Layer Stripping

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Abstract. We describe a rigorous layer stripping approach to inverse scattering for the Helmholtz equation in one dimension. In section 3, we show how the Ricatti ordinary differential equation, which comes from the invariant embedding approach to forward scattering, becomes an inverse scattering algorithm when combined with the principle of causality.

In section 4 we discuss a method of stacking and splitting layers. We first discuss a formula for combining the reflection coefficients of two layers to produce the reflection coefficient for the thicker layer built by stacking the first layer upon the second. We then describe an algorithm for inverting this procedure; that is, for splitting a reflection coefficient into two thinner reflection coefficients. We produce a strictly convex variational problem whose solution accomplishes this splitting.

Once we can split an arbitrary layer into two thinner layers, we proceed recursively until each reflection coefficients in the stack is so thin that the Born approximation holds (i.e. the reflection coefficient is approximately the Fourier transform of the derivative of the logarithm of the wave speed). We then invert the Born approximation in each thin layer.

1 Introduction

The layer stripping approach to inverse scattering is, in principle, very simple. It can be summarized as follows:

- <u>Born Approximation</u> A thin layer of a medium is easy to recognize from how it reflects an incoming wave. In many layer stripping methods, the layer is infinitesmally thin and the Born Approximation becomes a *trace formula*.
- <u>Causality Principle</u> The reflections from the thin layer nearest the receiver are sensed before the reflections from deeper within the medium.

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– Splitting The initial reflection guaranteed by the causality principle, combined with a specific model of wave propagation, provides enough information to determine the upper thin layer and to compute the response of the medium with that thin layer stripped away.

This approach has been investigated in many papers (e.g.[4], [3], [6], [8],[1], [2], [7]) As with any proposed method, the crucial question is stability. In more than one dimension this question is open, but in one dimension we can give an algorithm and a rigorous proof that it must succeed.

In our point of view, probably the most unexpected lesson here is the role of characterization. Most inverse problems have four fundamental parts: uniqueness, reconstruction, continuous dependence, and characterization. In our study of layer stripping, the characterization of the range of the scattering operator has consistently provided the insight which led to our reconstruction algorithms and the proper formulation of continuous dependence.

The next few subsections contain the results (some from [9], some from [10], and some new) for which we will provide proofs and elaborations in the next two sections.

1.1 The 1-D Helmholtz Equation and Travel Time

The one dimensional Helmholtz equation is:

$$\frac{d^2u}{dz^2} + \frac{\omega^2}{c^2(z)}u = 0 \tag{1}$$

We work on the negative half line or a subset thereof (a layer), $-\infty \leq B < z < T \leq 0$. The reflection and transmission coefficients are defined by the following conditions at the top and bottom of the layer:

$$u(z,\omega) \underset{T}{\sim} \sqrt{c_T}\left(\mathrm{e}^{\frac{-\mathrm{i}\omega(\mathrm{z}-\mathrm{T})}{\mathrm{c_T}}} + \mathrm{r}(\omega)\mathrm{e}^{\frac{\mathrm{i}\omega(\mathrm{z}-\mathrm{T})}{\mathrm{c_T}}}\right) \tag{2}$$

$$u(z,\omega) \underset{B}{\sim} \sqrt{c_B}t(\omega)\mathrm{e}^{\frac{-\mathrm{i}\omega(\mathrm{z}-\mathrm{B})}{\mathrm{c_B}}} \tag{3}$$

where the symbol $\underset{B}{\sim}$ means "has the same Cauchy data at $z = B$ as". That is, $u \underset{B}{\sim} v$ means that $u(B,\omega) = v(B,\omega)$ and that $u'(B,\omega) = v'(B,\omega)$. This is equivalent to the hypothesis that (1) holds on the whole line and that $c(z)$ is continuous and constant outside the layer. In the case that $B = -\infty$ we understand (3) as a limit.

Because only variations in the wave speed produce reflections, it is convenient to introduce

$$\alpha = -\frac{1}{2}\frac{dc}{dz}$$

At the detector, we observe the reflected waves parameterized by the time it takes the wave to reach them and return. It is convenient to replace the physical depth, z , by the travel time depth, x.

$$x(z) = \int_0^z \frac{dz}{c}$$

In travel time coordinates, (1) becomes

$$u'' + 2\alpha(x)u' + \omega^2 u = 0 \tag{4}$$

and the definition of reflection and transmission for the layer $B < x < T$ changes slightly (but r and t remain the same):

$$u(x,\omega) \underset{T}{\sim} \mathrm{e}^{-\mathrm{i}\omega(\mathrm{x}-\mathrm{T})} + \mathrm{r}(\omega)\mathrm{e}^{\mathrm{i}\omega(\mathrm{x}-\mathrm{T})} \tag{5}$$

$$u(x,\omega) \underset{B}{\sim} \mathrm{e}^{\int_{\mathrm{B}}^{\mathrm{T}} \alpha}\mathrm{t}(\omega)\mathrm{e}^{-\mathrm{i}\omega(\mathrm{x}-\mathrm{B})} \tag{6}$$

Our scattering theory will study the map S between α and r

$$\alpha(x) \overset{S}{\mapsto} r(\omega)$$

and our inverse scattering algorithm will produce $\alpha$from r. Before proceeding further, we discuss the recovery of $c(z)$ from $\alpha(x)$. First note that, with $T = 0$,

$$z'(x) = c(z(x)) = \mathrm{e}^{-2\int_0^{\mathrm{x}} \alpha}$$

so that

$$z(x) = \int_0^x \mathrm{e}^{-2\int_0^{\mathrm{x}'} \alpha} \mathrm{dx}'$$

is monotonone and therefore invertible on its range. Therefore,

$$c(z) = \mathrm{e}^{-2\int_0^{\mathrm{x(z)}} \alpha} \tag{7}$$

Our inverse scattering theory works with $\alpha \in L^2$. Thus, for some α's, (7) will produce a $c(z)$ defined only on a finite interval, with $c = 0$ at the bottom of that interval. This is as it should be. For example, if $\alpha \equiv 0.5$ (not exactly L^2, but easy to compute) then

$$c(z) = 1 + z$$

This corresponds to a medium whose wave speed decreases to zero as z approaches -1. In this medium, it takes an infinitely long time for a wave to reach $z = 1$ and no wave penetrates deeper than that. Our inversion can therefore do no better than to return the wave speed at depths above 1.

1.2 Characterization and continuous dependence

The Fourier transform of a function in $L^2(-\infty, 0)$ extends to be analytic in the complex upper half plane. The set of such analytic functions from the linear Hardy space, $H^2(C^+)$. The norm on $H^2(C^+)$ is defined to be

$$\|\rho\|_{H^2} = \sup_{b>0} \|\rho(\cdot + ib)\|_{L^2} \tag{8}$$

The Fourier transforms of a real-valued functions belong to

$$\mathcal{H}^2(C^+) = \{\rho \in H^2(C^+) \mid \rho(-\overline{\omega}) = \overline{\rho(\omega)}\} \tag{9}$$

The range of the (nonlinear) scattering map is also a Hardy space, $\mathcal{H}^E(C^+)$. $\mathcal{H}^E(C^+)$ is not linear, but it is a complete metric space (see section 2). We define

$$E(r) := \int e(r) d\omega := \int (-\log(1 - |r|^2)) d\omega \tag{10}$$

and $\mathcal{H}^E(C^+)$ to be the subset of $\mathcal{H}^2(C^+)$ such that

$$\mathcal{H}^E(C^+) = \left\{\rho \in \mathcal{H}^2(C^+) \mid \sup_{b>0} E(\rho(\cdot + ib)) < \infty\right\}$$

Our basic results on characterization of the range of the scattering map are stated below. They tell us how to recognize a reflection coefficient and how to recognize a reflection coefficient of a finite width layer (a layer of width W means an $\alpha \in L^2(-W, 0)$; when we say that r has width W, we mean that it is the reflection coefficient of a layer of width W).

Theorem 1 (Characterization of Reflection Coefficients).

- *The scattering map is a homeomorphism from $L^2(-\infty, 0)$ onto $\mathcal{H}^E(C^+)$.*
- *The nonlinear Plancherel equality holds*

$$E(r) = \pi ||\alpha||_{L^2}^2$$

- *An $r \in \mathcal{H}^E(C^+)$ has width W if and only if*

$$\frac{r}{t} \in \mathrm{e}^{-\mathrm{i}\omega\mathrm{W}} \mathcal{H}^2(\mathrm{C}^+) \cap \mathrm{e}^{\mathrm{i}\omega\mathrm{W}} \mathcal{H}^2(\mathrm{C}^-) \tag{11}$$

The second condition involves the transmission coefficient, t, which can be computed from r, as long as we know $r(\omega)$ for all real ω. t is the $\mathrm{e}^{\mathrm{i}\omega\mathrm{W}}$ times the unique outer function with modulus $\sqrt{1 - |r^2|}$ (see (114)).

1.3 Stacking and Splitting Layers

Suppose that we stack two layers, one with width W_1 and a second with width W_2, the resulting layer is

$$\alpha_{12} = \begin{cases} \alpha_1(x) & 0 > x > -W_1 \\ \alpha_2(x+W_1) & -W_1 > x > -(W_1+W_2) \end{cases} \tag{12}$$

and the resulting reflection coefficient is given by the formula

$$r_{12} = r_1 \circ \frac{t_1}{\bar{t}_1} r_2 \tag{13}$$

where $\circ$ represents the formula for composition of conformal s of the unit disk onto itself.

$$a \circ b := \frac{a+b}{1+\overline{a}b} \tag{14}$$

Notice that, according to the Plancherel equality, the E-norm of the layer-composition (13) is the sum of the E-norms of the reflection coefficients of the layers.

$$E(r) = E(r_1) + E(r_2) \tag{15}$$

Our inverse scattering algorithm is based on inverting (13).

Theorem 2 (Layer Splitting Decomposition). *Let $r \in \mathcal{H}^E(C^+)$, and let $W_1 > 0$. Then the strictly convex variational problem*

$$\min_{\substack{\rho \in \mathcal{H}^E(C^+) \\ r-\rho \in \mathrm{e}^{2\mathrm{i}\omega \mathrm{W}_1}\mathcal{H}^2(\mathrm{C}^+)}} E(\rho) \tag{16}$$

has a unique minimizer r_1, and r_1 is the first factor in the unique layer decomposition of r

$$r = r_1 \circ \frac{t_1}{\bar{t}_1} r_2 \tag{17}$$

such that $r_1, r_2 \in \mathcal{H}^E(C^+)$ and r_1 has width W_1. Moreover, the rest of the decomposition, namely t_1 and r_2, can be computed from formulas (136) *and* (137).

1.4 Thin Layers and the Born Approximation

Repeated application of theorem 2 allows us to split a reflection coefficient into a composition of layers of small width. Once the width is small enough, we may resort to the Born approximation or linear inverse scattering, which tells us that the reflection coefficient is approximately the Fourier transform of α at 2ω.

Theorem 3 (Born Approximation). *Let r have width W, then*

$$\|r(\omega) - \hat{\alpha}(2\omega)\|_{L^\infty} \leq 4\|\alpha\|^3_{L^2(-W,0)} W^{\frac{3}{2}} \tag{18}$$

$$\|r(\omega) - \hat{\alpha}(2\omega)\|^2_L \leq \|\alpha\|^2_{L^2(-W,0)} W^{\frac{1}{2}} \tag{19}$$

$$|\log(t) - i\omega| \leq W^{\frac{1}{2}} \|\alpha\|_{L^2} \tag{20}$$

1.5 Complete Layer Decomposition

Combining theorems 3 and 2, we may compute α from r by solving a sequence of convex variational problems ((16)) and then inverting a sequence of Fourier transforms.The theorem below is a corollary of the last two subsections

Theorem 4. *Let $r \in \mathcal{H}^E(C^+)$ and $\{W_i\}$ be a sequence of positive real numbers and $\{S_i\}$ their partial sums. Then r has a unique infinite decomposition:*

$$r = r_1 \circ \frac{t_1}{\overline{t}_1} \left(r_2 \circ \frac{t_2}{\overline{t}_2} (r_3 \circ \ldots \right. \tag{21}$$

The individual terms in the sum

$$a = \sum_{i=1}^{\infty} \left(\mathrm{e}^{2\mathrm{i}\omega \mathrm{S_i}} \frac{\mathrm{r_i}}{\mathrm{t_i}}(2\omega) \right)^{\vee} \tag{22}$$

are supported on disjoint intervals of width W_i and a converges to $\alpha = S^{-1}r$ in L^2 as the width of the W_i approach zero.

We will elaborate on the previous subsections in the next three sections.

2 The Geometry of $H^E(C^+)$

2.1 The Hardy Spaces $\mathcal{H}^p$

We recall, following [5], that for $1 \leq p \leq \infty$

$$H^p(\mathbb{C}^\pm) = \{\rho \mid \rho \text{ holomorphic in } \mathbb{C}^\pm \text{ and } \sup_{b>0} \|\rho(\cdot + ib)\|_{L^p} < \infty\}$$

All such functions have, and are uniquely determined by, their boundary values on the real axis. We will always demand an additional symmetry :

$$\mathcal{H}^p(\mathbb{C}^\pm) = \{\rho \in H^p(\mathbb{C}^\pm) \mid \rho(-\overline{\omega}) = \overline{\rho(\omega)}\} \tag{23}$$

We will make use primarily of $\mathcal{H}^2$. In fact, $\mathcal{H}^2(\mathbb{C}^\pm)$ are exactly the Fourier transforms of real valued L^2 functions supported on the negative (resp. positive) half line (see [5]). With $\mathcal{L}^2$ denoting L^2 functions with $f(-\omega) = \overline{f(\omega)}$, we have

$$\mathcal{L}^2(\mathbb{R}) = \mathcal{H}^2(\mathbb{C}^+) \oplus \mathcal{H}^2(\mathbb{C}^-) \tag{24}$$

We let $P^{\pm}$ denote the projections onto $\mathcal{H}^2(\mathbb{C}^{\pm})$ along $\mathcal{H}^2(\mathbb{C}^{\mp})$. P^+ is called the *Riesz transform.* We shall often write

$$f = f^+ + f^- \tag{25}$$

denoting $P^{\pm}f$ by $f^{\pm}$. We speak of f^+ as the *causal* part of f because it is the Fourier transform of a function supported in the past, and to f^- as the *a-causal* part, because it depends on the future. A reflection coefficient must be causal because reflections cannot arrive at the detector before they have originated from the source.

2.2 The Hardy Space $\mathcal{H}^E$

We shall define $\mathcal{H}^E(\mathbb{C}^+)$ like any other Hardy space :

$$\mathcal{H}^E(\mathbb{C}^+) = \{r \mid r \text{ holomorphic in } \mathbb{C}^+,\ \sup_{b>0} E(r) < \infty, \text{ and } r(-\overline{\omega}) = \overline{r(\omega)}\}$$

where the L^p norm is replaced by

$$E(r) = \int (-\log(1-|r|^2))d\omega \tag{26}$$

$$= \sum_{k=1}^{\infty} \frac{\int |r|^{2k}}{k} \tag{27}$$

An immediate consequence of (27) is:

Lemma 1. *E(r) is strictly convex and positive.*

We can use E to define a metric to measure the distance between two reflection coefficients and hence view $\mathcal{H}^E(C^+)$ as a metric space.

$$D_E^2(r,s) := E(-r \circ s) \tag{28}$$

We will call D_E the E-distance or the distance in the E-metric. A little motivation for the above definition is probably in order. Let

$$e(r) = -log(1-|r|^2) \tag{29}$$

$$p(r) = log\left(\frac{1+|r|}{1-|r|}\right) \tag{30}$$

$$d_e(r,s) = e(-r \circ s) \tag{31}$$

$$d_p(r,s) = p(-r \circ s) \tag{32}$$

For the moment, let r and s denote complex numbers in the unit disk. Then $p(r)$ is the Poincaré distance from r to the origin; $d_p(r,s)$ is the Poincaré distance from r to s. The definition (32) can also be described as follows: Choose a conformal map, F, of the unit disk which maps r to the origin, then measure the distance between $F(s)$ and the origin. This definition makes the Poincaré distance conformally invariant. The analogous definition gives the e-metric (and hence the E-metric) the same property.

Our reflection coefficients will take values in the Poincaré disk. Furthermore, the formula (13) shows that when we add a layer, the new reflection coefficient is formed by applying a conformal map to the old one, so that, in a conformally invariant metric, adding the same top layer to two different layers will not change the E-distance between their reflection coefficients.

Lemma 2. *The metrics, d_e, and therefore D_E, are conformally invariant; i.e. for any conformal F of the unit disk onto itself*

$$d_e(a,b) = d_e(F(a), F(b)).$$

Proof A conformal of the unit disk, $F(z)$ has the form

$$F(z) = e^{i\theta} \frac{a-z}{1-\overline{a}z} \tag{33}$$

where $\theta \in \mathbb{R}$ and a belongs to the unit disk. We use the notation F_a to refer to the in (33) with $\theta = 0$. Now

$$\begin{aligned} d_e(b,c) &= e(-b \circ c) \\ &= e(F_b(c)) \end{aligned}$$

while

$$d_e(G(b), G(c)) = e(F_{G(b)}(G(c))).$$

Now

$$F_{G(b)}(G(z)) : b \mapsto 0$$

so that, according to (33),

$$F_{G(b)}(G(z)) = e^{i\theta} F_b(z)$$

for some θ, so

$$\begin{aligned} d_e(G(b), G(c)) &= e(e^{i\theta} F_b(c)) \\ &= e(F_b(c)) \end{aligned}$$

■

Theorem 5 (Cauchy Schwartz and Triangle Inequalities).

$$|E(a,b)| \leq E(a)^{\frac{1}{2}} E(b)^{\frac{1}{2}} \tag{34}$$

$$D_E(r,s) \leq D_E(r,\tau) + D_E(\tau,s). \tag{35}$$

In addition, the Cauchy Schwartz inequality holds for the tails of the series expansion for $E(r)$, i.e.

$$|E_M(a,b)| \le E_M(a)^{\frac{1}{2}} E_M(b)^{\frac{1}{2}} \tag{36}$$

where

$$E_M(b) := \int \sum_{k=M+1}^{\infty} \frac{|b|^{2k}}{k} d\omega \tag{37}$$

<u>Proof</u>

$$\begin{aligned}
E(a,b) &= \int \log(1-\overline{a}b)d\omega \\
&= \int \sum_{k=1}^{\infty} \frac{\overline{a}b^{k}}{k} d\omega \\
&\le \int \left(\sum_{k=1}^{\infty} \frac{|a|^{2k}}{k}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \frac{|b|^{2k}}{k}\right)^{\frac{1}{2}} d\omega \\
&\le \left(\int \sum_{k=1}^{\infty} \frac{|a|^{2k}}{k} dw\right)^{\frac{1}{2}} \left(\int \sum_{k=1}^{\infty} \frac{|b|^{2k}}{k} d\omega\right)^{\frac{1}{2}} \\
&= E(a)^{\frac{1}{2}} E(b)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
D_E^2(a,b) &= E(-a \circ b) \\
&= E(a) + E(b) - 2ReE(a,b) \\
&\le E(a) + E(b) + 2E(a)^{\frac{1}{2}} E(b)^{\frac{1}{2}} \\
&= (E(a)^{\frac{1}{2}} + E(b)^{\frac{1}{2}})^2 \\
&= (D_E(a,0) + D_E(0,b))^2.
\end{aligned}$$

Now, given any C, choose F conformal and mapping 0 to C. Then

$$\begin{aligned}
D_E(a,b) &= D_E(F^{-1}(a), F^{-1}(b)) \\
&\le D_E(F^{-1}(a), 0) + D_E(0, F^{-1}(b)) \\
&= D_E(a,c) + D_E(c,b).
\end{aligned}$$

Finally, the last assertion follows from the proof of the first, on simply beginning the summations above at $k = M+1$ instead of $k+1$. ■

Corollary 1. *The unit disk, with the metric d_e, and $\mathcal{H}^E(\mathbb{C}^+)$, with the metric D_E, are complete metric spaces.*

Proof Suppose that a sequence $\{r_n\}$ is E-cauchy. According to (28)

$$\begin{aligned} D_E^2(r_n, r_m) &= E(-r_n \circ r_m) \\ &= \|\frac{r_n - r_m}{1 - \overline{r}_n r_m}\|_{L^2}^2 \\ &\geq \frac{1}{4}\|r_n - r_m\|_{L^2}^2 \end{aligned}$$

so that the sequence is L^2-cauchy, and therefore has an L^2 limit. The triangle inequality guarantees that $E(r_n)$ and hence $E(r)$ are bounded above. Another application of the triangle inequality shows that $D_E^2(r_n, r)$ is bounded above by $(E(r_n) + E(r))^2$ so that we may apply the dominated convergence theorem to conclude that $D_E^2(r_n, r)$ goes to zero. Thus the sequence converges in $\mathcal{H}^E$. ■

Theorem 6 (Weak and Strong Convergence). *Suppose that, for all $g \in \mathcal{H}^E(\mathbb{C}^+)$*

$$E(r_n, g) \to E(r, g)$$

and that

$$E(r_n) \to E(r)$$

then

$$D_E(r_n, r) \to 0.$$

In other words, weak convergence plus convergence of norms implies strong convergence.

Proof

$$\begin{aligned} D_E(r_n, r) &= E(r_n) + E(r) - 2\text{Re } E(r_n, r) \\ &\to E(r) + E(r) - 2\text{Re } E(r, r) \\ &= 0 \end{aligned}$$

■

3 The Ricatti Equation

We begin with a layer with top T and bottom B; for the moment, we assume that α is smooth with compact support and that B lies below the support of α. In [9], we worked with general $\alpha \in L^2$, but here we will work with more restricted α's and use our continuous dependence estimates to extend our results to all $\alpha \in L^2$. We return to the Helmholtz equation

$$u'' + 2\alpha(x)u' + \omega^2 u = 0 \tag{38}$$

and the conditions at the ends of the layer

$$u \underset{T}{\sim} \frac{\mathrm{e}^{-\int_{\mathrm{B}}^{\mathrm{T}} \alpha}}{t(T,\omega)} \left(\mathrm{e}^{-\mathrm{i}\omega(\mathrm{x}-\mathrm{T})} + \mathrm{r}(\omega)\mathrm{e}^{\mathrm{i}\omega(\mathrm{x}-\mathrm{T})}\right) \tag{39}$$

$$u \underset{B}{\sim} \mathrm{e}^{-\mathrm{i}\omega(\mathrm{x}-\mathrm{B})} \tag{40}$$

The condition (39) is equivalent to the pair of equations

$$u(T,\omega) = \frac{\mathrm{e}^{-\int_{\mathrm{B}}^{\mathrm{T}} \alpha}}{t(T,\omega)} \left(1 + r(T,\omega)\right) \tag{41}$$

$$u'(T,\omega) = -i\omega \frac{\mathrm{e}^{-\int_{\mathrm{B}}^{\mathrm{T}} \alpha}}{t(T,\omega)} \left(1 - r(T,\omega)\right) \tag{42}$$

and condition (40) insists that u is the unique solution to (38) with Cauchy data

$$u(B,\omega) = 1 \tag{43}$$

$$u'(B,\omega) = i\omega \tag{44}$$

Note that u is independent of B as long as B stays below the support of α. We shall now differentiate (41) and (42) with respect to T to derive the following differential equations for $r(T,\omega)$ and $t(T,\omega)$.

$$r' = 2i\omega r + \alpha(1 - r^2) \tag{45}$$

$$r(B,\omega) = 0 \tag{46}$$

$$t' = i\omega t - \alpha r t \tag{47}$$

$$t(B,\omega) = 1 \tag{48}$$

Strictly speaking, $'$ in the equations above should be differentiation with respect to T, not x; but as $u(x,\omega)$ does not depend on T,

$$\frac{d}{dT}\left(u\Big|_{x=T}\right) = \frac{\partial u}{\partial x}\Big|_{x=T} + \frac{\partial u}{\partial T}\Big|_{x=T} \tag{49}$$

$$= u'(T,\omega) + 0 \tag{50}$$

hence derivatives with respect to T and x are equivalent. One way to obtain (45) is to divided (42) by (41), obtaining

$$\frac{u'}{u} = -i\omega\left(\frac{1-r}{1+r}\right) \tag{51}$$

Differentiating (51) gives

$$
\begin{aligned}
2i\omega \frac{r'}{(1+r)^2} &= q\frac{u''}{u} - \left(\frac{u'}{u}\right)^2 \\
&= -\alpha\frac{u'}{u} + (i\omega)^2\frac{u}{u} - \left(\frac{u'}{u}\right)^2 \\
&= \alpha i\omega\left(\frac{1-r}{1+r}\right) + (i\omega)^2 - \left(-i\omega\left(\frac{1-r}{1+r}\right)\right)^2
\end{aligned}
$$

A little algebra now yields (45). Finally, differentiate (41) to obtain a formula for $u'(T,\omega)$ and set it equal to the formula for $u'(T,\omega)$ in (42). Then use (45) to arrive at (47).

3.1 Forward Scattering

We will establish our forward scattering via the Ricatti equation. We first prove:

Theorem 7. *Let* $\alpha \in C_0^\infty$ *with* $\operatorname{supp}(\alpha) \subset [B,0]$*, then there exists a unique solution* $r \in C([B,0],\mathcal{H}^E(C^+))$ *satisfying* (45) *and* (46)*. In addition,*

$$E(r) = \pi||\alpha||_{L^2}^2 \tag{52}$$

Proof

- Fix $\omega \in \mathbb{C}^+$ and prove local existence of a solution to the integral equation

$$r(x,\omega) = r(x_0) + \int_{x_0}^{x} \mathrm{e}^{2\mathrm{i}\omega(\mathrm{x}-\mathrm{y})}\alpha(\mathrm{y})\big(1-\mathrm{r}^2(\mathrm{y},\omega)\big)\mathrm{dy} \tag{53}$$

  by using the estimate

$$\left|\int_{x_0}^{x} \mathrm{e}^{2\mathrm{i}\omega(\mathrm{x}-\mathrm{y})}\alpha(\mathrm{y})\big(\mathrm{r}^2-\tilde{\mathrm{r}}^2(\mathrm{y},\omega)\big)\mathrm{dy}\right| \le ||\alpha||_{L^2}|x-x_0|^{\frac{1}{2}}|r+\tilde{r}||r-\tilde{r}| \tag{54}$$

  to show that the mapping defined by the right hand side of (53) is a contraction on a suitable ball (say $|r|<2$) if $|x-x_0|$ is small enough.

- Obtain global existence by noting that the a priori estimate $|r|<1$ follows from multiplying (45) by $\overline{r}$ and taking real parts to obtain

$$|r|^{2'} = \alpha(r+\overline{r})(1-|r|^2) \tag{55}$$

- Establish the large ω asymptotics of r by proving some bounds on r and its x-derivatives. We start by differentiating (45) to obtain the differential equation,

$$(r')' = (2i\omega - \alpha r)r' + \alpha'(1-r^2) \tag{56}$$

and the integral representation,

$$r'(x) = r'(B) + \int_B^x \mathrm{e}^{2\mathrm{i}\omega(\mathrm{x}-\mathrm{y})}\mathrm{e}^{2\mathrm{i}\int_\mathrm{x}^\mathrm{y}\alpha\mathrm{r}}\alpha'(\mathrm{y})\big(1-\mathrm{r}^2(\mathrm{y},\omega)\big)\mathrm{dy}$$

and, after noting that $r'(B) = \alpha(B) = 0$, the estimate

$$|r'(x)| \leq 2||\alpha||_{L^2}\mathrm{e}^{||\alpha||_{\mathrm{L}^2}|\mathrm{x}-\mathrm{B}|^{\frac{1}{2}}}|\mathrm{x}-\mathrm{B}|^{\frac{1}{2}} \tag{57}$$

The point of (57) is that $|r'(x)|$ is bounded independent of ω (but not x). That the same is true of $|r''(x)|$ and higher derivatives can be established in the same way. Now return to (56) and notice that every term except $2i\omega r'$ is bounded; so it must be bounded also. Hence

$$|r'(x)| \leq \frac{C}{\omega}$$

and, via integration, r must also satisfy this estimate. Next examine (45) to see that every term except $2i\omega r + \alpha$ is bounded by constant over ω, so that

$$r(x) = -\frac{\alpha}{2i\omega} + O\left(\frac{1}{\omega^2}\right) \tag{58}$$

– Establish the Plancherel equality (52) by dividing (55) by $1-|r|^2$.

$$-\log(1-|r|^2)' = \alpha(r+\overline{r}) \tag{59}$$

Next integrate both sides with respect to ω along the real axis. Note that since r is holomorphic in $\mathbb{C}^+$ and $\overline{r}$ in $\mathbb{C}^-$ with the asymptotics (58), we may perform a residue calculation to establish, for $b > 0$,

$$\int_{-\infty}^{\infty} r(\omega+ib)d\omega = \pi\alpha \tag{60}$$

$$\int_{-\infty}^{\infty} \overline{r(\omega+ib)}d\omega = 0 \tag{61}$$

Although neither of the integrals (60) nor (61) are continuous in b as it passes through zero, the sum of the two is continuous because (58) guarantees the integrability of $r+\overline{r}$. Thus we arrive at

$$\left(-\int \log(1-|r|^2)d\omega\right)' = \pi\alpha^2 \tag{62}$$

whence integration in x from B to 0 yields (52). ■

Theorem 8. *Suppose that $\alpha_n \longrightarrow \alpha$ in L^2. Then $r_n \longrightarrow r$ in $\mathcal{H}^E$.*

Proof

– $\underline{r_n^b \text{ is Cauchy in } L^2}$.Let $b > 0$ and let ω_b and $r^b(x,\omega)$ denote $\omega + ib$ and $\underline{r(x, \omega + ib)}$, respectively. Divide the interval $(-\infty, 0)$ into a finite number of intervals such that

$$||\alpha||_{L^2(x_k,x_{k+1})} < \frac{1}{4}\sqrt{b} \tag{63}$$

and note that, on each such interval, the same estimate will hold with α replaced by α_n, and $\frac{1}{4}$ replaced by a slightly larger constant, as long as n is large enough. We make use of the integral equation

$$r_n^b(x_{k+1}) = r_n^b(x_k) + \int_{x_k}^{x_{k+1}} \mathrm{e^{2i\omega_b(x_{k+1}-y)}\alpha(y)(1-(r_n^b)^2)dy} \tag{64}$$

and define

$$\rho_{nm}(x) = \sup_{x_k<y<x} ||r_n^b(y) - r_m^b(y)||_{L^2(d\omega)} \tag{65}$$

then

$$\begin{aligned}\rho_{nm}(x_{k+1}) \leq \rho_{nm}(x_k) + ||\hat{\alpha}_n - \hat{\alpha}_m||_{L^2(d\omega)} + \\ \int_{x_k}^{x_{k+1}} \mathrm{e^{-2b(x_{k+1}-y)}} \left|(\alpha_\mathrm{n} - \alpha_\mathrm{m})(\mathrm{r_n^b})^2 + \alpha_\mathrm{m}\left((\mathrm{r_n^b})^2 - (\mathrm{r_m^b})^2\right)\right|\end{aligned}$$

Applying the Cauchy-Schwartz inequlity a few more times,

$$\begin{aligned}\rho_{nm}(x_{k+1}) \leq \rho_{nm}(x_k) + (1 + \frac{1}{\sqrt{b}})||\alpha_n - \alpha_m||_{L^2(x_{k+1},x_k)} \\ + \frac{1}{\sqrt{b}}||\alpha_n||_{L^2(x_{k+1},x_k)}\rho_{nm}(x_{k+1})\end{aligned}$$

so that

$$\rho_{nm}(x_{k+1}) \leq \frac{1 + \frac{1}{\sqrt{b}}}{1 - \frac{||\alpha_n||_{L^2(x_{k+1},x_k)}}{\sqrt{b}}} \left(\rho_{nm}(x_k) + ||\alpha_n - \alpha_m||_{L^2(x_{k+1},x_k)}\right) \tag{66}$$

Applying (66) consecutively to each interval yields

$$\rho_{nm}(0) \leq K(b,\alpha)||\alpha_n - \alpha_m||_{L^2} \tag{67}$$

which implies that the r_n^b are Cauchy inL^2.

– $\underline{r_n^k \to r^k \text{ weakly in } L^2}$ As $|r_n|$ and $|r|$ are bounded by 1,

$$|(r_n^b)^k - (r^b)^k| \leq K(k)|r_n^b - r^b|$$

so that

$$\rho_n^b := (r_n^b)^k - (r^b)^k \xrightarrow{L^2} 0 \tag{68}$$

For any $h \in \mathcal{H}^2$

$$\begin{aligned}(\rho_n, h) &= (\rho_n, h^b) + (\rho_n, h - h^b) \\ &= (\rho_n^b, h) + (\rho_n, h - h^b)\end{aligned}$$

so that the second term can be made arbitrarily small by choice of b – remember that $\|\rho_n\|_{L^2}^2$ is bounded by $E(r_n)+E(r)$ which is bounded independently of n – and the first term on the right goes to zero because of (68).

– Weak E-convergence implies strong E-convergence by theorem 5. Verify the weak convergence as follows; let $g \in \mathcal{H}^E$ and E_M be defined as in (37)

$$E(r_n, g) - E(r, g) = \sum_{k=1}^{M} \frac{(r_n^k - r^k, g^k)}{k} + E_M(r_n, g) - E_M(r, g) \tag{69}$$

According to the Cauchy-Schwartz inequality, (36), and the (independent of n) bound on $E(r_n)$, a sufficiently large choice of M will make the last two terms arbitrarily small. Now each of the terms in the summation approach zero because of the weak L^2 convergence discussed in the previous paragraph. ■

An immediate consequence of the proof of theorem 8,which we will use later is below. We use the notation $\alpha_{(-\infty,x]}$ to denote α times the characteristic function of the interval $(-\infty, x]$.

Corollary 2. *If $\alpha \in L^2$, $r(x,\omega) = S\alpha_{(-\infty,x]}$, and $b > 0$, then $r^b(x,\omega)$ is the unique solution to the integral equation*

$$r^b(x,\omega) = \int_{-\infty}^{0} \mathrm{e}^{2\mathrm{i}\omega_\mathrm{b}(\mathrm{x}-\mathrm{y})}\alpha(\mathrm{y})\left(1 - (\mathrm{r}^\mathrm{b})^2\right)\mathrm{dy} \tag{70}$$

The last thing we do in this subsection is prove theorem 3 from the introduction.

Proof of theorem 3

We once again apply the Cauchy-Schwartz inequality to the integral equation

$$r(x) = \int_{-W}^{x} \mathrm{e}^{2\mathrm{i}\omega(\mathrm{x}-\mathrm{y})}\alpha(\mathrm{y})\left(1 - \mathrm{r}^2(\mathrm{y}))\right)\mathrm{dy} \tag{71}$$

$$= \mathrm{e}^{2\mathrm{i}\omega\mathrm{x}}\hat{\alpha}(2\omega) - \int_{-\mathrm{W}}^{0} \mathrm{e}^{2\mathrm{i}\omega(-\mathrm{y})}\alpha\mathrm{r}^2\mathrm{dy} \tag{72}$$

to obtain the L^2 estimate

$$||r - \hat{\alpha}(2\omega)||_{L^2} \le ||\alpha||_{L^2} W^{\frac{1}{2}} ||r||_{L^2} \tag{73}$$

$$\le ||\alpha||_{L^2} W^{\frac{1}{2}} E(r) \tag{74}$$

$$\le ||\alpha||_{L^2}^2 W^{\frac{1}{2}} \tag{75}$$

which is (19). For the L^∞ estimate, (18), we start with (71) to estimate

$$|r(y)| \leq 2|y|^{\frac{1}{2}}||\alpha||_{L^2}$$

and insert this estimate into (72) to obtain

$$\begin{aligned}|r - \hat{\alpha}(2\omega)| &\leq \int_{-W}^{0} \alpha(y) 4|y| ||\alpha||^2_{L^2(-w,y)} dy \\ &\leq \frac{4}{3} W^{\frac{3}{2}} ||\alpha||^3_{L^2}\end{aligned}$$

Finally, we integrate the differential equation (47) to obtain

$$t(x,\omega) = \mathrm{e}^{\mathrm{i}\omega(\mathrm{x}+\mathrm{W})} \mathrm{e}^{\int_{-\mathrm{W}}^{\mathrm{x}} \alpha(\mathrm{y})\mathrm{r}(\mathrm{y})\mathrm{dy}} \tag{76}$$

and apply the Cauchy-Schwartz inequality to the integral in the exponent to obtain (20). ■

3.2 Inverse Scattering

We will produce a solution to the inverse problem by solving the Ricatti equation (45), but this time with the initial data given at the top of the layer, which we take to be $x = 0$.

$$r' = 2i\omega r + \alpha(1 - r^2) \tag{77}$$

$$r(0,\omega) = r_0(\omega) \tag{78}$$

It will turn out that this single equation will provide an equation for α as well as r. We pass to the integral equation formulation:

$$r(x,\omega) = \mathrm{e}^{2\mathrm{i}\omega\mathrm{x}}\mathrm{r}(0,\omega) - \int_{\mathrm{x}}^{0} \mathrm{e}^{2\mathrm{i}\omega(\mathrm{x}-\mathrm{y})}\alpha(\mathrm{y})\mathrm{dy} + \int_{\mathrm{x}}^{0} \mathrm{e}^{2\mathrm{i}\omega(\mathrm{x}-\mathrm{y})}\alpha(\mathrm{y})\mathrm{r}^2(\mathrm{y})\mathrm{dy} \tag{79}$$

If we apply $P^{\pm}$, the orthogonal projectors from L^2 onto $\mathcal{H}^2(\mathbb{C}^{\pm})$ defined in (25), to (79), and use the facts that

$$P^+ r = r \tag{80}$$

$$\int_{x}^{0} \mathrm{e}^{2\mathrm{i}\omega(\mathrm{x}-\mathrm{y})}\alpha(\mathrm{y})\mathrm{dy} = \left(\alpha_{[x,0]}\right)^{\wedge} \tag{81}$$

$$P^+ \left(\alpha_{[x,0]}\right)^{\wedge} = 0 \tag{82}$$

Here

$$\alpha^{\wedge}_{[x,0]} = \int_{x}^{0} \mathrm{e}^{-2\mathrm{i}\omega\mathrm{y}}\alpha(\mathrm{y})\mathrm{dy}$$

$\alpha_{[x,0]}$ denotes α times the characteristic function of the interval $[x,0]$, and $\alpha^{\wedge}_{[x,0]}$ denotes its Fourier transform evaluated at 2ω. We obtain a pair of integral equations

$$r(x) = P^{+}\left(\mathrm{e}^{2\mathrm{i}\omega \mathrm{x}}\mathrm{r}(0) - \int_{\mathrm{x}}^{0} \mathrm{e}^{2\mathrm{i}\omega(\mathrm{x}-\mathrm{y})}\alpha(\mathrm{y})\mathrm{r}^2(\mathrm{y})\mathrm{dy}\right) \tag{83}$$

$$\left(\alpha_{[x,0]}\right)^{\wedge} = \mathrm{e}^{-2\mathrm{i}\omega \mathrm{x}}\mathrm{P}^{-}\left(\mathrm{e}^{2\mathrm{i}\omega \mathrm{x}}\mathrm{r}(0) - \int_{\mathrm{x}}^{0} \mathrm{e}^{2\mathrm{i}\omega(\mathrm{x}-\mathrm{y})}\alpha(\mathrm{y})\mathrm{r}^2(\mathrm{y})\mathrm{dy}\right) \tag{84}$$

for the pair of unknowns $(r(x,\omega), \alpha^{\wedge}_{[x,0]}(\omega))$.

Theorem 9. *If $r_0 \in \mathcal{H}^E(C^+)$, then there exists a unique solution pair $(r,\alpha) \in C((-\infty,0], \mathcal{H}^E(C^+) \oplus L^2(x,0))$ solving* (79) *(or, equivalently,* (83) *and* (84)*). Moreover,*

$$r(x,\omega) = S\alpha_{(-\infty,x]} \tag{85}$$

and the mapping S^{-1} is continuous.

Proof

– Local existence is proved by exhibiting the right hand side of (83) and (84) as a contraction on a ball in $\mathcal{H}^2 \cap \mathcal{H}^\infty \oplus L^2(x,0)$. To see this define

$$\Phi\begin{pmatrix} r \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & F^{-1}\mathrm{e}^{-2\mathrm{i}\omega \mathrm{x}} \end{pmatrix}\begin{pmatrix} P^{+} \\ P^{-} \end{pmatrix}\mathrm{e}^{2\mathrm{i}\omega \mathrm{x}}\Psi \tag{86}$$

with

$$\Psi\begin{pmatrix} r \\ \alpha \end{pmatrix} = \left(r(0) - \int_{x}^{0} \mathrm{e}^{-2\mathrm{i}\omega \mathrm{y}}\alpha(\mathrm{y})\mathrm{r}^2(\mathrm{y})\mathrm{dy}\right) \tag{87}$$

and F^{-1} denoting the inverse Fourier transform. It is straight forward to see that

$$\Psi : \mathcal{H}^2(C^+) \cap \mathcal{H}^\infty \oplus L^2(x,0) \longrightarrow \mathcal{H}^2(C^+) \cap \mathcal{H}^\infty \oplus L^2(x,0) \tag{88}$$

and is a contraction in L^2 norm on an appropriate ball as long as $|x|$ is small enough.

That Φ is a contraction in L^2 norm is an immediate consequence as both $P^{\pm}$ are bounded from L^2 to itself. What requires some discussion is the assertion that the first component of Φ, $P^{+}\mathrm{e}^{2\mathrm{i}\omega \mathrm{x}}\Psi$, remains in $\mathcal{H}^\infty(\mathbb{C}^+)$, even though, in general, $P^{\pm}$ are not bounded on L^∞. Now

$$P^{+}\mathrm{e}^{2\mathrm{i}\omega \mathrm{x}}\Psi = \mathrm{e}^{2\mathrm{i}\omega \mathrm{x}}\Psi - \mathrm{P}^{-}\mathrm{e}^{2\mathrm{i}\omega \mathrm{x}}\Psi \tag{89}$$

so it is enough to bound $G := P^{-}\mathrm{e}^{2\mathrm{i}\omega \mathrm{x}}\Psi$ in the L^∞ norm. This follows because

$$G \in \mathrm{e}^{2\mathrm{i}\omega \mathrm{x}}\mathcal{H}^2(\mathrm{C}^+) \cap \mathcal{H}^2(\mathbb{C}^-) \tag{90}$$

i.e. G is the Fourier transform a function, g, with support in the interval $(0,-x)$. Therefore

$$G = \int_0^{-x} \mathrm{e}^{-2\mathrm{i}\omega\mathrm{y}} \mathrm{gdy} \tag{91}$$

$$|G| \le |x|^{\frac{1}{2}} ||g||_{L^2} \tag{92}$$

$$\le |x|^{\frac{1}{2}} \pi ||G||_{L^2} \tag{93}$$

$$\tag{94}$$

so its L^∞ norm is bounded by the a constant times its L^2 norm.

– Global existence follows from the Plancherel equality

$$E(r(x_1)) - E(r(x_0)) = \pi \int_{x_0}^{x_1} \alpha^2 dx \tag{95}$$

which itself follows from integrating (62) between x_0 and x_1. Thus (95) implies that E(r(x)) decreases as x decreases toward $-\infty$, providing an a priori estimate which allows us to extend our interval of existence.

As a consequence, we have produced a pair (r, α) satisfying (79), in particular, for $N = 1, 2, 3, \ldots$, and $b > 0$ r^b satisfies the integral equation

$$r^b(x, \omega) = r^b(-N, \omega) e^{2i\omega_b(x+N)} + \int_{-N}^{x} e^{2i\omega_b(x-y)} \alpha(1 - (r^b)^2) dy$$

Now, since $\|r^b(x, \cdot)\|_{L^2(d\omega)} \le E(r^b(x, \cdot)) \le E(r^0)$ and $|r^b(x, \omega)|$ is bounded above by one, we may fix x and let $N \to \infty$. We obtain

$$r^b(x, \omega) = 0 + \int_{-\infty}^{x} e^{2i\omega_b(x-y)} \alpha(1 - (r^b)^2) dy \tag{96}$$

which shows, according to corollary 2 that $r = S\alpha$.

– The continuity of S^{-1}
We start by dividing up the half line into k intervals of length W such that

$$\sqrt{E(r)W} < \frac{1}{2} \tag{97}$$

On the k'th interval, $[x_k, x_{k+1}]$, both r and r_n satisfy an integral equation analogous to (86), namely

$$\Phi \begin{pmatrix} r(x) \\ \alpha^{\wedge}_{[x_k, x_{k+1}]} \end{pmatrix} = \begin{pmatrix} P^+ \\ P^- \end{pmatrix} \mathrm{e}^{2\mathrm{i}\omega\mathrm{x}} \Psi(\mathrm{x}) \tag{98}$$

with

$$\Psi_k(r) = r(x_k) - \int_{x_k}^{x_{k+1}} \mathrm{e}^{-2\mathrm{i}\omega\mathrm{y}} \alpha(\mathrm{y}) \mathrm{r}^2(\mathrm{y}) \mathrm{dy} \tag{99}$$

Define

$$\rho_{n,k} = \sup_{x_k < y < x_{k+1}} ||r_n(y) - r(y)||_{L^2(d\omega)} \tag{100}$$

$$A_{n,k} = ||\alpha_n(y) - \alpha(y)||_{L^2([x_k, x_{k+1}])} \tag{101}$$

and subtract the integral equation (98) for r_n from that for r, then

$$\begin{aligned} A_{n,k} + \rho_{n,k} &\leq 2||\Psi_k(r) - \Psi_k(r_n)||_{L^2(d\omega)} \\ &\leq 2\rho_{n,k-1} + 2||\alpha||_{l2([x_k, x_{k+1}])} W^{\frac{1}{2}} \rho_{n,k} + 2||r||_{L^2(d\omega)} W^{\frac{1}{2}} A_{n,k} \end{aligned}$$

so that, since $\sqrt{E(r)}$ dominates the L^2 norms of both α and r,

$$A_{n,k} \leq \frac{\rho_{n,k-1}}{1 - 2\sqrt{WE(r)}}$$

$$\rho_{n,k} \leq \frac{\rho_{n,k-1}}{1 - 2\sqrt{WE(r)}}$$

which yields recursively

$$A_{n,k} \leq \left(\frac{1}{1 - 2\sqrt{WE(r)}} \right)^k ||r(0) - r_n(0)||_{L^2(d\omega)} \tag{102}$$

which shows that α_n converges in L^2 of every finite interval, hence weakly in $L^2(-\infty, 0)$, to α. Now the convergence of norms guaranteed by the Plancherel equality yields strong L^2 convergence and the continuity of S^{-1}. ■

4 Layer Stacking and Splitting

4.1 Stacking Layers

There are several ways to deduce the formulas (13) for stacking layers directly from the Helmholtz equation. As we have already deduced the Ricatti equations, we shall start with the observation that, as a consequence of (45) and (47), $\frac{r}{t}$ and $\frac{1}{t}$ satisfy the linear system of equations:

$$\left(\frac{r}{t}\right)' = i\omega \left(\frac{r}{t}\right) + \alpha \left(\frac{1}{t}\right) \tag{103}$$

$$\left(\frac{1}{t}\right)' = \alpha \left(\frac{r}{t}\right) - i\omega \left(\frac{1}{t}\right) \tag{104}$$

For single layer with reflection and transmission coefficients r_1 and t_1, the solution to (103) with intimal data at $x = B$

$$\begin{pmatrix} \frac{r}{t}(B) \\ \frac{1}{t}(B) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{105}$$

is

$$\begin{pmatrix} \frac{r}{t}(T) \\ \frac{1}{t}(T) \end{pmatrix} = \begin{pmatrix} \frac{r_1}{t_1} \\ \frac{1}{t_1} \end{pmatrix} \tag{106}$$

Now if $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ solve (103) so does $\begin{pmatrix} \overline{v_2} \\ \overline{v_1} \end{pmatrix}$, hence the fundamental solution matrix to (103), mapping data from the bottom to the top of the layer is

$$M_1 = \begin{pmatrix} \frac{1}{\overline{t_1}} & \frac{r_1}{t_1} \\ \frac{\overline{r_1}}{\overline{t_1}} & \frac{1}{t_1} \end{pmatrix} \tag{107}$$

If we stack layer 1 atop layer 2, then we start with initial data $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ at the bottom of layer 2, the value of the solution to (103) at the top of layer 2 (which is the same as the bottom of layer 1) is $\begin{pmatrix} \frac{r_2}{t_2} \\ \frac{1}{t_2} \end{pmatrix}$ so that, at the top of layer 1

$$\begin{pmatrix} \frac{r_{12}}{t_{12}} \\ \frac{1}{t_{12}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\overline{t_1}} & \frac{r_1}{t_1} \\ \frac{\overline{r_1}}{\overline{t_1}} & \frac{1}{t_1} \end{pmatrix} \begin{pmatrix} \frac{r_2}{t_2} \\ \frac{1}{t_2} \end{pmatrix} \tag{108}$$

which can then be unwound to produce

$$r_{12} = \frac{r_1 + \frac{t_1}{\overline{t_1}} r_2}{1 + \overline{r_1} \frac{t_1}{\overline{t_1}} r_2} \tag{109}$$

$$t_{12} = \frac{t_1 t_2}{1 + \overline{r_1} \frac{t_1}{\overline{t_1}} r_2} \tag{110}$$

We remark that stacking layers is associative, that is, stacking layer 2 on top of layer 3, and then layer 1 on top of the 2-3 stack, had better yield the same thing as stacking layer 1 atop layer 2 and then putting the 1-2 layer atop layer 3. The corresponding formula for reflection coefficients takes the form

$$r_{123} = r_1 \circ \frac{t_1}{\overline{t_1}} \left(r_2 \circ \frac{t_2}{\overline{t_2}} r_3 \right) \tag{111}$$

$$= \left(r_1 \circ \frac{t_1}{\overline{t_1}} r_2 \right) \circ \frac{t_{12}}{\overline{t_{12}}} r_3 \tag{112}$$

We return our attention to (109). Our goal is to start with any $r \in \mathcal{H}^E(C^+)$, choose a width, W_1, and produce a factorization as in (109), with r_1 the reflection coefficient of a layer of width W_1. Once we accomplish this step, we can repeat it on r_2, eventually representing r as a composition of reflection coefficients, each of width W_1. If W_1 is small enough, the Born approximation will yield a good approximation for α; if not we may subdivide each layer again into thinner layers until the Born approximation applies.

Before stating our main decomposition result, we recall some basic relationships between t and r.

Lemma 3. *For a layer,* r_1 *of width* W_1

$$|t_1|^2 = 1 - |r_1|^2 \tag{113}$$

$$t_1 = \mathrm{e}^{\mathrm{i}\omega \mathrm{W}_1} \mathrm{e}^{\mathrm{P}^+ \log(1-|\mathrm{r}_1|^2)} \tag{114}$$

$$\frac{r_1}{t_1} \in \mathrm{e}^{\mathrm{i}\omega \mathrm{W}_1} \mathcal{H}^2(\mathrm{C}^-) \cap \mathrm{e}^{-\mathrm{i}\omega \mathrm{W}_1} \mathcal{H}^2(\mathrm{C}^+) \tag{115}$$

<u>Proof</u> We start with (47), and multiply both sides by $\overline{t_1}$ to obtain

$$|t_1|^{2\prime} = -\alpha_1(r_1 + \overline{r_1}) = (1 - |r_1|^2)' \tag{116}$$

where the last equality makes use of (55). Now integrate both sides, using (48) and (46) to get (113). On the other hand, we may integrate (47) directly to obtain

$$t_1 = \mathrm{e}^{\mathrm{i}\omega \mathrm{W}_1} \mathrm{e}^{\int_{-\mathrm{W}_1}^{0} \alpha_1 \mathrm{r}_1 \mathrm{dy}} \tag{117}$$

If we let

$$\tau_1 = \mathrm{e}^{-\mathrm{i}\omega \mathrm{W}_1} \mathrm{t}_1 \tag{118}$$

$$\log(\tau_1) = \int_{-W_1}^{0} \alpha_1 r_1 dy \in \mathcal{H}^2(C^+) \tag{119}$$

$$\log(\tau_1) = 2P^+ \mathrm{Re} \log(\tau_1) \tag{120}$$

$$= P^+ \log(|\tau_1|^2) \tag{121}$$

$$= P^+ \log(1 - |r_1|^2) \tag{122}$$

from which (114) follows. Note also that

$$\mathrm{e}^{-||\alpha_1||_{\mathrm{L}^2} \mathrm{W}_1^{\frac{1}{2}}} \le |\tau_1| \le \mathrm{e}^{||\alpha_1||_{\mathrm{L}^2} \mathrm{W}_1^{\frac{1}{2}}} \tag{123}$$

so that $\frac{r_1}{\tau_1} \in \mathcal{H}^2(C^+)$ which together with (118) gives the second inclusion in (115). To see the first inclusion, let $r_2 \in \mathcal{H}^E(C^+)$ be the reflection coefficient of α_2 and let r be given by (109). Then

$$\pi \left(||\alpha_1||^2 + ||\alpha_2||^2 \right) = E(r_1) + E(r_2) + \int \log(1 + \frac{\overline{r_1}}{\overline{t_1}} t_1 r_2) d\omega \tag{124}$$

but the Plancherel equality tells us that the first two terms on the right exactly equal the two terms on the left, so that

$$\int \log(1 + \frac{\overline{r_1}}{\overline{t_1}} t_1 r_2) d\omega = 0 \tag{125}$$

Since we may choose r_2 arbitrarily small, we must have

$$\int \frac{\overline{r_1}}{\overline{t_1}} t_1 r_2 d\omega = 0 \tag{126}$$

but r_2 can be an arbitrary function in $\mathcal{H}^E(C^+)$, so that $\frac{\overline{r_1}}{\overline{t_1}} t_1 \in \mathcal{H}^E(C^+)$. Since $\frac{1}{t_1} \in \mathrm{e}^{\mathrm{i}\omega \mathrm{W}_1} \mathcal{H}^\infty(\mathbb{C}^+)$, we have $\frac{\overline{r_1}}{\overline{t_1}} \in \mathrm{e}^{-\mathrm{i}\omega \mathrm{W}_1} \mathcal{H}^2(\mathrm{C}^+)$. Taking complex conjugates gives the remainder of (115). ■

Theorem 10. *Let $r \in \mathcal{H}^E(C^+)$ and $W_1 > 0$. The following are equivalent*

1. r_1 *is the unique minimizer of the strictly convex variational problem*

$$\min_{\substack{\rho \in \mathcal{H}^E(C^+) \\ r-\rho \in \mathrm{e}^{2\mathrm{i}\omega \mathrm{W}_1}\mathcal{H}^2(\mathrm{C}^+)}} E(\rho) \tag{127}$$

2. r_1 *satisfies*

$$r - \rho \in \mathrm{e}^{2\mathrm{i}\omega \mathrm{W}_1}\mathcal{H}^2(\mathrm{C}^+) \tag{128}$$

$$\frac{r_1}{t_1} \in \mathrm{e}^{\mathrm{i}\omega \mathrm{W}_1}\mathcal{H}^2(\mathrm{C}^-) \cap \mathrm{e}^{-\mathrm{i}\omega \mathrm{W}_1}\mathcal{H}^2(\mathrm{C}^+) \tag{129}$$

3. *The unique layer decomposition of* r *into a stack of two layers with top layer of width* W_1 *is*

$$r = r_1 \circ \frac{t_1}{\overline{t_1}} r_2 \tag{130}$$

Proof

– 3 implies 2 Any reflection coefficient, r_1 of with W satisfies (129) according to (115). If we use the equality (113), we may rewrite (130) as

$$r = r_1 + t_1^2 \left(1 + \frac{\overline{r_1}}{\overline{t_1}} t_1 r_2\right)^{-1} r_2` \tag{131}$$

which makes it apparent that the second term belongs to $\mathrm{e}^{2\mathrm{i}\omega \mathrm{W}}\mathcal{H}^2(\mathrm{C}^+)$, establishing (128).

– 2 implies 1 Let $\sigma \in \mathrm{e}^{2\mathrm{i}\omega \mathrm{W}}\mathcal{H}^2(\mathrm{C}^+)$ and compute

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} E(r_1 + \epsilon\sigma) = \int \frac{\overline{r_1}}{\overline{t_1}} \frac{\sigma}{t_1} \tag{132}$$

$$= 0 \tag{133}$$

because

$$\frac{\sigma}{t_1} \in \mathrm{e}^{\mathrm{i}\omega \mathrm{W}_1}\mathcal{H}^2(\mathrm{C}^+) \tag{134}$$

we conclude that

$$\frac{\overline{r_1}}{\overline{t_1}} \in \mathrm{e}^{-\mathrm{i}\omega \mathrm{W}_1}\mathcal{H}^2(\mathrm{C}^+) \tag{135}$$

so thatr_1 is a critical point for the variational problem. But as the variational problem is strictly convex, that critical point can only be the unique minimizer.

– 1 implies 3 In theory, we know that the layer decomposition (130) exists and is unique because we have proved existence and uniqueness of the inverse problem in theorem 9. The point here is that we can compute r_1 by performing the convex minimization. Once we have r_1 in hand, according to (113), we can produce

$$t_1 = \mathrm{e}^{\mathrm{i}\omega \mathrm{W}_1} \mathrm{e}^{\mathrm{P}^+ \log(1-|\mathrm{r}_1|^2)} \tag{136}$$

and then

$$r_2 = \frac{\bar{t}_1}{t_1} \left(-r_1 \circ r\right) \tag{137}$$

so that the entire layer decomposition is in hand.

∎

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