, D. and Trudinger, N.S.: *Elliptic Partial Differential Equations of Second Order* Springer 1998.

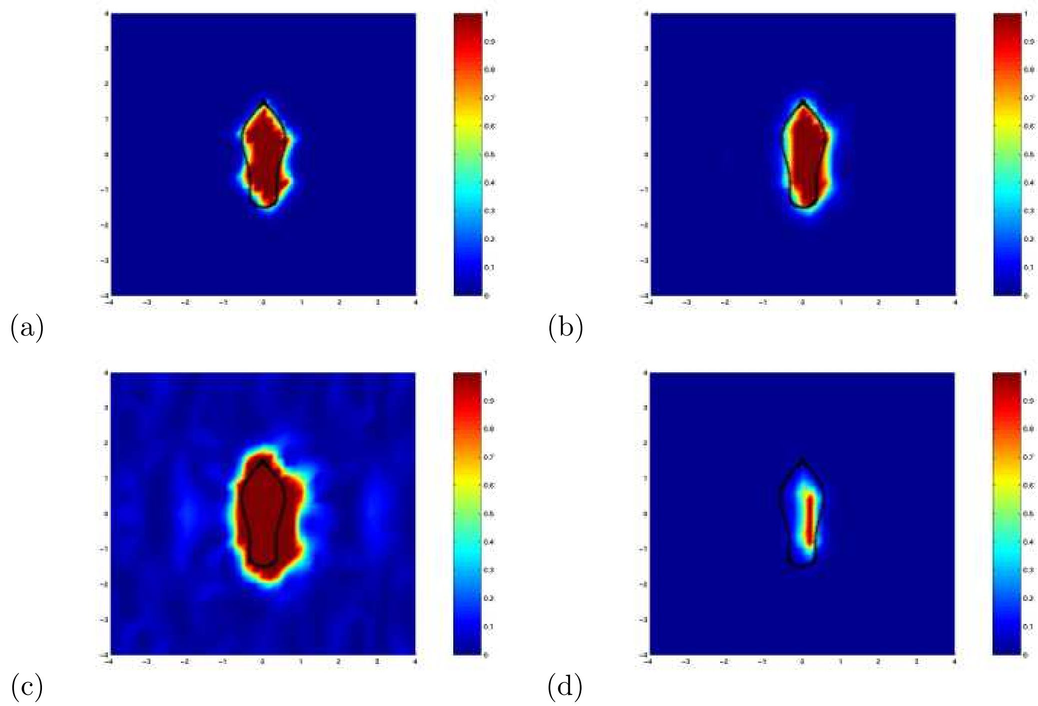


Figure 3: We demonstrate the dependence of the reconstructions on the size of the sampling domain G_0 and on the cut-off parameter c. For the first two images we choose $p = 2$, $c =$1e10, $L = 12$, $\alpha =$1e-7 and consider circles G(z) with radius $r = 4$ (a) and $r = 6$ (b). The reconstructions become slightly smoother for larger r, but they do not change significantly. However, if we change the cut-off parameter c from 1e10 to 1e9 (c) or to 1e11 (d), this changes the reconstructed size of the object. For c too small, the object is over-estimated. For c too large, the object is under-estimated.

[8] Ikehata, M.: "Reconstruction of an obstacle from the scattering amplitude at a fixed frequency." Inverse Problems 14 (1998), no. 4, 949–954.

[9] Ikehata, M.: "Enclosing a polygonal cavity in a two-dimensional bounded domain from Cauchy data", Inverse Problems 15 (October 1999) 1231-1241.

[10] Kirsch, A: "Characterization of the shape of a scattering obstacle using the spectral data of the far field operator." *Inverse problems* **14**(6), 1489-1512 (1998).

[11] Kress, R.: *Linear Integral Equations.* Springer-Verlag 1989.

[12] Kusiak, S. and Sylvester, J.: " The scattering support." (submitted for publication).

[13] Luke, Potthast: "Limited aperture image processing in inverse acoustic obstacle scattering." (submitted for publication)

[14] Luke, D.R. and Potthast, R.: "The no response test - a sampling method for inverse scattering problems." SIAM J. Appl. Math, to appear.

[15] Panich, O.I.:, "On the question of the solvability of the exterior boundary-value problems for the wave equation and Maxwell's equations. Usp. Mat Nauk **20A**, 221-226 (1965) (Russian).

[16] Potthast, R.: "A fast new method to solve inverse scattering problems." Inverse Problems 12, 731-742 (1996).

[17] Potthast, R.: "A point-source method method for inverse acoustic and electromagnetic obstacle scattering problems." IMA Jour. Appl. Math. 61, 119-140 (1998).

[18] Potthast, R.: *Point sources and multipoles in inverse scattering theory.* Habilitation Thesis, Göttingen 1999.

[19] Potthast, R.: Stability estimates and reconstructions in inverse scattering using singular sources. J. Comp. Appl. Math. 114, 247-274 (2000).

[20] Potthast, R.: *Point sources and multipoles in inverse scattering theory.* Chapman & Hall (2001).

[21] Potthast, R.: 'A set-handling approach for the no-response test and related methods.' submitted for publication.