LINEAR AND NONLINEAR INVERSE SCATTERING

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Abstract.
In this paper we discuss one dimensional scattering and inverse scattering for the Helmholtz equation on the half line from the point of view of the layer stripping. By full or nonlinear scattering, we mean the transformation between the index of refraction (actually half of its logarithmic derivative) and the reflection coefficient. We refer to this mapping as nonlinear scattering, because the mapping itself is nonlinear. Another appropriate name is multiple scattering, as this model includes the affects of multiple reflections.

By linear scattering we mean the Born, or single scattering, approximation. This is the Frechet derivative of the full scattering transform at the constant index of refraction, which can be calculated to be exactly the Fourier transform.

In [6], we introduced a variant of layer stripping based on causality and the Riesz transform, rather than on trace formulas – see [2], [3],[4],or [7],for other approaches to layer stripping. A by-product of our layer stripping formalism was the discovery of a nonlinear Plancherel equality, which plays a role in the analysis of the inverse scattering problem, analogous to that played by the linear Plancherel equality in developing the theory of the Fourier transform.

That linear-nonlinear analogy sets the theme for this work. In the next section, we review the main results of [6], including a brief derivation of the nonlinear Plancherel equality. We show how this equality suggests a natural metric for measuring the distance between reflection coefficients and show that the scattering and inverse scattering maps become homeomorphisms when we use this metric.

In section 2, we exhibit a nonlinear Riesz transform , which plays for the nonlinear inverse scattering problem the same role in enforcing causality as the linear Riesz transform [1] plays in signal processing. We then construct a numerical inverse scattering algorithm [2] based on the nonlinear Riesz transform. A rough statement of the results there (compare theorems 2.1 and 2.2 with theorems 2.3 and 2.4) is that the inverse scattering transform is a Fourier decomposition, with the addition of Fourier modes replaced by the composition of these modes as conformal maps of the unit disk. Notice that such a composition rule preserves the nonlinear constraint that the modulus of the reflection coefficient remain less than one.

In the final section we prove two linear and nonlinear Paley-Wiener theorems and a Shannon sampling theorem. One of our Paley- Weiner theorems (theorem 3.2) appears to be new even in the linear case.

Key words. Riesz Transform, inverse scattering, Helmhotz equation

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1. Introduction.

We begin with the Helmholtz equation,

$$\frac{d^2u}{dz^2} + \omega^2 n^2(z)u = 0 \tag{1.1}$$

and assume that the index of refraction is constant and equal to 1 in $z \geq 0$ As in [6], we transform into travel time coordinates by introducing the new independent

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[1]The Riesz transform is perhaps less well known to the signal processing community than the Hilbert transform, which is the principal value integral $Hf(\omega) = \frac{1}{\pi}\int \frac{f(\eta)}{\eta-\omega}d\eta$. The Riesz transform is a projection built from the Hilbert transform via the formula, $P^+ = \frac{1}{2}(I + iH)$.

[2]A matlab GUI implementation is available at www.math.washington.edu/∼ sylvest/

variable

$$x = \int_0^z n(s)ds.$$

as well as the variation in the index of refraction with respect to travel time:

$$\alpha(x) = \frac{n'(x)}{2n(x)}$$

This definition of α is differs from the one used in [6] by a factor of two. This has the effect of making the linear scattering map exactly the Fourier transform of α, rather than the Fourier transform of $\alpha/2$, and thus simplifies a lot of the following notation. It is worth mentioning an equivalent definition of α, namely,

$$\alpha(x) = \frac{1}{2}\frac{dc(z)}{dz}\bigg|_{z=z(x)}$$

That is, α is just half the derivative of the local wave-speed ($c = \frac{1}{n}$) with respect to the depth, z, viewed as a function of the travel time depth, x. This formula has the practical advantage that there is an exact correspondence between piecewise linear $c(z)$ and piecewise constant $\alpha(x)$. In travel time coordinates, (1.1) becomes

$$u'' + 2\alpha(x)u' + \omega^2 u = 0 \tag{1.2}$$

There is a unique solution to (1.2) which has the asymptotics

$$u(x,\omega) \sim e^{-i\omega x}$$

as $x \to -\infty$. Because $n \equiv 1$ for $x > 0$, $\alpha \equiv 0$ for $x > 0$; so that, for $x > 0$, u has the representation

$$u(x,\omega) = \frac{1}{t(\omega)}\left(e^{-i\omega x} + r_0(\omega)e^{i\omega x}\right) \tag{1.3}$$

Equation (1.3) serves as a definition for the reflection coefficient, $r_0(\omega)$, as well as the transmission coefficient $t(\omega)$. We confine our attention to $r_0(\omega)$ here, and we denote the scattering map by S

$$\alpha \overset{S}{\mapsto} r_0(\omega)$$

The layer stripping approach is based on the observations that:

1) $r_0(\omega)$ can be naturally extended to $r(x,\omega)$, defined for any $x \leq 0$.
2) $r(x,\omega)$ satisfies a Ricatti type ordinary differential equation.

To see 1), check first that

$$r_0(\omega) = \frac{-i\omega + \frac{u(0,\omega)'}{u(0,\omega)}}{-i\omega - \frac{u(0,\omega)'}{u(0,\omega)}} \tag{1.4}$$

and then define

$$r(x,\omega) = \frac{-i\omega + \frac{u(x,\omega)'}{u(x,\omega)}}{-i\omega - \frac{u(x,\omega)'}{u(x,\omega)}} \tag{1.5}$$

A straight forward calculation, using (1.2) will verify that

$$r' = 2i\omega r + \alpha(1 - r^2) \tag{1.6}$$
$$r(-\infty,\omega) = 0 \tag{1.7}$$
$$r(0,\omega) = r_0(\omega) \tag{1.8}$$

In [6], the forward scattering problem is treated by analysis of the initial value problem, (1.6-1.7), while inverse scattering is treated by studying the initial value problem (1.6-1.8). Before summarizing this approach, we examine this formulation of single scattering. The Frechet derivative, at $\alpha \equiv 0$, of the scattering map S is called the Born (or single scattering) approximation. To calculate this, let $\alpha = \varepsilon a$ and $\rho = \frac{dr}{d\varepsilon}\big|_{\varepsilon=0}$; then differentiating (1.6-1.7-1.8) yields

$$\rho' = 2i\omega\rho + a \tag{1.9}$$
$$\rho(-\infty,\omega) = 0 \tag{1.10}$$
$$\rho(0,\omega) = \rho_0(\omega) \tag{1.11}$$

which has the explicit solution

$$\rho(x,\omega) = \int_{-\infty}^{x} e^{2i\omega(x-y)} a(y)dy \tag{1.12}$$
$$= (H_{y<0}a(y+x))^{\wedge} \tag{1.13}$$

where $H_{y<0}$ denotes the indicator function of the left half line. At $x = 0$, this is just the Fourier transform (we use $-2i\omega$ in the exponent for convenience) of $H_{y<0}a$. Notice that, for each x, $\rho(x,\omega)$ belongs to the Hardy space, $\mathcal{H}^2(\mathbb{C}^+)$.

We recall, following [5], that

$$H^2(\mathbb{C}^+) = \{\rho \mid \rho \text{ holomorphic in } \mathbb{C}^+ \text{ and } \sup_{b>0} \|\rho(\cdot + ib)\|_{L^2} < \infty\}$$

As a is real valued, ρ will have an additional symmetry, so we define

$$\mathcal{H}^2(\mathbb{C}^+) = \{\rho \in H^2(\mathbb{C}^+) \mid \rho(-\overline{\omega}) = \overline{\rho(\omega)}\} \tag{1.14}$$

With $\mathcal{H}^2(\mathbb{C}^-)$ defined similarly and $\mathcal{L}^2$ denoting L^2 functions with $f(-\omega) = \overline{f(\omega)}$, we have

$$\mathcal{L}^2(\mathbb{R}) = \mathcal{H}^2(\mathbb{C}^+) \oplus \mathcal{H}^2(\mathbb{C}^-) \tag{1.15}$$

In fact, $\mathcal{H}^2(\mathbb{C}^\pm)$ are exactly the Fourier transforms of real valued L^2 functions supported on the negative (resp. positive) half line (see [5]). Thus the initial value problem (1.9-1.10) has a unique solution among the continuous maps from the interval $(-\infty, 0)$ into $\mathcal{H}^2(\mathbb{C}^+)$.

If we turn to the initial value problem (1.9 – 1.11), we see that it has the explicit solution

$$\rho(x,\omega) = \rho_0(\omega)e^{2i\omega x} + \int_0^x e^{2i\omega(x-y)}a(y)dy \tag{1.16}$$

Now, while $\rho(x,\omega) \in \mathcal{H}^2(\mathbb{C}^+)$ for every x, neither of the two terms in (1.16) do. In fact, the second term is in $\mathcal{H}^2(\mathbb{C}^-)$. If we apply the projections P^+ and P^-, which project orthogonally onto the first and second factors in (1.15), to (1.16), we obtain

$$\rho(x,\omega) = P^+\rho_0(\omega)e^{2i\omega x} \tag{1.17}$$

$$(H_{x<y<0}a)^\wedge = e^{-2i\omega x}P^-\rho_0(\omega)e^{2i\omega x} \tag{1.18}$$

We may restate (1.12) and (1.17–1.18) as theorems about initial value problems for (1.9); namely,

Theorem 1.1 (Single Scattering).
<u>*Forward Scattering*</u> *For any $a(x)$ in $L^2(-\infty,0)$, there exists a unique solution $\rho(x,\omega)$ in $C((-\infty,0);\mathcal{H}^2(\mathbb{C}^+))$ satisfying (1.9) and (1.10).*

<u>*Inverse Scattering*</u> *For any $\rho_0(\omega)$ in $\mathcal{H}^2(\mathbb{C}^+)$, there exists a unique pair $(a(x),\rho(x,\omega))$ in $L^2(-\infty,0)\oplus C((-\infty,0);\mathcal{H}^2(\mathbb{C}^+))$ satisfying (1.9) and (1.11).*

The point of the theorem is that, when we add causality (i.e. $\rho_0(\omega)\in\mathcal{H}^2(\mathbb{C}^+)$) to (1.9), the upward propagation for the ODE (1.9) remains an evolution for ρ, but the downward initial value problem becomes an evolution for both ρ and a. Thus the inverse problem is solved by merely propagating the differential equation in the other direction. The main theorem in [6] is that the same is true for the full scattering problem. That is:

Theorem 1.2 (Multiple Scattering).
<u>*Forward Scattering*</u> *For any $\alpha(x)$ in $L^2(-\infty,0)$, there exists a unique solution $r(x,\omega)$ in $C((-\infty,0);\mathcal{H}^E(\mathbb{C}^+))$ satisfying (1.6) and (1.7).*

<u>*Inverse Scattering*</u> *For any $r_0(\omega)$ in $\mathcal{H}^E(\mathbb{C}^+)$, there exists a unique pair $(\alpha(x),r(x,\omega))$ in $L^2(-\infty,0)\oplus C((-\infty,0);\mathcal{H}^E(\mathbb{C}^+))$ satisfying (1.6) and (1.8).*

The only difference between the two theorems is that the second makes use of the the Hardy space, $\mathcal{H}^E(\mathbb{C}^+)$, defined by

$$\mathcal{H}^E(\mathbb{C}^+) = \{r \mid r \text{ holomorphic in } \mathbb{C}^+,\ \sup_{b>0} E(r) < \infty, \text{ and } r(-\overline{\omega}) = \overline{r(\omega)}\}$$

which replaces the L^2 norm with

$$E(r) := \int e(r)d\omega := \int (-\log(1-|r|^2))d\omega \tag{1.19}$$

The appearance of the space $\mathcal{H}^E(\mathbb{C}^+)$ is a consequence of the Plancherel theorem.

Theorem 1.3 (Plancherel Theorems).
For $\rho(x,\omega)$ satisfying (1.9) and (1.10), and for $r(x,\omega)$ satisfying (1.6) and (1.7),

$$\int_{-\infty}^{\infty} |\rho(x,\omega)|^2 d\omega = \pi \int_{-\infty}^{x} |a|^2 dx \tag{1.20}$$

$$\int_{-\infty}^{\infty} -\log(1-|r(x,\omega)|^2)\ d\omega = \pi \int_{-\infty}^{x} |\alpha|^2 dx \tag{1.21}$$

Warning 1. *The reader should be warned that, while* (1.20) *does not depend on a being real valued,* (1.21) *and Theorem 1.2 do depend on the reality of α. We are assuming the reality of α in the rest of this paper.*

For a careful proof of theorem 1.3 we refer to [6], but we include a sketch below:

Proof:
We start with (1.6), multiply by $\overline{r}$, and take real parts to obtain

$$|r|^{2'} = \alpha\,(r+\overline{r})\,(1-|r|^2) \tag{1.22}$$

Dividing by $1-|r|^2$ gives

$$-\log(1-|r|^2)' = \alpha\,(r+\overline{r}) \tag{1.23}$$

The formal expansion $r(x,\omega) = \frac{\alpha(x)}{2i\omega} + \mathrm{O}\left(\frac{1}{\omega^2}\right)$ for large ω, suggests that

$$\int_{-\infty}^{\infty} r(\omega) d\omega = \frac{\pi\alpha(x)}{2}$$

so that, integrating (1.23) with respect to ω gives

$$\left(-\int_{-\infty}^{\infty} \log(1-|r(x,\omega)|^2) d\omega\right)' = \pi|\alpha|^2.$$

Integrating with respect to x gives (1.21). We may obtain (1.20) by an analogous argument, or from (1.21) by setting $\alpha = \varepsilon a$ and letting $\varepsilon \to 0$. ■

The Plancherel equality (1.21) suggests a metric to measure distance between two reflection coefficients and hence view $\mathcal{H}^E(C^+)$ as a complete metric space. In this topology, the scattering map S becomes a homeomorphism. We need a little notation to describe this metric:

$$E(r,s) := \int e(r,s) d\omega = \int (-\log(1-\overline{r}s)) d\omega \tag{1.24}$$

$$\begin{array}{rcl} D_E^2(r,s) & := & E(-r \circ s) \\ & := & E\left(\frac{s-r}{1-\overline{r}s}\right) \\ & =: & \int d_e(r,s) d\omega \\ & = & \int e\left(\frac{s-r}{1-\overline{r}s}\right) d\omega \end{array} \tag{1.25}$$

We will call (1.24) the E-inner product, and (1.25) the distance in the E-metric. As justification for the previous definitions, we offer the theorem below [3]:

THEOREM 1.4. *The scattering map is a homeomorphism from* $L^2(\mathbb{R}^-)$ *onto* $\mathcal{H}^E(\mathbb{C}^+)$, *i.e. if* $r_n = S\alpha_n$, *then*

$$\|\alpha_n - \alpha\|_{L^2} \to 0$$

if and only if

$$D_E(r_n, r) \to 0.$$

The following results verify that D_E is indeed a metric and provide sufficient infrastructure to prove theorem 1.4 and facilitate the analysis necessary in section 3.

LEMMA 1.5. *The metrics,* d_e*, and therefore* D_E*, are conformally invariant; i.e. for any conformal map* F *of the unit disk onto itself*

$$d_e(r, s) = d_e(F(r), F(s)).$$

Proof A conformal map of the unit disk, $F(z)$ has the form

$$F(z) = e^{i\theta}\frac{a - z}{1 - \overline{a}z} \tag{1.26}$$

where $\theta \in \mathbb{R}$ and a belongs to the unit disk. We use the notation F_a to refer to the map in (1.26) with $\theta = 0$. Now

$$\begin{aligned} d_e(b, c) &= e(-b \circ c) \\ &= e(F_b(c)) \end{aligned}$$

while

$$d_e(G(b), G(c)) = e(F_{G(b)}(G(c))).$$

Now

$$F_{G(b)}(G(z)) : b \mapsto 0$$

so that, according to (1.26),

$$F_{G(b)}(G(z)) = e^{i\theta}F_b(z)$$

for some θ, so

$$\begin{aligned} d_e(G(b), G(c)) &= e(e^{i\theta}F_b(c)) \\ &= e(F_b(c)) \end{aligned}$$

■

[3] A weaker version of continuous dependence was stated in [6]. L. Päivärinta has pointed out an error – a term is missing in (83) – which invalidates the proof given there. Theorem 1.4 provides a correction to that proof, as well as a stronger result.

Theorem 1.6 (Cauchy Schwartz and Triangle Inequalities).

$$|E(r,s)| \le E(r)^{\frac{1}{2}} E(s)^{\frac{1}{2}} \tag{1.27}$$

$$D_E(r,s) \le D_E(r,\tau) + D_E(\tau,s). \tag{1.28}$$

Proof

$$\begin{aligned} E(a,b) &= \int \log(1-\overline{a}b)dw \\ &= \int \sum_{k=1}^{\infty} \frac{\overline{a}b^k}{k} dw \\ &\le \int \left(\sum_{k=1}^{\infty} \frac{|a|^{2k}}{k}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \frac{|b|^{2k}}{k}\right)^{\frac{1}{2}} dw \\ &\le \left(\int \sum_{k=1}^{\infty} \frac{|a|^{2k}}{k} dw\right)^{\frac{1}{2}} \left(\int \sum_{k=1}^{\infty} \frac{|b|^{2k}}{k} dw\right)^{\frac{1}{2}} \\ &= E(a)^{\frac{1}{2}} E(b)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} D_E^2(a,b) &= E(-a \circ b) \\ &= E(a) + E(b) - 2ReE(a,b) \\ &\le E(a) + E(b) + 2E(a)^{\frac{1}{2}} E(b)^{\frac{1}{2}} \\ &= (E(a)^{\frac{1}{2}} + E(b)^{\frac{1}{2}})^2 \\ &= (D_E(a,0) + D_E(0,b))^2. \end{aligned}$$

Now, given any C, choose F conformal and mapping 0 to C. Then

$$\begin{aligned} D_E(a,b) &= D_E(F^{-1}(a), F^{-1}(b)) \\ &\le D_E(F^{-1}(a), 0) + D_E(0, F^{-1}(b)) \\ &= D_E(a,c) + D_E(c,b). \end{aligned}$$

■

Theorem 1.7 (Weak and Strong Convergence). *Suppose that, for all* $g \in \mathcal{H}^E(\mathbb{C}^+)$

$$E(r_n, g) \to E(r,g)$$

and that

$$E(r_n) \to E(r)$$

then

$$D_E(r_n, r) \to 0.$$

In other words, weak convergence plus convergence of norms implies strong convergence.

Proof

$$D_E(r_n, r) = E(r_n) + E(r) - 2\text{Re } E(r_n, r)$$

which converge to

$$\to E(r) + E(r) - 2\text{Re } E(r, r)$$
$$= 0$$

■

We shall see in §3 that the topology induced by E occurs naturally when we consider a reflection coefficient as a mapping from $\mathbb{C}^+$ into the Poincaré disk. In fact, $d_e(r, s)$ is only slightly different from the Poincaré metric on the unit disk. Specifically,

LEMMA 1.8. *Let $d_e = d_e(a, b)$ denote e-distance between any two points in the Poincaré disk and $d_p = d_p(a, b)$ denote the corresponding Poincaré distance, then*

$$C_1 \left[d_p H_{d_p \geq \log 5} + d_p^2 H_{d_p \leq \log 5}\right] < d_e < C_2 \left[d_p H_{d_p \geq \log 5} + d_p^2 H_{d_p \leq \log 5}\right] \tag{1.29}$$

with $H_{d_p < \log(5)}$ representing the indicator function of the set $\{d_p < \log(5)\}$, i.e the Poincare ball of radius log(5). In particular, the e-distance can tend to zero if and only if the Poincare distance does.

Proof

The Poincaré distance is defined by:

$$d_p(a, 0) := log(\frac{1 + |a|}{1 - |a|}) \tag{1.30}$$

$$d_p(a, b) := d_p((F_b)a, 0) \tag{1.31}$$

We refer to [1] for details about the Poincaré metric. We will establish below the equivalence of the d_e and d_P.

For $0 < r < 1$

$$d_e(r, 0) = \log(\frac{1}{1 - r}) - \log(1 + r)$$
$$d_p(r, 0) = \log(\frac{1}{1 - r}) + \log(1 + r)$$

For $0 < r < \frac{2}{3}$, expanding the series for log gives

$$r - \frac{r^2}{2} < \log(1 + r) < r - \frac{r^2}{2} + \frac{r^3}{3}$$
$$r + \frac{r^2}{2} < \log(\frac{1}{1 - r}) < r + \frac{r^2}{2} + \frac{r^3}{3(1 - r)}$$

so that

$$\begin{aligned}\frac{7}{9}r^2 \ < \ r^2 - \frac{r^3}{3} \ <d_e(r,0)< \ r^2 + \frac{r^3}{3(1-r)} \ < \ \frac{5}{3}r^2 \\ 2r <d_p(r,0)< \ 2r + \frac{r^3}{3}(1+\frac{1}{1-r}) \ < \ \frac{70}{27}r\end{aligned}$$

which gives

$$\frac{81}{700}d_p^2(r,0) \ <d_e(r,0)< \ \frac{5}{12}d_p^2(r,0)$$

For $0 < r < \frac{2}{3}$. For $\frac{2}{3} < r < 1$, alias $d_p(r,0) > \log(5)$,

$$\begin{aligned}\log(1+r) \ &< \ \log(2) \\ \log(3) \ &< \ \log(\frac{1}{1-r}) \\ (1-\frac{\log(2)}{\log(3)})\log(\frac{1}{1-r}) \ &< \ d_e(r,0) \\ d_p(r,0) \ &< \ (1+\frac{\log(2)}{\log(3)})\log(\frac{1}{1-r}) \\ \frac{1-\frac{\log(2)}{\log(3)}}{1+\frac{\log(2)}{\log(3)}}d_p(r,0) \ < \ d_e(r,0) \ &< \ d_p(r,0)\end{aligned}$$

Combining the estimates for $r < \frac{2}{3}$ and $r > \frac{2}{3}$, we obtain (1.29). ■

COROLLARY 1.9. *The unit disk, with the metric d_e, and $\mathcal{H}^E(\mathbb{C}^+)$, with the metric D_E, are complete metric spaces.*

Proof The first statement follows immediately from the completeness of the Poincaré metric. The second from noting that, if a sequence, r_n, is Cauchy in the D_E metric, then the function $d_e(r_n(\omega), r_m(\omega))$ goes to zero in $L^2(d\omega) + L^1(d\omega)$, hence almost everywhere. The completeness of d_e now allows us to produce a limit at almost every ω in the standard way, and the dominated convergence theorem ensures that it will be in $\mathcal{H}^E(\mathbb{C}^+)$. ■

Proof of theorem 1.4 We rely on Propositions 2.1 and 3.1 of [S-W-C] for the weak convergence results in the lemma below. We use the notation $r^b(\omega) = r(\omega + ib)$.

LEMMA 1.10. *If α_n converge to α in $L^2(dx)$, then, for every $b > 0$, r_n^b converges to r^b in $L^2(d\omega)$. If r_n converge to r in $\mathcal{H}^E(\mathbb{C}^+)$ (i.e. $D_E(r_n, r) \to 0$), then, for every $M < 0$, α_n converges to α in $L^2(M, 0; dx)$.*

Proof of the lemma Propositions 2.1 and 3.1 produce $r^b(x,\omega)$ (respectively $\alpha(x)$) on intervals, (x_0, x_1), as fixed points of certain contractions Φ — the actual reflection coefficient of the lemma is $r(0,\omega)$. Any time we produce a function by such a contraction mapping argument, some continuous dependence statement comes along for free. The lemma asserts just that continuous dependence.
In the statements below r is always a function of ω, as well as x, although we shall frequently not indicate the dependence on ω explicitly. Specifically, proposition 2.1

of [6] asserts that $\Phi(r; r(x_0), \alpha)$ defined by

$$\Phi(r; r(x_0), \alpha) = r(x_0, \omega)e^{2i(\omega+ib)(x-x_0)} + \int_{x_0}^{x} e^{2i(\omega+ib)(x-y)}\alpha(y)(1 - r^2(y, \omega))dy$$

is a contraction on $r(x, \omega)$ in the space of continuous maps from the interval (x_0, x_1) into $\mathcal{H}^2(\mathbb{C}^+)$ functions bounded by 1 (x_0 may be $-\infty$, in which case $r^0 = 0$).
We use the semi-colon in $\Phi(r; r(x_0), \alpha)$ to indicate that we are regarding Φ as a mapping r's to r's, while the map depends on the parameters $r(x_0)$ and α.
Now, if we have α_n approaching α, then

$$r_n^b = \Phi(r_n^b; \alpha_n, r_n^b(x_0))$$

while

$$r^b = \Phi(r^b; \alpha, r^b(x_0))$$

so that

$$\begin{aligned} r^b - r_n^b = \quad & [\Phi(r^b; \alpha, r^b(x_0)) - \Phi(r_n^b; d, r^b(x_0))] \\ & +[\Phi(r_n^b; \alpha, r^b(x_0)) - \Phi({}^b r_n; \alpha_n, r_n^b(x_0))] \end{aligned}$$

Hence,

$$\begin{aligned} \sup_{x_0<x<x_1} \|r^b - r_n^b \quad & \|_{L^2(d\omega)} \leq \theta \sup_{x_0<x<x_1} \|r^b - r_n^b\|_{L^2(d\omega)} \\ & +\|\alpha - \alpha_n\|_{L^2(dx)}(1 + \|\alpha_n\|_{L^2(dx)}) + \|r^b(x_0) - r_n^b(x_0)\|_{L^2(d\omega)} \end{aligned}$$

hence

$$\begin{aligned} \sup_{x_0<x<x_1} \|r^b \quad & -r_n^b\|_{L^2(d\omega)} \leq \\ & \tfrac{1}{1-\theta}\left(\|\alpha - \alpha_n\|_{L^2(dx)}(1 + \|\alpha_n\|_{L^2(dx)}) + \|r^b(x_0) - r_n^b(x_0)\|_{L^2(d\omega)}\right) \end{aligned}$$

where $\theta < 1$ depends only on $\|\alpha_n\|_{L^2(x_0,x_1;dx)}$, which must be sufficiently small. Combining this estimate over a finite number of intervals yields the convergence of the r_n^b asserted in the first paragraph of the lemma.
The second statement is obtained analogously from Proposition 3.1. Here the contraction is

$$\Phi = \Phi(\alpha, r; r(x_0))$$

where Φ is contraction mapping from pairs (α, r) to pairs (α, r) depending on the parameter $r(x_0)$ (see equation (94) in [S-W-C]). We don't include the details here. ∎

Once we have lemma 1.10, we can use our Plancherel equality to strengthen the convergence. The lemma implies that

$$\rho_n^b = r_n^b - r^b \xrightarrow{L^2} 0.$$

To see that this implies weak L^2 convergence of the r_n, we let $g \in \mathcal{H}^2(\mathbb{C}^+)$, and compute the $\mathcal{H}^2(\mathbb{C}^+)$ inner product of g and ρ_n.

$$\begin{aligned}(\rho_n, g) &= (\rho_n^b, g) + (\rho_n - \rho_n^b, g)\\ &= (\rho_n^b, g) + (\rho_n, g - g^b)\\ |(\rho_n, g)| &\leq \|\rho_n^b\|_{L^2}\|g\|_{L^2} + \|\rho_n\|_{L^2}\|g - g^b\|_{L^2}\end{aligned} \tag{1.32}$$

Since $\|\rho_n\|_{L^2}$ are uniformly bounded by $E^{\frac{1}{2}}(r_n) + E^{\frac{1}{2}}(r) = \|\alpha\|_{L^2} + \|\alpha_n\|_{L^2}$, which converges to $2\|\alpha\|_{L^2}$, we may first choose b and then n to make the last and then the first terms in (1.32) small.

Now, the previous argument can be applied verbatim with r_n replaced by its k-th power, r_n^k, to establish the weak L^2 convergence, $r_n^k \to r^k$ for any positive integer k. Thus, for $g \in H^E$ each term in the series expansion for $E(r_n, g)$ converges to the corresponding term in the series for $E(r, g)$.
Our nonlinear Cauchy Schwartz inequality (1.27)

$$E(r_n, g) \leq E(r_n)^{\frac{1}{2}} E^{\frac{1}{2}}(g)$$

gives the uniform and absolute convergence of both series, establishing weak convergence in $\mathcal{H}^E$, i.e.

$$E(r_n, g) \to E(r, g)$$

which then gives strong convergence via theorem 1.7.
To see the reverse implication, suppose that $r_n \to r$ in $\mathcal{H}^E$. The triangle inequality (1.28) tells us that

$$D_E(r_n, 0) \leq D_E(r, 0) + D_E(r, r_n)$$

so that

$$\begin{aligned}D_E(r_n, 0) - D_E(r, 0) &\leq D_E(r, r_n)\\ E(r_n) - E(r) &\leq D_E(r, r_n)\end{aligned}$$

so that $E(r_n) \to E(r)$. The Plancherel equality then implies that $\|\alpha_n\|_{L^2} \to \|\alpha\|_{L^2}$. The convergence of norms is enough to strengthen the weak L^2 convergence asserted in lemma 1.10 to strong L^2 convergence. ∎

2. Linear and Nonlinear Riesz Transforms. Time dependent inverse scattering algorithms make strong use of causality, the fact that reflections from the top of the medium arrive back at the sensor sooner than those which arise from lower depths. When a signal is given as a function of frequency, causality is reflected in the analyticity of the data in the complex upper half plane. In the frequency domain, the splitting into the past and future is provided by the linear Riesz transform. Specifically, the linear space $\mathcal{L}^2(\mathbb{R})$ is the direct sum of the two Hardy spaces, $\mathcal{H}^2(\mathbb{C}^+)$ and $\mathcal{H}^2(\mathbb{C}^-)$:

$$\mathcal{L}^2(\mathbb{R}) = \mathcal{H}^2(\mathbb{C}^+) \oplus \mathcal{H}^2(\mathbb{C}^-) \tag{2.1}$$

We let $P^{\pm}$ denote the projections onto $\mathcal{H}^2(\mathbb{C}^{\pm})$ along $\mathcal{H}^2(\mathbb{C}^{\mp})$. P^+ is the *linear Riesz transform.* We shall often write

$$f = f^+ + f^- \tag{2.2}$$

denoting $P^{\pm}f$ by $f^{\pm}$. We speak of f^+ as the *causal* part of f because it is the Fourier transform of a function supported in the past, and to f^- as the *acausal* part, because it depends on the future and is therefore inconsistent with the idea of causality — the signal represents the present which cannot depend on the future..
In the next subsection we will describe how the Riesz transform can be used as the basic building block for a numerical algorithm to invert the Fourier transform (i.e. a linear inverse scattering algorithm).

In the subsection after that, where we show that this linear algorithm becomes a full nonlinear inverse scattering algorithm when we replace the linear Riesz transform by the nonlinear Riesz transform.
toggl We shall use the notation $\circ$ to represent the formula for composition of conformal maps of the unit disk onto itself.

$$a \circ b := \frac{a+b}{1+\overline{a}b} \tag{2.3}$$

and replace the linear splitting in (2.2) by the nonlinear splitting:

$$f = f^+ \circ f^- \tag{2.4}$$

where $f^{\pm} \in \mathcal{H}^E(\mathbb{C}^{\pm})$. The *nonlinear projection* $\mathcal{P}^+ f = f^+$ is the *nonlinear Riesz transform.* We shall see that it is genuinely a nonlinear version of an orthogonal projection (see theorem 2.5). This nonlinear projection and its properties form the heart of our inversion scheme.

We can view the Fourier transform as decomposing a function in $L^2(\mathbb{R})$ into a sum of complex exponentials. In our nonlinear decomposition, the additive group operation, $+$, is replaced by $\circ$, the formula for the composition of conformal maps of the unit disk. Thus our decomposition will decompose a function which is pointwise less than or equal to one (as are all reflection coefficients) into a composition of complex exponentials. Where the Fourier decomposition is orthogonal with respect to L^2, our nonlinear decomposition will turn out to be orthogonal with respect to the E inner product defined in (1.24). All of this will be made specific in the next two subsections.

2.1. Inverting the Fourier transform via the Riesz transform. In this section we describe a numerical algorithm which uses the Riesz transform as the basic step for inverting the Fourier transform. The primary reason for introducing it here is to serve as a model against which to compare the nonlinear inversion procedure of the next section. As a method for inverting the linear Fourier transform, it is computationally very expensive. Our own implementation, however, has proved incredibly robust and accurate, especially for functions which decay slowly at infinity (e.g. $L^2(\mathbb{R})$ and no better).

A typical strategy for numerically computing the Fourier transform (or inverse Fourier transform) of a function, a, is the following:

- Approximate a by a piecewise constant function (Δ is a small negative parameter), i.e.

$$a(x) \sim \sum_{j=1}^{\infty} a(j\Delta) H_{j\Delta < x < (j-1)\Delta}$$

- Compute the Fourier transform of the step function by hand, i.e.

$$\hat{H}_{j\Delta < x < (j-1)\Delta} = \mathrm{e}^{-2\mathrm{i}\omega(\mathrm{j}-1)\Delta} \left(\frac{\mathrm{e}^{-2\mathrm{i}\omega\Delta} - 1}{2\mathrm{i}\omega} \right) = \mathrm{e}^{-2\mathrm{i}\omega(\mathrm{j}-\frac{1}{2})\Delta} \frac{\sin \omega\Delta}{\omega}$$

- Add up the pieces

$$\hat{a}(\omega) \sim \sum_{j=1}^{\infty} a(j\Delta) \mathrm{e}^{-2\mathrm{i}\omega(\mathrm{j}-\frac{1}{2})\Delta} \frac{\sin \omega\Delta}{\omega}$$

Our strategy for computing the forward linear scattering map is exactly this, although we prefer to describe it in terms of the differential equation, (1.9), to provide the analogy to the nonlinear case. We have a different strategy for inverting, namely,

- Use the Riesz transform to break up $\hat{a}$ into a sum of functions whose inverse Fourier transforms have support in small intervals

$$\hat{a}(\omega) = \sum_{j=1}^{\infty} \hat{a}_{j\Delta}(\omega)$$

- Solve for the constant $A_{j\Delta}$ so as to best fit each $\hat{a}_{j\Delta}(\omega)$ with the Fourier transform of a step function, i.e.

$$\hat{a}_{j\Delta} \sim A_{j\Delta} \mathrm{e}^{-2\mathrm{i}\omega(\mathrm{j}-\frac{1}{2})\Delta} \frac{\sin \omega\Delta}{\omega}$$

- Now a is approximately the sum of step functions

$$a(x) \sim \sum_{j=1}^{\infty} A_{j\Delta} H_{j\Delta < x < (j-1)\Delta}$$

  so that $a(j\Delta) \sim A_{j\Delta}$.

We begin a more explicit description with the exact solution to (1.9) in terms of the initial data at some point x_j.

$$\rho(x,\omega) = \rho(x_j) e^{2i\omega(x-x_j)} + \int_{x_j}^{x} e^{2i\omega(x-y)} a(y) dy \tag{2.5}$$

Next, we fix a negative small parameter , Δ — this will be our step size —, set $x_j = j\Delta$, and evaluate (2.5) at $x_{j-1} = (j-1)\Delta$. That is;

$$\rho_{(j-1)\Delta} = e^{-2i\omega\Delta}\rho_{j\Delta} + e^{2i\omega(j-1)\Delta}\hat{a}_{j\Delta} \tag{2.6}$$

where $\rho_{j\Delta}$ denotes the $\rho(j\Delta, \omega)$ and

$$\hat{a}_{j\Delta} = \int_{j\Delta}^{(j-1)\Delta} e^{-2i\omega y} a(y)dy \tag{2.7}$$

Notice that, for small Δ,

$$\hat{a}_{j\Delta} \sim a(j\Delta)\mathrm{e}^{-2\mathrm{i}\omega(\mathrm{j}-\frac{1}{2})\Delta}\frac{\sin\omega\Delta}{\omega} \tag{2.8}$$

so that once we have the $\hat{a}_{j\Delta}$'s in hand, the pointwise values of a are easy to approximate.

We would like to focus on (2.6), rearranging slightly to

$$\rho_{(j-1)\Delta} = e^{-2i\omega\Delta}\left(e^{2i\omega j\Delta}\hat{a}_{j\Delta} + \rho_{j\Delta}\right) \tag{2.9}$$

The relation (2.9) can be used to define a forward step or an inverse step. the **forward step** is the mapping

$$(\hat{a}_{j\Delta}, \rho_{j\Delta}) \overset{F_j}{\mapsto} \rho_{(j-1)\Delta}$$

while the **inverse step** is the mapping

$$\rho_{(j-1)\Delta} \overset{I_j}{\mapsto} (\hat{a}_{j\Delta}, \rho_{j\Delta})$$

The implementation of the inverse step requires the linear Riesz transform. We must note that

$$\rho_{j\Delta} \in \mathcal{H}^2(\mathbb{C}^+) \tag{2.10}$$
$$\hat{a}_{j\Delta} \in e^{-2i\omega j\Delta}\mathcal{H}^2(\mathbb{C}^+) \cap e^{-2i\omega(j-1)\Delta}\mathcal{H}^2(\mathbb{C}^-) \tag{2.11}$$

and define I_j via

$$\rho_{j\Delta} = P^+ e^{2i\omega\Delta}\rho_{(j-1)\Delta} \tag{2.12}$$
$$\hat{a}_{j\Delta} = e^{-2i\omega j\Delta}P^- e^{2i\omega\Delta}\rho_{(j-1)\Delta} \tag{2.13}$$

If we start with a finite sequence of $\{a_{j\Delta}\}_{j=1}^M$, we can set $\rho_{(M+1)\Delta} = 0$ and iterate the forward step to produce a $\rho \in \mathcal{H}^2(\mathbb{C}^+)$. Conversely, if we start with a $\rho \in \mathcal{H}^2(\mathbb{C}^+)$ we can produce an infinite sequence of $a_{j\Delta}$'s by iterating the inverse step. We need to worry a little about convergence in order to let M approach infinity, but the Plancherel equality provides plenty of control for that. Notice that (2.11) implies that supp $(a_{j\Delta})$ is contained in the interval $[j\Delta, (j-1)\Delta]$.

We summarize the previous discussion with the next two theorems.

Theorem 2.1 (Linear Layer Stripping Decomposition).
For any $\Delta < 0$, any $\rho \in \mathcal{H}^2(\mathbb{C}^+)$ has a unique decomposition

$$\rho = \hat{a}_{1\Delta} + \hat{a}_{2\Delta} + \hat{a}_{3\Delta} + \dots \tag{2.14}$$

with

$$\hat{a}_{j\Delta} \in \mathrm{e}^{-2\mathrm{i}\omega(\mathrm{j}-1)\Delta}\mathcal{H}^2(\mathbb{C}^+) \;\cap\; \mathrm{e}^{-2\mathrm{i}\omega\mathrm{j}\Delta}\mathcal{H}^2(\mathbb{C}^-) \tag{2.15}$$

Moreover,

$$\|\rho\|^2 = \sum_{j=1}^{\infty} \|\hat{a}_{j\Delta}\|^2 \tag{2.16}$$

and conversely, any sum of the form

$$\sum_{j=1}^{\infty} \hat{a}_{j\Delta} \tag{2.17}$$

converges to a $\rho \in \mathcal{H}^2(\mathbb{C}^+)$, provided only that (2.15) holds and the sum on the right hand side of (2.16) is finite.

Theorem 2.2 (Linear convergence Theorem).
Let ρ be the Fourier transform of $a \in L^2(\mathbb{R}^-)$, the functions $a_{j\Delta}(\omega)$ be defined as in Theorem 2.1, and let

$$A_{j\Delta} = \frac{\int_{(j-1)\Delta}^{j\Delta} \hat{a}_{j\Delta}(\omega)\mathrm{e}^{2\mathrm{i}\omega(\mathrm{j}-\frac{1}{2})\Delta}\frac{\sin\omega\Delta}{\omega}\mathrm{d}\omega}{\int_{(j-1)\Delta}^{j\Delta} |\frac{sin\omega\Delta}{\omega}|^2 d\omega}$$

so that $A_{j\Delta}\mathrm{e}^{-2\mathrm{i}\omega(\mathrm{j}-\frac{1}{2})\Delta}\frac{\sin\omega\Delta}{\omega}$ is the best L^2 approximation to $a_{j\Delta}(\omega)$. Then the following limit exists in the L^2 topology,

$$\lim_{\Delta\to 0}\sum_{j=1}^{\infty} A_{j\Delta}H_{j\Delta<x<(j-1)\Delta} = a$$

2.2. Inverse scattering via the nonlinear Riesz transform. The nonlinear equivalents of theorems 2.1 and 2.2 are:

Theorem 2.3 (Nonlinear Layer Stripping Decomposition). *Let $r \in \mathcal{H}^E(\mathbb{C}^+)$ and $\Delta < 0$ with $|\Delta E(r)| < \frac{\pi}{4}$. Then r has a unique decomposition*

$$r = \hat{\alpha}_{1\Delta} \circ \hat{\alpha}_{2\Delta} \circ \hat{\alpha}_{3\Delta} \circ \dots \tag{2.18}$$

with

$$\hat{\alpha}_{j\Delta} \in \mathrm{e}^{-2\mathrm{i}\omega(\mathrm{j}-1)\Delta}\mathcal{H}^{\mathrm{E}}(\mathbb{C}^+) \;\cap\; \mathrm{e}^{-2\mathrm{i}\omega\mathrm{j}\Delta}\mathcal{H}^{\mathrm{E}}(\mathbb{C}^-) \tag{2.19}$$

Moreover,

$$E(r) = \sum_{j=1}^{\infty} E(\hat{\alpha}_{j\Delta}) \tag{2.20}$$

and conversely, any infinite composition of the form

$$\mathop{\circ}_{j=1}^{\infty} \hat{\alpha}_{j\Delta} \tag{2.21}$$

converges to an $r \in \mathcal{H}^E(\mathbb{C}^+)$, provided only that (2.19) holds and the sum on the right hand side of (2.20) is finite.

Theorem 2.4 (Nonlinear convergence Theorem).
Let r be the scattering data of $\alpha \in L^2(\mathbb{R}^-)$, the functions $\hat{\alpha}_{j\Delta}(\omega)$ be defined as in Theorem 2.3, and let

$$A_{j\Delta} = \frac{\int_{(j-1)\Delta}^{j\Delta} \hat{\alpha}_{j\Delta}(\omega) \mathrm{e}^{2\mathrm{i}\omega(\mathrm{j}-\frac{1}{2})\Delta} \frac{\sin\omega\Delta}{\omega} \mathrm{d}\omega}{\int_{(j-1)\Delta}^{j\Delta} |\frac{sin\omega\Delta}{\omega}|^2 d\omega}$$

so that $A_{j\Delta}\mathrm{e}^{-2\mathrm{i}\omega(\mathrm{j}-\frac{1}{2})\Delta}\frac{\sin\omega\Delta}{\omega}$ is the best L^2 approximation to $\alpha_{j\Delta}(\omega)$. Then the following limit exists in the L^2 topology,

$$\lim_{\Delta\to 0} \sum_{j=1}^{\infty} A_{j\Delta} H_{j\Delta<x<(j-1)\Delta} = \alpha$$

Warning 2. *The composition, $\circ$, in (2.18) is neither commutative nor associative. It should always be read from right to left, i.e*

$$a \circ b \circ c \circ d = a \circ (b \circ (c \circ d))$$

Before giving the proof of either theorem, we will explain the approximations that we use to produce our algorithm. The motivating factor behind the approximation we choose is the Plancherel equality. We are discretizing the nonlinear scattering map which obeys the nonlinear Plancherel equality, (1.21). We choose our discretization so that it obeys a similar Plancherel equality, namely (2.20). Notice that, as Δ approaches zero, (2.20) becomes (1.21).

Our point of departure is the integral equation equivalent to (1.6) with initial data at the point $x = \Delta$.

$$r(x,\omega) = r(\Delta,\omega)\mathrm{e}^{2\mathrm{i}\omega(\mathrm{x}-\Delta)} + \int_\Delta^{\mathrm{x}} \mathrm{e}^{2\mathrm{i}\omega(\mathrm{x}-\mathrm{y})}\alpha(\mathrm{y})\big(1-\mathrm{r}^2(\mathrm{y},\omega)\big)\mathrm{dy} \tag{2.22}$$

As we did in the linear case, we fix a negative small parameter , Δ, for our step size, and evaluate (2.22) at $x = 0$, rewriting slightly and suppressing the ω dependence. Namely,

$$r(0) = r(\Delta)\mathrm{e}^{-2\mathrm{i}\omega\Delta} + \int_\Delta^0 \mathrm{e}^{-2\mathrm{i}\omega\mathrm{y}}\alpha(\mathrm{y})\mathrm{dy} - \int_\Delta^0 \mathrm{e}^{-2\mathrm{i}\omega(\Delta-\mathrm{y})}\alpha(\mathrm{y})\mathrm{r}^2(\mathrm{y})\mathrm{dy} \tag{2.23}$$

We approximate $r^2(y)$ in the integral above by

$$r^2(y) \sim \mathrm{e}^{2\mathrm{i}\omega(\mathrm{y}-0)}\mathrm{r}(0)\mathrm{e}^{2\mathrm{i}\omega(\mathrm{y}-\Delta)}\mathrm{r}(\Delta) \tag{2.24}$$

which is the product of the forward and backward approximations

$$r(y) \sim \mathrm{e}^{2\mathrm{i}\omega(\mathrm{y}-0)}\mathrm{r}(0)$$
$$r(y) \sim \mathrm{e}^{2\mathrm{i}\omega(\mathrm{y}-\Delta)}\mathrm{r}(\Delta)$$

Inserting this approximation into (2.23), we obtain an approximate discrete relationship

$$r(0) \sim r(\Delta)\mathrm{e}^{-2\mathrm{i}\omega\Delta} + \int_{\Delta}^{0} \mathrm{e}^{-2\mathrm{i}\omega\mathrm{y})}\alpha(\mathrm{y})\mathrm{dy} - \mathrm{e}^{-2\mathrm{i}\omega\Delta}\mathrm{r}(0)\mathrm{r}(\Delta)\int_{\Delta}^{0} \mathrm{e}^{2\mathrm{i}\omega\mathrm{y})}\alpha(\mathrm{y})\mathrm{dy}$$

Recalling that α is real valued and using the notation from (2.7)

$$r(0) \sim r(\Delta)\mathrm{e}^{-2\mathrm{i}\omega\Delta} + \hat{\alpha}_\Delta - \mathrm{e}^{-2\mathrm{i}\omega\Delta}\mathrm{r}(0)\mathrm{r}(\Delta)\overline{\hat{\alpha}_\Delta} \tag{2.25}$$

We denote by R_0 and R_Δ the quantities that satisfy that discrete relationship exactly, so that

$$R_0 = R_\Delta \mathrm{e}^{-2\mathrm{i}\omega\Delta} + \hat{\alpha}_\Delta - \mathrm{e}^{-2\mathrm{i}\omega\Delta}\mathrm{R}_0\mathrm{R}_\Delta\overline{\hat{\alpha}_\Delta} \tag{2.26}$$

Equation (2.26) can be solved for R_0, yielding

$$R_0 = \frac{\hat{\alpha}_\Delta + \mathrm{e}^{-2\mathrm{i}\omega\Delta}\mathrm{R}_\Delta}{1 + \mathrm{e}^{-2\mathrm{i}\omega\Delta}\mathrm{R}_\Delta\overline{\hat{\alpha}_\Delta}} \tag{2.27}$$

$$= \hat{\alpha}_\Delta \circ \mathrm{e}^{-2\mathrm{i}\omega\Delta}\mathrm{R}_\Delta \tag{2.28}$$

where $\circ$ is as defined in (2.3). The relation (2.28) applies to the points 0 and Δ; the same calculation can be applied to the points $(j-1)\Delta$ and $j\Delta$. The resulting expression is

$$\begin{aligned} R_{(j-1)\Delta} &= \mathrm{e}^{2\mathrm{i}\omega(\mathrm{j}-1)\Delta}\hat{\alpha}_{\mathrm{j}\Delta} \circ \mathrm{e}^{-2\mathrm{i}\omega\Delta}\mathrm{R}_{\mathrm{j}\Delta} \\ &= \mathrm{e}^{-2\mathrm{i}\omega\Delta}\left(\mathrm{e}^{2\mathrm{i}\omega\mathrm{j}\Delta}\hat{\alpha}_{\mathrm{j}\Delta} \circ \mathrm{R}_{\mathrm{j}\Delta}\right) \end{aligned} \tag{2.29}$$

which are the nonlinear analogs of (2.6) and (2.9).

Our nonlinear **forward step** is the mapping

$$(\hat{\alpha}_{j\Delta}, R_{j\Delta}) \overset{\mathcal{F}_j}{\mapsto} R_{(j-1)\Delta} \tag{2.30}$$

while our **inverse step** is the mapping

$$R_{(j-1)\Delta} \overset{\mathcal{I}_j}{\mapsto} (\hat{\alpha}_{j\Delta}, R_{j\Delta}) \tag{2.31}$$

The implementation of this inverse step requires the nonlinear Riesz transform. We must note that

$$R_{j\Delta} \in \mathcal{H}^E(\mathbb{C}^+)$$
$$\hat{\alpha}_{j\Delta} \in e^{-2i\omega(j-1)\Delta}\mathcal{H}^E(\mathbb{C}^-) \cap e^{-2i\omega j\Delta}\mathcal{H}^E(\mathbb{C}^+)$$

and define I_j via

$$R_{j\Delta} = \mathcal{P}^{+} e^{2i\omega\Delta} R_{(j-1)\Delta} \tag{2.32}$$
$$\hat{\alpha}_{j\Delta} = e^{-2i\omega j\Delta} \mathcal{P}^{-} e^{2i\omega\Delta} R_{(j-1)\Delta} \tag{2.33}$$

The nonlinear operators $\mathcal{P}^{\pm}$ are defined by the following theorem (i.e $\mathcal{P}^{\pm}F := g^{\pm}$ in the factorization below).

Theorem 2.5 (The Nonlinear Riesz Transform). *Let $\Delta < 0$ and $F \in \mathrm{e}^{2\mathrm{i}\omega\Delta}\mathcal{H}^{\mathrm{E}}(\mathbb{C}^{+})$ with $|\Delta E(F)| < \frac{\pi}{4}$. Then F has a unique factorization into*

$$F = g^{-} \circ g^{+} \tag{2.34}$$

with

$$g^{-} \in \mathcal{H}^{E}(\mathbb{C}^{-}) \cap \mathrm{e}^{2\mathrm{i}\omega\Delta}\mathcal{H}^{\mathrm{E}}(\mathbb{C}^{+}) \tag{2.35}$$

and

$$g^{+} \in \mathcal{H}^{E}(\mathbb{C}^{+}) \tag{2.36}$$

Moreover,

$$E(F) = E(g^{+}) + E(g^{-}) \tag{2.37}$$

Proof

Let $g \in \mathrm{e}^{2\mathrm{i}\omega\Delta}\mathcal{H}^{2}(\mathbb{C}^{+})$ and $g^{\pm} = P^{\pm}g$. We note that it is possible to estimate $\|g^{-}\|_{L^{\infty}}$ in terms of $\|g^{-}\|_{L^{2}}$. Specifically, since any $g^{-} \in \mathcal{H}^{2}(\mathbb{C}^{-}) \cap \mathrm{e}^{2i\omega\Delta}\mathcal{H}^{2}(\mathbb{C}^{+})$ is the Fourier transform of a function $\gamma \in L^{2}(0,\Delta)$:

$$g^{-}(\omega) = \int_{\Delta}^{0} \mathrm{e}^{-2i\omega x}\gamma(x)dx \tag{2.38}$$
$$\|g^{-}\|_{L^{\infty}} \le \Delta^{\frac{1}{2}}\|\gamma\|_{L^{2}} = \left(\frac{\Delta}{\pi}\right)^{\frac{1}{2}} \|g^{-}\|_{L^{2}} \tag{2.39}$$

We intend to produce $g = g^{-} + g^{+}$ as the unique fixed point of the map

$$g \overset{\Phi}{\mapsto} h = \Big(F\ (1 + \overline{g^{-}}g^{+})\Big) \tag{2.40}$$

We will show that Φ is a contraction on the ball

$$B_F = \{g \in \mathrm{e}^{2\mathrm{i}\Delta\omega}\mathcal{H}^{2}(\mathbb{C}^{+}) \mid \|\mathrm{g}\|^{2} \le 4\mathrm{E(F)}\} \tag{2.41}$$

To see this, let $h_1 = \Phi(g_1)$ and $h_2 = \Phi(g_2)$; then

$$
\begin{aligned}
\|h_1 - h_2\|_{L^2} &\le \|F\|_{L^\infty}\left(\|g_1^- - g_2^-\|_{L^\infty}\|\frac{g_1^+ + g_2^+}{2}\|_{L^2} + \|g_1^+ - g_2^+\|_{L^2}\|\frac{g_1^- + g_2^-}{2}\|_{L^\infty}\right)\\
&\le \|F\|_{L^\infty}\left(\frac{\Delta}{4\pi}\right)^{\frac{1}{2}}\\
&\times \left(\|g_1^- - g_2^-\|_{L^2}\|g_1^+ + g_2^+\|_{L^2} + \|g_1^+ - g_2^+\|_{L^2}\|g_1^- + g_2^-\|_{L^2}\right)\\
&\le \left(\frac{\Delta}{4\pi}\right)^{\frac{1}{2}}\frac{1}{2}\\
&\times \left(\varepsilon\|g_1^- - g_2^-\|_{L^2}^2 + \frac{1}{\varepsilon}\|g_1^+ + g_2^+\|_{L^2}^2 + \varepsilon\|g_1^+ - g_2^+\|_{L^2}^2 + \frac{1}{\varepsilon}\|g_1^- + g_2^-\|_{L^2}^2\right)\\
&\le \left(\frac{\Delta}{4\pi}\right)^{\frac{1}{2}}\frac{1}{2}\left(\varepsilon\|g_1 - g_2\|_{L^2}^2 + \frac{1}{\varepsilon}\|g_1 + g_2\|_{L^2}^2\right)\\
&\le \left(\frac{\Delta}{4\pi}\right)^{\frac{1}{2}}\|g_1 - g_2\|_{L^2}\|g_1 + g_2\|_{L^2}\\
&\le \left(\frac{4\Delta E}{\pi}\right)^{\frac{1}{2}}\|g_1 - g_2\|_{L^2}
\end{aligned}
$$

To see that Φ preserves B_F, we take $g_1 = 0$ in the estimate above, and apply the contraction estimate.

$$
\begin{aligned}
\|h_2\|_{L^2} &\le \|h_1\|_{L^2} + \|h_2 - h_1\|_{L^2} && (2.42)\\
&\le \|F\|_{L^2} + \left(\frac{\Delta}{4\pi}\right)^{\frac{1}{2}}\|g_1\|_{L^2}^2 && (2.43)\\
&\le E^{\frac{1}{2}} + \left(\frac{\Delta}{4\pi}\right)^{\frac{1}{2}} 4E && (2.44)\\
&\le E^{\frac{1}{2}}\left(1 + \left(\frac{4\Delta E}{\pi}\right)^{\frac{1}{2}}\right) && (2.45)\\
&\le (4E)^{\frac{1}{2}} && (2.46)
\end{aligned}
$$

We remark that, as a consequence of (2.37) any pair (g^-, g^+), which implements the splitting in (2.34), must belong to B_F, and hence be the unique fixed point of Φ.
It remains only to verify (2.37). It is a direct calculation that

$$
\begin{aligned}
1 - |g^- \circ g^+|^2 &= \frac{(1 - |g^-|^2)(1 - |g^+|^2)}{|1 + \overline{g^-}g^+|^2}\\
\int_{-\infty}^{\infty}\log(1 - |g^- \circ g^+|^2) &= \int_{-\infty}^{\infty}\log(1 - |g^-|^2)\\
&+ \int_{-\infty}^{\infty}\log(1 - |g^+|^2)\\
&- \int_{-\infty}^{\infty}\log(|1 + \overline{g^-}g^+|^2)
\end{aligned}
$$

The last term is twice real part of

$$\begin{aligned}
\int_{-\infty}^{\infty}\log(1+\overline{g^-}g^+) &= \sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\frac{(\overline{g^-}g^+)^k}{k} \\
&= \sum_{n=1}^{\infty}\frac{\left((g^-)^k,(g^+)^k\right)}{k} \\
&= 0
\end{aligned}$$

The final equality follows from the orthogonality of $\mathcal{H}^2(\mathbb{C}^+)$ and $\mathcal{H}^2(\mathbb{C}^-)$. ■

We need one more lemma before we proceed with the proof of theorem 2.3.

Lemma 2.6 (Uniqueness of Forward Solutions). *Suppose that*

$$\hat{\alpha}_{j\Delta} \in \mathrm{e}^{-2\mathrm{i}\omega(\mathrm{j}-1)\Delta}\mathcal{H}^{\mathrm{E}}(\mathbb{C}^+) \ \cap\ \mathrm{e}^{-2\mathrm{i}\omega\mathrm{j}\Delta}\mathcal{H}^{\mathrm{E}}(\mathbb{C}^-)$$

and

$$\sum_{1}^{\infty} E(\hat{\alpha}_{j\Delta}) < \infty$$

then the infinite composition

$$\mathop{\circ}_{n=j}^{\infty}\hat{\alpha}_{n\Delta} := \lim_{N\to\infty}\mathop{\circ}_{n=j}^{N}\hat{\alpha}_{n\Delta} \tag{2.47}$$

exists and $\mathrm{e}^{2\mathrm{i}\omega(\mathrm{j}-1)\Delta}\mathop{\circ}\limits_{\mathrm{n}=\mathrm{j}}^{\infty}\hat{\alpha}_{\mathrm{n}\Delta}$ *is the unique* $\mathcal{H}^E(\mathbb{C}^+)$ *valued solution to*

$$R_{(j-1)\Delta} = (\mathrm{e}^{2\mathrm{i}\omega(\mathrm{j}-1)\Delta}\hat{\alpha}_{\mathrm{j}\Delta})\circ\mathrm{e}^{-2\mathrm{i}\omega\Delta}\mathrm{R}_{\mathrm{j}\Delta} \tag{2.48}$$

We leave the proof for the time being, and instead use the lemma for the

<u>Proof of theorem 2.3</u>
The nonlinear Riesz transform tells us that, for small enough Δ,

$$\mathrm{e}^{2\mathrm{i}\omega\Delta}\mathrm{R}_{(\mathrm{j}-1)\Delta} = \mathrm{g}^-\circ\mathrm{g}^+ \tag{2.49}$$

with the factors g^- and g^+ being unique. We can then define $R_{j\Delta}$ and $\hat{\alpha}_{j\Delta}$ by

$$g^- = \mathrm{e}^{2\mathrm{i}\omega\mathrm{j}\Delta}\hat{\alpha}_{\mathrm{j}\Delta} \tag{2.50}$$

$$g^+ = R_{j\Delta} \tag{2.51}$$

and apply (2.49) recursively as follows

$$\begin{aligned}
\mathrm{e}^{2\mathrm{i}\omega\Delta}\mathrm{R}_{0\Delta} &= \mathrm{e}^{2\mathrm{i}\omega\Delta}\hat{\alpha}_{\Delta}\circ\mathrm{R}_{\Delta} \\
R_{0\Delta} &= \hat{\alpha}_{\Delta}\circ\mathrm{e}^{-2\mathrm{i}\omega\Delta}\mathrm{R}_{\Delta} \\
&= \hat{\alpha}_{\Delta}\circ\mathrm{e}^{-2\mathrm{i}\omega2\Delta}\left(\mathrm{e}^{2\mathrm{i}\omega\Delta}\mathrm{R}_{\Delta}\right) \\
&= \hat{\alpha}_{\Delta}\circ\mathrm{e}^{-2\mathrm{i}\omega2\Delta}\left(\mathrm{e}^{2\mathrm{i}\omega2\Delta}\hat{\alpha}_{2\Delta}\circ\mathrm{R}_{2\Delta}\right) \\
&= \hat{\alpha}_{\Delta}\circ\hat{\alpha}_{2\Delta}\circ\mathrm{e}^{-2\mathrm{i}\omega3\Delta}\left(\mathrm{e}^{2\mathrm{i}\omega\Delta}\mathrm{R}_{2\Delta}\right) \\
&= \hat{\alpha}_{\Delta}\circ\hat{\alpha}_{2\Delta}\circ\hat{\alpha}_{3\Delta}\circ\mathrm{e}^{-2\mathrm{i}\omega4\Delta}\left(\mathrm{e}^{2\mathrm{i}\omega\Delta}\mathrm{R}_{3\Delta}\right)
\end{aligned}$$

etcetera. We have used the identity

$$\mathrm{e}^{2\mathrm{i}\omega\mathrm{M}}(\mathrm{a}\circ\mathrm{b}) = \mathrm{e}^{2\mathrm{i}\omega\mathrm{M}}\mathrm{a}\circ\mathrm{e}^{2\mathrm{i}\omega\mathrm{M}}\mathrm{b} \tag{2.52}$$

several times. Notice that, by construction, each $\hat{\alpha}_{j\Delta}$ satisfies (2.19) and that

$$\sum_{j=1}^{N} E(\hat{\alpha}_{j\Delta}) = E(R_{0\Delta}) - E(R_{N\Delta}) \leq E(R_{0\Delta})$$

so that we have produced a sequence$\{\hat{\alpha}_{j\Delta}\}_{j=1}^{\infty}$ which satisfy the hypotheses of lemma 2.6 and a sequence $\{R_{j\Delta}\}_{j=1}^{\infty}$ which belong to $\mathcal{H}^E(\mathbb{C}^+)$ and satisfy (2.48). The uniqueness part of the lemma asserts that

$$R_{j\Delta} = \mathrm{e}^{2\mathrm{i}\omega(\mathrm{j}-1)\Delta} \mathop{\circ}_{\mathrm{n}=\mathrm{j}}^{\infty} \hat{\alpha}_{\mathrm{n}\Delta}$$

which, for $j = 1$, is (2.18). The equality (2.20) then follows from a recursive application of (2.37). The inclusions (2.19) follow from (2.35) together with (2.50), and (2.21) is asserted as part of lemma 2.6. Thus, pending the proof of that lemma, the proof of theorem 2.3 is complete. ■

Proof of lemma 2.6

We first establish the convergence of (2.47). To do this we compute the e-distance between two partial compositions, making use of its invariance under conformal mappings,

$$\begin{aligned} d_e\left(\mathop{\circ}_{n=1}^{N} \hat{\alpha}_{n\Delta}\ ,\ \mathop{\circ}_{n=1}^{N+M} \hat{\alpha}_{n\Delta})\right) &= d_e\left(F_{\hat{\alpha}_\Delta}(\mathop{\circ}_{n=1}^{N} \hat{\alpha}_{n\Delta})\ ,F_{\hat{\alpha}_\Delta}(\mathop{\circ}_{n=1}^{N+M} \hat{\alpha}_{n\Delta})\right) \\ &= d_e\left(\mathop{\circ}_{n=2}^{N} \hat{\alpha}_{n\Delta}\ ,\ \mathop{\circ}_{n=2}^{N+M} \hat{\alpha}_{n\Delta}\right) \end{aligned}$$

Continuing in this way, we reach

$$d_e\left(\mathop{\circ}_{n=1}^{N} \hat{\alpha}_{n\Delta}\ ,\ \mathop{\circ}_{n=1}^{N+M} \hat{\alpha}_{n\Delta}\right) = d_e\left(0\ ,\ \mathop{\circ}_{n=N+1}^{N+M} \hat{\alpha}_{n\Delta}\right)$$

so that

$$\begin{aligned} \int_{-\infty}^{\infty} d_e\left(\mathop{\circ}_{n=1}^{N} \hat{\alpha}_{n\Delta}, \mathop{\circ}_{n=1}^{N+M} \hat{\alpha}_{n\Delta}\right) d\omega &= E\left(\mathop{\circ}_{n=N+1}^{N+M} \hat{\alpha}_{n\Delta}\right) \\ D_E\left(\mathop{\circ}_{n=1}^{N} \hat{\alpha}_{n\Delta}, \mathop{\circ}_{n=1}^{N+M} \hat{\alpha}_{n\Delta}\right) &= \sum_{n=N+1}^{N+M} E(\hat{\alpha}_{n\Delta}) \end{aligned}$$

which approaches zero because we have assumed that the infinite sum converges. Thus the sequence is Cauchy in the D_E metric and completeness (Corollary 1.9) implies the existence of the limit. Once the limit exists, then the fact that $\mathrm{e}^{2\mathrm{i}\omega(\mathrm{j}-1)\Delta} \mathop{\circ}_{\mathrm{n}=\mathrm{j}}^{\infty} \hat{\alpha}_{\mathrm{n}\Delta}$ satisfies (2.48) is a direct calculation. To complete the proof of the lemma, we must show that the solution is unique.

To see this, suppose that there are two solutions to (2.48), R_j and S_j — we drop the subscript Δ to make the text a little more readable —, then a direct calculation shows that

$$R_{(j-1)} - S_{(j-1)} = \left[\frac{\mathrm{e}^{-2\mathrm{i}\omega\Delta} - \left(\frac{\mathrm{R_{(j-1)}+S_{(j-1)}}}{2}\right)\overline{(\mathrm{e}^{2\mathrm{i}\omega\mathrm{j}\Delta}\hat{\alpha}_\mathrm{j})}}{1 + \left(\frac{R_j+S_j}{2}\right)\overline{(\mathrm{e}^{2\mathrm{i}\omega\mathrm{j}\Delta}\hat{\alpha}_\mathrm{j})}} \right] (R_j - S_j) \tag{2.53}$$

Now $\overline{(\mathrm{e}^{2\mathrm{i}\omega\mathrm{j}\Delta}\hat{\alpha}_\mathrm{j})} \in \mathcal{H}^E(\mathbb{C}^+)$, so every term in (2.53) extends to be holomorphic in the upper half plane. In addition,

$$\left|\frac{R_{(j-1)} + S_{(j-1)}}{2}\right| \leq 1$$
$$\left|\frac{R_j + S_j}{2}\right| \leq 1$$

Moreover, as $j \to \infty$,

$$E(\overline{(\mathrm{e}^{2\mathrm{i}\omega\mathrm{j}\Delta}\hat{\alpha}_\mathrm{j})}) = E(\hat{\alpha}_j) \to 0 \tag{2.54}$$

Let $\omega \in \mathbb{C}^+$, Im $\omega > 0$, be fixed but arbitrary, (2.54) implies that $\overline{(\mathrm{e}^{2\mathrm{i}\omega\mathrm{j}\Delta}\hat{\alpha}_\mathrm{j}(\omega))}$ tends to zero as $j \to \infty$, so that the quantity in square brackets in (2.53) will be arbitrarily close to $\mathrm{e}^{-2\mathrm{i}\omega\Delta}$, and, in particular, of modulus uniformly less than some epsilon less than one, for infinitely many j. Moreover, (2.53) implies that $(R_j - S_j)(\omega)$ vanishes for no j or for all j. But if some $(R_j - S_j)(\omega) \neq 0$, then $(R_{j+1} - S_{j+1})(\omega)$ will have modulus strictly bigger by a factor of $\frac{1}{\epsilon}$ according to the preceding paragraph. Eventually, $(R_{j+M} - S_{j+M})(\omega)$ will have arbitrarily large modulus, which cannot be as both R_{j+M} and S_{j+M} belong to $\mathcal{H}^E(\mathbb{C}^+)$ and hence are less than one in the entire upper half plane. Thus every $(R_j - S_j)$ must vanish at every ω in the open upper half plane, i.e. $R_j = S_j$. This completes the proof of lemma 2.6. ■

Finally, we begin the proof of theorem 2.4.

Proof of theorem 2.4

We shall need a little bit of notation for the proof. We let $r(x,\omega)$ and $\alpha(x)$ denote the *true solution*, whose existence is asserted in theorem 1.2. We shall use superscript notation when referring to the true solution. In particular,

$$\begin{aligned} r^{j\Delta} &= r(j\Delta,\omega) \\ \alpha^{j\Delta} &= \alpha(j\Delta)H_{j\Delta<x<(j-1)\Delta} \end{aligned}$$

We will let $\hat{\alpha}^{j\Delta}$ denote the Fourier transform of $\alpha^{j\Delta}$.

Subscript notation will be used to refer to the approximate solutions,$R_{j\Delta}$ and $\alpha_{j\Delta}$ generated by the discrete algorithm described in theorem 2.3. The $\alpha_{j\Delta}$'s denote the inverse Fourier transforms of the $\hat{\alpha}_{j\Delta}$'s.

The bulk of the proof will be to establish local convergence,i.e.

LEMMA 2.7 (Local Convergence). *Let $r_0(\omega) \in \mathcal{H}^E(\mathbb{C}^+)$, $M < 0$, and $\varepsilon > 0$, then there exists $\delta > 0$ such that, for all $x \in (M,0)$ and $0 > \Delta > -\delta$*

$$\sup_{1 \le j \le [\frac{M}{\Delta}]} \|r^{j\Delta} - R_{j\Delta}\|_{L^2(d\omega)} \le \varepsilon \tag{2.55}$$

$$\|\alpha - \sum_{j=1}^{[\frac{M}{\Delta}]} \alpha_{j\Delta}\|_{L^2(-M,0)} \le \varepsilon \tag{2.56}$$

In the lemma, $[\frac{M}{\Delta}]$ denotes the integer part of $\frac{M}{\Delta}$, i.e the number of steps of size Δ it takes to reach M. Once we have the local convergence lemma, we can utilize our nonlinear Plancherel equalities to make our convergence estimates global. We begin by choosing M so that

$$\|\alpha\|_{L^2(-\infty,M)} \le \varepsilon$$

and $|\Delta|$ so small that (2.56) holds. The two Plancherel equalities yield

$$E(r_0) = \|\alpha\|^2_{L^2(M,0)} + \|\alpha\|^2_{L^2(-\infty,M)}$$
$$E(r_0) = \sum_{j=1}^{[\frac{M}{\Delta}]} E(\hat{\alpha}_{j\Delta}) + \sum_{[\frac{M}{\Delta}]+1}^{\infty} E(\hat{\alpha}_{j\Delta})$$

By choosing $|\Delta|$ even smaller if necessary, we can guarantee that

$$\sum_1^{[\frac{M}{\Delta}]} E(\hat{\alpha}_{j\Delta}) - \sum_1^{[\frac{M}{\Delta}]} \|\hat{\alpha}_{j\Delta}\|^2 \le \varepsilon$$

Lemma 2.7 guarantees us that

$$\sum_1^{[\frac{M}{\Delta}]} \left(\|\hat{\alpha}_{j\Delta}\|^2 - \|\alpha\|^2_{L^2(M,0)} \right) \le \varepsilon$$

which allows us to conclude that

$$\sum_{[\frac{M}{\Delta}]+1}^{\infty} E(\hat{\alpha}_{j\Delta}) \le 3\varepsilon$$

so that both the true α and the approximate $\alpha_{j\Delta}$'s have small norm, independently of Δ, outside $(-M,0)$. Hence

$$\|\alpha - \sum_1^{\infty} \alpha_{j\Delta}\| \le 4\varepsilon$$

That we can further approximate the $\alpha_{j\Delta}$'s by $A_{j\Delta}H_{j\Delta<x<(j-1)\Delta}$ follows from the density of piecewise constant functions in L^2, so this completes the proof of theorem 2.4. ∎

Proof of lemma 2.7

We begin with the integral equation for the true solution

$$r^{j\Delta} = r^{(j-1)\Delta} e^{2i\omega\Delta} + e^{2i\omega j\Delta}\hat{\alpha}^{j\Delta} + e^{2i\omega j\Delta}\int_{j\Delta}^{(j-1)\Delta} \mathrm{e}^{-2i\omega y}\alpha^{j\Delta}\mathrm{r}^2(\mathrm{y})\mathrm{dy}$$

and its discrete counterpart

$$\begin{aligned} R_{j\Delta} \;=\; & R_{(j-1)\Delta}e^{2i\omega\Delta} + \mathrm{e}^{2i\omega j\Delta}\hat{\alpha}_{j\Delta} \\ & + \mathrm{e}^{2i\omega j\Delta}\int_{j\Delta}^{(j-1)\Delta} \mathrm{e}^{-2i\omega y}\alpha_{j\Delta}\mathrm{R}_{j\Delta}\mathrm{e}^{2i\omega(y-j\Delta)}\mathrm{R}_{(j-1)\Delta}\mathrm{e}^{2i\omega(y-(j-1)\Delta)}\mathrm{dy} \end{aligned}$$

We subtract the two to obtain:

$$D_j = D_{(j-1)}e^{2i\omega\Delta} + A_j + S_j^1 + S_j^2 + S_j^3 + S_j^4 \tag{2.57}$$

where

$$\begin{aligned}
D_j &= r^{j\Delta} - R_{j\Delta} \\
A_j &= \hat{\alpha}^{j\Delta} - \hat{\alpha}_{j\Delta} \\
S_j^1 &= \mathrm{e}^{2i\omega j\Delta}\int_{j\Delta}^{(j-1)\Delta} \mathrm{e}^{-2i\omega y}(\alpha^{j\Delta}(\mathrm{y}) - \alpha_{j\Delta}(\mathrm{y}))\mathrm{r}^2(\mathrm{y})\mathrm{dy} \\
S_j^2 &= \mathrm{e}^{2i\omega j\Delta}\int_{j\Delta}^{(j-1)\Delta} \mathrm{e}^{-2i\omega y}\alpha_{j\Delta}\left(\mathrm{r}^2(\mathrm{y}) - \mathrm{r}^{j\Delta}\mathrm{e}^{2i\omega(y-j\Delta)}\mathrm{r}^{(j-1)\Delta}\mathrm{e}^{2i\omega(y-(j-1)\Delta)}\right)\mathrm{dy} \\
S_j^3 &= \mathrm{e}^{-2i\omega(j-1)\Delta}\int_{j\Delta}^{(j-1)\Delta} \mathrm{e}^{2i\omega y}\alpha_{j\Delta}(\mathrm{y})\left(\mathrm{R}_{(j-1)\Delta} - \mathrm{r}^{(j-1)\Delta}\right)\left(\frac{\mathrm{R}_{j\Delta} + \mathrm{r}^{j\Delta}}{2}\right)\mathrm{dy} \\
S_j^4 &= \mathrm{e}^{-2i\omega(j-1)\Delta}\int_{j\Delta}^{(j-1)\Delta} \mathrm{e}^{2i\omega y}\alpha_{j\Delta}(\mathrm{y})\left(\mathrm{R}_{j\Delta} - \mathrm{r}^{j\Delta}\right)\left(\frac{\mathrm{R}_{(j-1)\Delta} + \mathrm{r}^{(j-1)\Delta}}{2}\right)\mathrm{dy}
\end{aligned}$$

The relevant estimates of $S^1 \ldots S^4$ — all norms are $L^2(d\omega)$ — which we will need in the sequel are listed below. They can all be obtained by applying the Cauchy-Schwartz inequality to the relevant definition above, and estimating r and R in L^∞ by 1 or in $L^2(d\omega)$ by $E^{\frac{1}{2}} = E(r_0)^{\frac{1}{2}}$. We also estimate the L^2 norm of $\alpha_{j\Delta}$ by the the L^2 norm of its Fourier transform $\hat{\alpha}_{j\Delta}$.

In the estimate for S^2, it is necessary to first rewrite the difference which appears inside the parentheses as the sum of two terms, analogous to those which appear in S_3 and S_4.

$$\|S_j^1\| \le |\Delta E|^{\frac{1}{2}}\|A_j\| \le |\Delta E|^{\frac{1}{2}}\left(\|\hat{\alpha}_{j\Delta}\| + \|\hat{\alpha}^{j\Delta}\|\right) \tag{2.58}$$

$$\|S_j^2\| \le 4|\Delta|^{\frac{1}{2}}\|\hat{\alpha}^{j\Delta}\|\|\hat{\alpha}_{j\Delta}\|(1 + |\Delta E|^{\frac{1}{2}}) \tag{2.59}$$

$$\|S_j^3\| \le |\Delta E|^{\frac{1}{2}}\|\hat{\alpha}_{j\Delta}\|\;\|D^{j-1}\| \tag{2.60}$$

$$\|S_j^4\| \le |\Delta E|^{\frac{1}{2}}\|\hat{\alpha}_{j\Delta}\|\;\|D^{j}\| \tag{2.61}$$

whence follows

$$\begin{aligned}\sum_{j=1}^{[M/\Delta]}\|S_j^2\| &\le |\Delta|^{\frac{1}{2}}(1+|\Delta E|^{\frac{1}{2}})\sum_{j=1}^{[M/\Delta]}\|\hat{\alpha}^{j\Delta}\|^2\\ &\le |\Delta|^{\frac{1}{2}}(1+|\Delta E|^{\frac{1}{2}})E\\ &= |\Delta|^{\frac{1}{2}}K_E\end{aligned}\tag{2.62}$$

where K_E will be used to denote an assortment of different constants, all of which share the property that they depend only on $E=E(r_0)$. We shall also need

$$\begin{aligned}\sum_{j=1}^{[M/\Delta]}|\Delta E|^{\frac{1}{2}}\|\hat{\alpha}^{j\Delta}\| &\le |\Delta E|^{\frac{1}{2}}\left(\sum_{j=1}^{[M/\Delta]}\|\hat{\alpha}^{j\Delta}\|^2\right)^{\frac{1}{2}}\left(\tfrac{M}{\Delta}\right)^{\frac{1}{2}}\\ &\le EM^{\frac{1}{2}}\end{aligned}\tag{2.63}$$

and

$$\sum_{j=1}^{[M/\Delta]}\left(\|S_j^1\|+\|S_j^2\|+\|S_j^3\|+\|S_j^4\|\right)^2\le|\Delta|K_E\tag{2.64}$$

We begin our main estimate with a slight rearrangement of (2.57),

$$D_j+A_j=D_{(j-1)}e^{2i\omega\Delta}+S_j^1+S_j^2+S_j^3+S_j^4$$

The observation that D_j and A_j are orthogonal gives the estimate

$$\begin{aligned}\|D_j\|^2 \le\ & \|D_{j-1}\|^2+\left(2\|D_{j-1}\|\ \|S_1\|\|-\|A_j\|^2\right)\\ &+2\|D_{j-1}\|\left(\|S_j^2\|+\|S_j^3\|+\|S_j^4\|\right)\\ &+\left(\|S_j^1\|+\|S_j^2\|+\|+S_j^3\|+\|S_j^4\|\right)^2\end{aligned}\tag{2.65}$$

We can employ(2.60), (2.61), and (2.58) to establish

$$\begin{aligned}\|D_{j-1}\|\|S_3\| &\le |\Delta E|^{\frac{1}{2}}\|\hat{\alpha}_{j\Delta}\|\ \|D_{j-1}\|^2\\ \|D_{j-1}\|\|S_4\| &\le |\Delta E|^{\frac{1}{2}}\|\hat{\alpha}_{j\Delta}\|\ \|D_{j-1}\|\ \|D_j\|\\ &\le \tfrac{1}{2}|\Delta E|^{\frac{1}{2}}\|\hat{\alpha}_{j\Delta}\|\ \left(\|D_{j-1}\|^2+\|D_j\|^2\right)\\ \left(2\|D_{j-1}\|\ \|S_1\|-\|A_j\|^2\right) &\le \|A_j\|\left(2|\Delta E|^{\frac{1}{2}}\|D_{j-1}\|-\|A_j\|\right)\\ &\le 4|\Delta E|\|D_{j-1}\|^2\end{aligned}$$

which combine with (2.65) to give

$$\begin{aligned}\left(1-|\Delta E|^{\frac{1}{2}}\|\hat{\alpha}_{j\Delta}\|\right)\|D_j\|^2 &\le \left(1+4|\Delta E|+\tfrac{3}{2}|\Delta E|^{\frac{1}{2}}\|\hat{\alpha}_{j\Delta}\|\right)\|D_{j-1}\|^2+T_j\\ \|D_j\|^2 &\le \left(\frac{1+4|\Delta E|+\frac{3}{2}|\Delta E|^{\frac{1}{2}}\|\hat{\alpha}_{j\Delta}\|}{1-|\Delta E|^{\frac{1}{2}}\|\hat{\alpha}_{j\Delta}\|}\right)\|D_{j-1}\|^2+\frac{T_j}{1-|\Delta E|^{\frac{1}{2}}\|\hat{\alpha}_{j\Delta}\|}\end{aligned}\tag{2.66}$$

where

$$T_j=\left(2\|D_{j-1}\|\ \|S_j^2\|+\left(\|S_j^1\|+\|S_j^2\|+\|S_j^3\|+\|S_j^4\|\right)^2\right)\tag{2.67}$$

A consequence of (2.62) is that

$$\sum_{j=1}^{\infty}\|D_{j-1}\|\|S_j^2\|\le K_E|\Delta|^{\frac{1}{2}}$$

which combines with (2.64) to show that

$$\sum_{j=1}^{\infty} T_j \le K_E |\Delta|^{\frac{1}{2}} \tag{2.68}$$

so that we may sum (2.66) to reach

$$\begin{aligned} \|D_N\|^2 &\le \prod_{j=1}^{N} \left(\frac{1+4|\Delta E|+\frac{3}{2}|\Delta E|^{\frac{1}{2}}\|\hat{\alpha}_{j\Delta}\|}{1-|\Delta E|^{\frac{1}{2}}\|\hat{\alpha}_{j\Delta}\|} \right) \sum_{j=1}^{N} \frac{T_j}{1-|\Delta E|^{\frac{1}{2}}\|\hat{\alpha}_{j\Delta}\|} \\ &\le \exp\left(\sum_{j=1}^{N} \left(4|\Delta E| + \frac{5}{2}|\Delta E|^{\frac{1}{2}}\|\hat{\alpha}_{j\Delta}\| \right) \right) K_E |\Delta|^{\frac{1}{2}} \\ &\le \exp\left(4EM + 2(EM)^{\frac{1}{2}} \right) K_E |\Delta|^{\frac{1}{2}} \end{aligned} \tag{2.69}$$

as long as $N \le [M/\Delta]$. This gives (2.55), the first assertion of lemma 2.7. The other assertion, (2.56), follows from first rewriting (2.65) as

$$\begin{aligned} \|A_j\|^2 \le \|D_{j-1}\|^2 - \|D_j\|^2 \\ +2\|D_{j-1}\| \left(\|S_j^2\| + \|S_j^3\| + \|S_j^4\|\right) + \left(\|S_j^1\| + \|S_j^2\| + \|S_j^3\| + \|S_j^4\|\right)^2 \end{aligned}$$

and summing

$$\begin{aligned} \sum_{j=1}^{N} \|A_j\|^2 &\le \|D_0\|^2 - \|D_N\|^2 + \sup_{0\le j\le N} \|D_j\| \sum_{j=1}^{N} \left(\|S_j^2\| + \|S_j^3\| + \|S_j^4\|\right) \\ &\quad + \sum_{j=1}^{N} \left(\|S_j^1\| + \|S_j^2\| + \|S_j^3\| + \|S_j^4\|\right)^2 \\ &\le \|D_0\|^2 + \sup_{0\le j\le N} \|D_j\| K_E M^{\frac{1}{2}} + |\Delta| K_E \end{aligned}$$

but $\|D_0\|^2 = 0$ and we have proved that the $\|D_j\| \to 0$ as Δ does. Thus

$$\sum_{j=1}^{N} \|A_j\|^2 \to 0$$

which establishes (2.56) and finishes the proof. ■

3. The Paley-Wiener Theorems. In the context of remote sensing, an important question is to determine the extent of an inhomogeneity, That is, from the scattering data, determine the support of α or a. A classical theorem of Paley and Wiener states that the support of a can be determined from the growth rate of the the extension of ρ_0 to $\mathbb{C}$ along the imaginary axis. Our first theorem is restatement of this fact, along with an extension to the multiple scattering case.
The second theorem states that it is possible to determine the width of an inhomogeneity from the modulus of the scattering data, i.e either $|\rho(\omega)|$ or $|r(\omega)|$. *When we write $\rho(\omega)$ or $r(\omega)$, we mean $\rho_0(\omega)$ or $r_0(\omega)$, respectively.*
We will use subscripts, e. g. r_α and ρ_a, to denote the dependence of r and ρ on α and a. We say that $a \in L^2(B,T)$ if a is a square integrable function on $(-\infty, 0)$ which is

zero outside the interval (B, T) (B stands for Bottom and T for Top).

Theorem 3.1 (Paley-Wiener 1).

$$a \in L^2(B,0) \Leftrightarrow \rho_a \in \mathrm{e}^{-2i\omega B}\mathcal{H}^2(\mathbb{C}^-) \tag{3.1}$$

$$a \in L^2(-\infty,T) \Leftrightarrow \rho_a \in \mathrm{e}^{-2i\omega T}\mathcal{H}^2(\mathbb{C}^+) \tag{3.2}$$

$$\alpha \in L^2(-\infty,T) \Leftrightarrow r_\alpha \in \mathrm{e}^{-2i\omega T}\mathcal{H}^E(\mathbb{C}^+) \tag{3.3}$$

$$\alpha \in L^2(B,0) \Leftrightarrow \frac{r_\alpha}{1-|r_\alpha|^2} \in \mathrm{e}^{-2i\omega B}\mathcal{H}^2(\mathbb{C}^-) \tag{3.4}$$

$$\alpha \in L^2(B,T) \Rightarrow \frac{r_\alpha}{1-|r_\alpha|^2} \in \mathrm{e}^{-2i\omega(2T-B)}\mathcal{H}^2(\mathbb{C}^+) \tag{3.5}$$

Theorem 3.2 (Paley-Wiener 2). *If α and $a \in L^2$ and have compact support, then i), ii), and iii) are equivalent:*

$$\begin{array}{ll} i) & a \in L^2(B,T) \text{ for some } T \text{ and } B \text{ with } T-B=W \\ ii) & |\rho_a|^2 \in \mathrm{e}^{2i\omega W}\mathcal{H}^1(\mathbb{C}^-) \\ iii) & |\rho_a|^2 \in \mathrm{e}^{-2i\omega W}\mathcal{H}^1(\mathbb{C}^+) \end{array} \tag{3.6}$$

$$\begin{array}{ll} i) & \alpha \in L^2(B,T) \text{ for some } T \text{ and } B \text{ with } T-B=W \\ ii) & \frac{|r_\alpha|^2}{1-|r_\alpha|^2} \in \mathrm{e}^{2i\omega W}\mathcal{H}^1(\mathbb{C}^-) \\ iii) & \frac{|r_\alpha|^2}{1-|r_\alpha|^2} \in \mathrm{e}^{-2i\omega W}\mathcal{H}^1(\mathbb{C}^+) \end{array} \tag{3.7}$$

Our proofs will depend on the Plancherel equalities. We use $E(r)$ to denote the left hand side of (1.21). We shall need the Gateaux derivatives of r and ρ in what follows. We will use $s_{\alpha\beta}$ to denote the derivative of r in the direction β at α, i.e.

$$s_{\alpha\beta} := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} r_{\alpha+\varepsilon\beta} \tag{3.8}$$

In a complex Hilbert space, it is possible to compute the inner product of two vectors from norms of complex linear combinations of the two. This is called polarization. If we state these formulas in terms of derivatives, we see the natural analogy for our *nonlinear norm*, $E(r)$.

Lemma 3.3 (polarization formulas).

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[\|a+\varepsilon b\|^2 - \|a+i\varepsilon b\|^2\right] = 2i\,(a,b) \tag{3.9}$$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[\|\rho_{a+\varepsilon b}\|^2 - \|\rho_{a+i\varepsilon b}\|^2\right] = 2i\,(\rho_a,\rho_b) \tag{3.10}$$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[E(r_{\alpha+\varepsilon\beta}) - E(r_{\alpha+i\varepsilon\beta})\right] = 2i\left(\frac{r_\alpha}{1-|r_\alpha|^2}, s_{\alpha\beta}\right) \tag{3.11}$$

The proof of Lemma 1 is a simple calculation which we do not include. An immediate consequence is

LEMMA 3.4 (polarized plancherel equalities).

$$(a,b) = (\rho_a, \rho_b) \tag{3.12}$$

$$(\alpha,\beta) = \left(\frac{r_\alpha}{1-|r_\alpha|^2}, s_{\alpha\beta}\right) \tag{3.13}$$

We intend to use (3.12) and (3.13) to obtain our Paley-Wiener theorems. In order to make use of (3.13), we need to examine the span of the $s_{\alpha\beta}$'s. We shall show

LEMMA 3.5 (Properties of the $s_{\alpha\beta}$'s).

$$\beta \in L^2(M,B) \Rightarrow s_{\alpha\beta} \in \mathrm{e}^{-2i\omega B}\mathcal{H}^2(\mathbb{C}^+) \tag{3.14}$$

$$\alpha \in L^2(B,0) \Rightarrow \underset{\beta\in L^2(-\infty,B)}{\mathrm{Span}} \{s_{\alpha\beta}\} = \mathrm{e}^{-2i\omega B}\mathcal{H}^2(\mathbb{C}^+) \tag{3.15}$$

$$\begin{bmatrix} \alpha \in L^2(-\infty,T) \\ \beta \in L^2(N,0) \\ N \geq T \end{bmatrix} \Rightarrow s_{\alpha\beta} - \hat{\beta} \in \mathrm{e}^{-2i\omega(2T-N)}\mathcal{H}^2(\mathbb{C}^+) \tag{3.16}$$

We delay the proof of the lemma for a bit and proceed first with the proof of the Paley- Wiener theorems.

Proof of Paley-Wiener 1

We begin with the proof of (3.1). If $a \in L^2(B,0)$, then a brief glance at (1.12) shows that

$$\rho_a(x,\omega) = \int_B^x e^{2i\omega(x-y)}a(y)dy \tag{3.17}$$

$$= e^{2i\omega(x-B)} \int_0^{x-B} e^{-2i\omega y}a(y+B)dy \tag{3.18}$$

so that, taking $x=0$ gives, $\rho_a \in \mathrm{e}^{-i\omega B}\mathcal{H}^2(\mathbb{C}^-)$. A similar computation shows that, if $a \in L^2(-\infty,T)$, then $\rho_a \in \mathrm{e}^{-i\omega T}\mathcal{H}^2(\mathbb{C}^+)$

Now, if $\rho_a \in \mathrm{e}^{-i\omega B}\mathcal{H}^2(\mathbb{C}^-)$, then ρ_a is perpendicular to all of $\mathrm{e}^{-i\omega B}\mathcal{H}^2(\mathbb{C}^+)$, which includes ρ_b for every $b \in L^2(-\infty,B)$. Now use the Plancherel equality (3.12)

$$0 = (\rho_a,\rho_b) = (a,b) \tag{3.19}$$

to see that a is perpendicular to $L^2(-\infty,B)$. Thus we have established (3.1); (3.2) is analogous.

To establish (3.3); we recall from [6] that $r_\alpha(x,\cdot)$ is the unique $\mathcal{H}^E(\mathbb{C}^+)$ valued solution to the integral equation

$$r_\alpha(x,\omega) = \int_{-\infty}^{x} e^{2i\omega(x-y)}\alpha(y)\left(1-r_\alpha^2(y,\omega)\right)dy \tag{3.20}$$

If $\alpha \in L^2(-\infty,T)$

$$r_\alpha(x,\omega) = e^{2i\omega(x-T)}\int_{-\infty}^{T} e^{2i\omega(T-y)}\alpha(y)\left(1-r_\alpha^2(y,\omega)\right)dy \tag{3.21}$$

$$= e^{2i\omega(x-T)}r_\alpha(T,\omega) \in e^{2i\omega(x-T)}\mathcal{H}^E(\mathbb{C}^+) \tag{3.22}$$

To see the converse, assume that $r_\alpha(0,\omega) = e^{-2i\omega T}f$ for some $f \in \mathcal{H}^E(\mathbb{C}^+)$. According to theorem 1.2 (see [6]), f is a reflection coefficient, $r_{\tilde{\alpha}}$, and $\alpha(x)$ must be zero for $x > T$ and $\tilde{\alpha}(x-T)$ for $x \leq T$.

We now move to the proof of (3.4). Suppose first that $\alpha \in L^2(B,0)$. Then the Plancherel equality (3.13) implies that $\frac{r_\alpha}{1-|r_\alpha|^2}$ is perpendicular to $\underset{\beta\in L^2(-\infty,B)}{\text{Span}}\{s_{\alpha\beta}\}$, which, according to (3.15) is exactly $\mathrm{e}^{-2i\omega B}\mathcal{H}^2(\mathbb{C}^+)$. On the other hand, if $\frac{r_\alpha}{1-|r_\alpha|^2}$ is perpendicular to $\mathrm{e}^{-2i\omega B}\mathcal{H}^2(\mathbb{C}^+)$, then it is perpendicular to every $s_{\alpha\beta}$ with $\beta \in L^2(M,B)$ according to (3.14). Hence the Plancherel equality (3.13) implies that α is perpendicular to every $\beta \in L^2(M,B)$.

To prove (3.5), we again use the Plancherel equality (3.13), this time in conjunction with (3.16). For $\beta \in L^2((N,0)\text{with}N \geq T$,

$$0 = (\alpha,\beta) = \left(\frac{r_\alpha}{1-|r_\alpha|^2}, s_{\alpha\beta}\right) \tag{3.23}$$

Since $\alpha \in L^2(B,T)$, $\frac{r_\alpha}{1-|r_\alpha|^2}$ is perpendicular to $\mathrm{e}^{-i\omega B}\mathcal{H}^2(\mathbb{C}^+)$ by (3.4).If we choose $N = 2T - B > T$, then (3.16) implies that $\frac{r_\alpha}{1-|r_\alpha|^2}$ will be perpendicular to $s_{\alpha\beta} - \hat{\beta}$, i.e.

$$\left(\frac{r_\alpha}{1-|r_\alpha|^2}, s_{\alpha\beta}\right) = \left(\frac{r_\alpha}{1-|r_\alpha|^2}, \hat{\beta}\right)$$

As these $\hat{\beta}$'s exactly span $\mathrm{e}^{-2i\omega(2T-B)}\mathcal{H}^2(\mathbb{C}^-)$, $\frac{r_\alpha}{1-|r_\alpha|^2} \in \mathrm{e}^{-2i\omega(2T-B)}\mathcal{H}^2(\mathbb{C}^+)$. This finishes the proof of the theorem. ■

Proof of Paley-Wiener 2

First note that the statement "$|f|^2 \in \mathrm{e}^{2i\omega W}\mathcal{H}^1(\mathbb{C}^-)$" means that function $|f|^2$, which is defined on the real axis and takes real values there, has an analytic extension to the lower half plane and has certain growth properties there.

The equivalence of *ii)* and *iii)* in (3.6) is an immediate consequence of the fact that $|\rho|^2$ is real valued on the real axis. A function $f \in \mathrm{e}^{2i\omega W}\mathcal{H}^1(\mathbb{C}^-)$ if and only if $\overline{f} \in \mathrm{e}^{-2i\omega W}\mathcal{H}^1(\mathbb{C}^+)$, so real valued f belong to both spaces or neither. The same comment applies to (3.7).

We return to (3.6). First, if $a \in L^2(B,T)$, then $\rho_a \in \mathrm{e}^{-2i\omega B}\mathcal{H}^2(\mathbb{C}^-)$ and $\overline{\rho}_a \in \mathrm{e}^{2i\omega T}\mathcal{H}^2(\mathbb{C}^-)$, so that the product of the two belongs to $\mathrm{e}^{2i\omega(T-B)}\mathcal{H}^1(\mathbb{C}^-)$. For the converse, we assume that a has compact support, so that ρ_a is holomorphic in all of $\mathbb{C}$. We must show that the width of that support is exactly W. We will need some basic facts about the factorization of functions in $\mathcal{H}^2(\mathbb{C}^+)$ into inner and outer factors. The facts we state below can be found in [5].

Every function $f \in \mathcal{H}^2(\mathbb{C}^+)$ can be factored into the product of an inner and an outer function. The inner function can be factored further into a complex exponential, a Blashcke product, and a singular inner function. Each of the factors is unique, the inner factors have modulus one on the real line and have modulus less than one in $\mathbb{C}^+$. The outer factor belongs to $\mathcal{H}^2(\mathbb{C}^+)$ and has no zeros in the open upper half plane. In addition, the outer factor depends only on $|f(\omega)|$ on the real line. Specifically,

$$
\begin{aligned}
f &= I_f \ \times \ O_f && (3.24)\\
&= \mathrm{e}^{-2i\omega T_f} \ \times \ B_f(w) \ \times \ S_f(w) \ \times \ O_f(w) && (3.25)
\end{aligned}
$$

where T_f is the smallest (most negative) real number such that $\mathrm{e}^{2i\omega T_f} f \in \mathcal{H}^2(\mathbb{C}^+)$. The Blaschke product, B_f is given by

$$
B_f(\omega) = \prod \epsilon_n \frac{\omega - \omega_n}{\omega - \overline{\omega_n}} \tag{3.26}
$$

where the product is over the roots of f in the open upper half plane and $\epsilon_n = \pm 1$ depending on whether the roots are bigger or smaller than one in modulus. The singular inner factor is necessarily absent when f extends to be analytic in a neighborhood of the real line.

Now our $\rho_a \in \mathcal{H}^2(\mathbb{C}^+)$ has such a factorization:

$$
\begin{aligned}
\rho_a &= I_{\rho_a} \ \times \ O_{\rho_a} && (3.27)\\
&= \mathrm{e}^{-2i\omega T} \ \times \ B^+_{\rho_a} \times \ O_{\rho_a} && (3.28)
\end{aligned}
$$

As ρ_a is holomorphic in $\mathbb{C}$, the singular inner factor is not present. Our hypothesis is that

$$
\begin{aligned}
0 &= \left(|\rho_a|^2, \mathrm{e}^{2i\omega(t+W)}\right) \quad \text{for all} \quad t<0 && (3.29)\\
&= \left(\rho_a, \rho_a \mathrm{e}^{2i\omega(t+W)}\right) && (3.30)\\
&= \left(\frac{\mathrm{e}^{2i\omega(T-W)}\rho_a}{B^+}, O_{\rho_a}\mathrm{e}^{2i\omega t}\right) && (3.31)
\end{aligned}
$$

Since O_{ρ_a} is outer, $O_{\rho_a}\mathrm{e}^{2i\omega t}$ spans all of $\mathcal{H}^2(\mathbb{C}^+)$ and we may conclude from (3.31) that $\frac{\mathrm{e}^{2i\omega(T-W)}\rho_a}{B^+} \in \mathcal{H}^2(\mathbb{C}^-)$. But $\frac{1}{B^+} =: B^-$ is a Blaschke product with roots in the lower half plane, so that if we factor $\frac{\mathrm{e}^{2i\omega(T-W)}\rho_a}{B^+} \in \mathcal{H}^2(\mathbb{C}^-)$, B^- must appear as part of the Blaschke factor in its $\mathcal{H}^2(\mathbb{C}^-)$ inner function. Therefore we may conclude that $\mathrm{e}^{2i\omega(T-W)}\rho_a$ is also in $\mathcal{H}^2(\mathbb{C}^-)$. Summarizing, we have proved that

$$
\rho_a \in \mathrm{e}^{-2i\omega T}\mathcal{H}^2(\mathbb{C}^+) \ \cap \ \mathrm{e}^{-2i\omega(T-W)}\mathcal{H}^2(\mathbb{C}^-) \tag{3.32}
$$

which puts α in $L^2(B,T)$ according to the first Paley-Wiener theorem.

We shall obtain (3.7) by a completely analogous computation. If $\alpha \in L^2(B,T)$, we may obtain *iii*) from *i*) by simply multiplying (3.3) by the complex conjugate of (3.4). For the converse, we begin by noting that r_α has a factorization as

$$r_\alpha = I_{r_\alpha} \times O_{r_\alpha} \tag{3.33}$$
$$= \mathrm{e}^{-2i\omega T_{r_\alpha}} \times B^+_{r_\alpha} \times O_{r_\alpha} \tag{3.34}$$

There is no singular inner factor because, under our hypothesis that α has compact support, r_α extends to be meromorphic in all of $\mathbb{C}$. To see this, note that $\frac{r_\alpha}{1-|r_\alpha|^2}$ is entire via (3.4) and (3.5) and

$$\frac{1}{1-|r_\alpha|^2} = 1 + \frac{|r_\alpha|^2}{1-|r_\alpha|^2} \tag{3.35}$$

is entire by hypothesis. r_α is the quotient of these two and therefore meromorphic. Now, as $\frac{|r_\alpha|^2}{1-|r_\alpha|^2} \in \mathrm{e}^{2i\omega W}\mathcal{H}^1(\mathbb{C}^-)$,

$$0 = \left(\frac{|r_\alpha|^2}{1-|r_\alpha|^2}, \mathrm{e}^{2i\omega(t+W)}\right) \quad \forall t < 0 \tag{3.36}$$
$$= \left(\frac{r_\alpha}{1-|r_\alpha|^2}, r_\alpha \mathrm{e}^{2i\omega(t+W)}\right) \tag{3.37}$$
$$= \left(\frac{\mathrm{e}^{2i\omega(T-W)}\frac{r_\alpha}{1-|r_\alpha|^2}}{B^+_{r_\alpha}}, O_{r_\alpha}\mathrm{e}^{2i\omega t}\right) \tag{3.38}$$

Since O_{r_α} is outer, $O_{r_\alpha}\mathrm{e}^{2i\omega t}$ spans all of $\mathcal{H}^2(\mathbb{C}^+)$ and we may conclude from (3.38) that $\frac{\mathrm{e}^{2i\omega(T-W)}\frac{r_\alpha}{1-|r_\alpha|^2}}{B^+} \in \mathcal{H}^2(\mathbb{C}^-)$. As before, $\frac{1}{B^+} =: B^-$ is a Blaschke product with roots in the lower half plane, so that it appears in the factorization of $\frac{\mathrm{e}^{2i\omega(T-W)}\frac{r_\alpha}{1-|r_\alpha|^2}}{B^+}$ in $\mathcal{H}^2(\mathbb{C}^-)$. Therefore we may conclude that $\mathrm{e}^{2i\omega(T-W)}\frac{r_\alpha}{1-|r_\alpha|^2}$ is also in $\mathcal{H}^2(\mathbb{C}^-)$. Now an application of of (3.3) and (3.4) puts α in $L^2(B,T)$. ■

<u>Proof of Lemma 3.5</u>
We begin with (1.6), for a family of α's, i.e.

$$r'_{\alpha+\varepsilon\beta} = 2i\omega r_{\alpha+\varepsilon\beta} + \alpha(1 - r^2_{\alpha+\varepsilon\beta}) \tag{3.39}$$
$$r_{\alpha+\varepsilon\beta}(-\infty,\omega) = 0 \tag{3.40}$$

Differentiate with respect to ε and set $\varepsilon = 0$ to obtain

$$s'_{\alpha\beta} = 2i\omega s_{\alpha\beta} - \alpha r_\alpha s_{\alpha\beta} + \beta(1 - r^2_\alpha) \tag{3.41}$$
$$s_{\alpha\beta}(-\infty,\omega) = 0 \tag{3.42}$$

We can write the solution to this linear equation as

$$s_{\alpha\beta}(x,\omega) = \int_{-\infty}^{x} \mathrm{e}^{2i\omega(x-y)}\mathrm{e}^{-\int_y^x \alpha(t)r_\alpha(t,\omega)dt}\beta(y)\left(1 - r^2_\alpha(y,\omega)\right)dy \tag{3.43}$$

We shall consider β's in $L^2(M,0)$ with M finite. This is just to avoid fussing about the convergence of the integral above. In particular, we need to make note of the fact that $s_{\alpha\beta}(x,\omega) \in \mathcal{H}^2(\mathbb{C}^+)$ for every x. This follows from rewriting (3.43) as

$$\begin{array}{rcl} s_{\alpha\beta}(x,\omega) & = & \int_M^x \mathrm{e}^{2i\omega(x-y)}\beta(y)dy \\ & + & \int_M^x \mathrm{e}^{2i\omega(x-y)}(\mathrm{e}^{-\int_y^x \alpha(t)r_\alpha(t,\omega)dt} - 1)\ \beta(y)dy \\ & - & \int_M^x \mathrm{e}^{2i\omega(x-y)}\mathrm{e}^{-\int_y^x \alpha(t)r_\alpha(t,\omega)dt}\beta(y)r_\alpha^2(y,\omega)dy \end{array} \tag{3.44}$$

Each of the three terms is in $\mathcal{H}^2(\mathbb{C}^+)$ because β is in $L^2(dx)$ and $r_\alpha(x,.)$ is in $\mathcal{H}^2(\mathbb{C}^+) \cap \mathcal{H}^\infty(\mathbb{C}^+)$. If $\beta \in L^2(M,B)$ then (3.43) becomes

$$\begin{aligned} s_{\alpha\beta}(x,\omega) &= \mathrm{e}^{2i\omega(x-B)} \int_M^B \mathrm{e}^{2i\omega(B-y)}\mathrm{e}^{-\int_y^x \alpha(t)r_\alpha(t,\omega)dt}\beta(y)\left(1 - r_\alpha^2(y,\omega)\right)dy \\ &= \mathrm{e}^{2i\omega(x-B)}s_{\alpha\beta}(B,\omega) \end{aligned}$$

and (3.14) follows on setting $x = 0$.

If $\alpha \in L^2(B,0)$ and $\beta \in L^2(-\infty,B)$, then (3.43) simplifies to

$$s_{\alpha\beta}(x,\omega) = \mathrm{e}^{2i\omega(x-B)}\mathrm{e}^{-\int_B^x \alpha(t)r_\alpha(t,\omega)dt}\int_{-\infty}^B \mathrm{e}^{2i\omega(B-y)}\beta(y)dy \tag{3.45}$$

As β varies through $L^2(-\infty,B)$, $\int_{-\infty}^B \mathrm{e}^{2i\omega(B-y)}\beta(y)dy$ span $\mathcal{H}^2(\mathbb{C}^+)$. Multiplication by $\mathrm{e}^{-\int_B^x \alpha(t)r_\alpha(t,\omega)dt} \in \mathcal{H}^\infty(\mathbb{C}^+)$ maps $\mathcal{H}^2(\mathbb{C}^+)$ onto itself, so setting $x = 0$ in (3.45) proves (3.15).

It remains only to prove (3.16); in this case (3.43) simplifies to

$$s_{\alpha\beta}(x,\omega) = \int_N^x \mathrm{e}^{2i\omega(x-y)}\beta(y)dy \tag{3.46}$$

$$- \int_N^x \mathrm{e}^{2i\omega(x-y)}\beta(y)r_\alpha^2(T,y)\mathrm{e}^{2i\omega(y-T)}dy \tag{3.47}$$

At $x = 0$

$$s_{\alpha\beta}(0,\omega) = \hat{\beta} + \mathrm{e}^{-2i\omega 2T}r_\alpha^2(T,y)\int_N^0 \mathrm{e}^{2i\omega y}\beta(y)dy \tag{3.48}$$

$$= \hat{\beta} + \mathrm{e}^{-2i\omega(2T-N)}r_\alpha^2(T,y)\int_N^0 \mathrm{e}^{2i\omega(y-N)}\beta(y)dy \tag{3.49}$$

We need only note that $\int_N^0 \mathrm{e}^{2i\omega(y-N)}\beta(y)dy \in \mathcal{H}^2(\mathbb{C}^+)$ and $r_\alpha^2(T,\omega) \in \mathcal{H}^\infty(\mathbb{C}^+)$ to finish the proof of (3.16). ■

As a corollary of the Paley-Wiener theorems we obtain a nonlinear version of the Shannon Sampling theorem. We state both the linear and nonlinear versions below.

Theorem 3.6 (Shannon Sampling Theorem). *Suppose that*

> <u>Linear</u> $a(x) \in L^2(0,B)$, *then* $\rho_a(\omega)$ *can be exactly reconstructed from its sampled values,* $\{\rho_a(n\Delta)\}_{n\in Z}$, *as long as* $0 < \Delta \leq \frac{\pi}{B}$
>
> <u>Nonlinear</u> $\alpha(x) \in L^2(0,B)$, *then* $r_\alpha(\omega)$ *can be exactly reconstructed from its sampled values,* $r_\alpha(n\Delta)$, *as long as* $0 < \Delta \leq \frac{\pi}{2B}$

Remark
It is tempting to expect the same estimate for Δ in the nonlinear as in the linear case. We don't know if this is true.

Proof
The linear theorem is well known, so we include only a brief recap of the proof. It is customary to translate a so that $a \in L^2(-\frac{B}{2}, \frac{B}{2})$ and denote its Fourier transform by $\hat{a}$. Define the periodic function, $a_p(x)$ by

$$a_p(x) = \sum_{n=-\infty}^{\infty} a(x + n\frac{\pi}{\Delta}) \tag{3.50}$$

which has Fourier series

$$a_p(x) = \sum_{n=-\infty}^{\infty} A_n \mathrm{e}^{2in\Delta x} \tag{3.51}$$

The condition that $\Delta \leq \frac{\pi}{B}$ implies that

$$A_n = \Delta \rho_a(n\Delta) \tag{3.52}$$

Moreover,

$$a = a_p(x)\ H_{-\frac{B}{2}<x<\frac{B}{2}} \tag{3.53}$$

so that

$$\begin{aligned} \hat{a}(\omega) &= \hat{a}_p * \hat{H}_{-\frac{B}{2}<x<\frac{B}{2}} && (3.54)\\ &= \sum_{n=-\infty}^{\infty} \rho_a(n\Delta)\delta(\omega - n\Delta) *\ \mathrm{sinc}(\frac{\pi\omega}{\Delta}) && (3.55)\\ &= \sum_{n=-\infty}^{\infty} \rho_a(n\Delta)\ \mathrm{sinc}\left(\pi\left(\frac{\omega}{\Delta} - n\right)\right) && (3.56) \end{aligned}$$

Once we have the linear theorem in hand, we obtain the nonlinear theorem as a corollary. If $\alpha(x) \in L^2(0, B)$, then $\frac{r_\alpha}{1-|r_\alpha|^2}$ and $\frac{1}{1-|r_\alpha|^2} = 1 - \frac{|r_\alpha|^2}{1-|r_\alpha|^2}$ have Fourier transforms supported in the interval $(-B, B)$ according to (3.3), (3.5), and (3.7).
From the sampled values $r_\alpha(n\Delta)$ for $\Delta \leq \frac{1}{2B}$, we may compute the sampled values of both $\frac{r_\alpha}{1-|r_\alpha|^2}$ and $\frac{1}{1-|r_\alpha|^2}$. Applying the linear sampling theorem to these two functions yields formulas for each of them at any ω, and taking their ratio yields a formula for $r_\alpha(\omega)$. ∎

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