Layer Stripping for the Helmholtz Equation

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0 Abstract

We develop a new layer stripping technique for the inverse scattering problem for the one dimensional Helmholtz equation on the half line. The technique eliminates the use of "trace formulas", relying instead on a nonlinear plancherel equality which provides a simple and precise characterization of the reflection data. We prove both convergence of the algorithm and well posedness of the forward and inverse scattering problems.

0 Introduction

The subject of this paper is the inverse scattering problem for the Helmholtz equation on the half line. We have two new pieces to add to this extensively studied puzzle. Specifically, let n be positive and locally integrable with $\frac{d}{dy}\left(\frac{1}{\sqrt{n}}\right) \in L^2(-\infty, 0)$ (this is a pretty general class of n's, including all the rational functions without zero's or poles on the negative real axis); suppose that $n \equiv n_0$ is constant on $(0, +\infty)$. Then there is a unique solution $v(x)$ to

$$\frac{d^2 v}{dy^2} + \omega^2 n^2(y) v = 0 \tag{1}$$

$$v \sim e^{-i\omega \int_0^y n(s)ds} \text{ as } y \to -\infty \tag{2}$$

For $y > 0$, v may be written in the form

$$v(y) = \frac{1}{T(\omega)}\left(e^{-i\omega n_0 y} + R(\omega) e^{i\omega n_0 y}\right) \tag{3}$$

so that (3) defines the reflection coefficient, $R(\omega)$.

It has been known for a long time that $R(\omega)$ uniquely determines $n(x)$. The first new ingredient in this paper is a simple precise characterization of $R(\omega)$ and a nonlinear Plancherel type equality for inverse scattering. Specifically,

Theorem 0.1 *A function $R(\omega)$ is the reflection coefficient for (1) such that $(n^{-1/2})' \in L^2(-\infty, 0)$, if and only if, $R(-\omega) = \overline{R(\omega)}$ and R extends analytically to the upper half plane with*

$$E(R) := -\int_{-\infty}^{\infty} \log(1 - |R|^2) d\omega < \infty \tag{4}$$

Moreover,

$$E(R) = \frac{\pi}{16} \int_{-\infty}^{0} |(n^{-1/2})'|^2 dy \tag{5}$$

The appearance of $(n^{-1/2})'$ in (5), as well as the asymptotic condition (2), becomes a little more transparent if we introduce travel time coordinates in (1), defining a new independent variable x via

$$x(y) = \int_0^y n(\tau) d\tau \tag{6}$$

We let

$$u(x) = v(y(x)) \tag{7}$$

$$\gamma(x) = n(y(x)) \tag{8}$$

$$\begin{array}{rcl} \alpha(x) & = & \gamma(x)^{-1}\frac{d\gamma}{dx} \\ & = & n^{-2}\frac{dn}{dy} \end{array} \tag{9}$$

then (1) becomes

$$\frac{1}{\gamma}(\gamma u')' + \omega^2 u = 0 \tag{10}$$

or

$$u'' + \alpha u' + \omega^2 u = 0 \tag{11}$$

with the solution of interest satisfying

$$u \sim e^{-i\omega x} \text{ as } x \to -\infty \tag{12}$$

For (11) and (12), the equality (5) becomes

$$-\int_{-\infty}^{\infty} \log(1 - |R(\omega)|^2) = \frac{\pi}{4}\int_{-\infty}^{0} |\alpha|^2 dx \tag{13}$$

where $R(\omega)$ requires a slightly different definition, replacing (3) with

$$u(x) = \frac{1}{T(\omega)}\left(e^{-i\omega x} + R(\omega)e^{i\omega x}\right) \tag{14}$$

but turns out to be the same function. To see why we refer to (13) as a nonlinear Plancherel equality, we recall that the linearized scattering map at $\alpha = 0$, known as the Born approximation, is just the Fourier transform.

In the limit as α, and hence r, approach zero, the equality (13) becomes

$$\int_{-\infty}^{\infty} |R(\omega)|^2 = \frac{\pi}{4}\int_{-\infty}^{0} |\alpha|^2 dx \tag{15}$$

which is the classical Plancherel equality.

We shall find it convenient to deal with (11) below. We remark that, once we have solved the inverse problem for $\alpha(x)$ and hence $\gamma(x)$; $x(y)$, and therefore $n(y)$, can be found by integrating the ordinary differential equation

$$\begin{array}{rcl} \frac{dx}{dy} & = & \gamma(x) \\ x(0) & = & 0 \end{array}$$

The second new feature in this paper is that it provides the first mathematically complete formulation of a stable layer stripping algorithm for a continuous medium. In addition, this formulation eliminates the use of trace formulas. All of the layer-stripping algorithms we know of, for continuous or discrete media, rely on trace formulas, and, with one major exception [2], these formulas are not stable enough to permit rigorous mathematical analysis of convergence and stability . Indeed, even in the case of discretely layered media [1], where exact recursive formulas eliminate the convergence question, there appears to be no discussion of stability.

For impulse-response data for the wave (i.e.time dependent Helmholtz) equation, the downward continuation algorithm has been analyzed successfully. The key element here is an *a priori* estimate in [6], which is replaced in our work by the exact equality (13), which we believe to be slightly stronger. In connection with the wave equation, one typically looks at response data for a finite time; while the data in the time harmonic problem is the Fourier transform of the infinitely long response. This difference precludes using one method to develop direct conclusions about the other.

Nevertheless, the layer stripping algorithm we present below can be transformed into the time dependent context, yielding wave splitting methods similar to those found in [3]. We expect the methods described in this paper can be used to provide a characterization of the data as well as rigorous mathematical foundations for these methods.

Somewhat analogous to downward continuation in the time harmonic case is the work of Deift and Trubowitz [4] for the Schrödinger equation, where a carefully chosen trace formula yields convergence and stability.

The layer stripping approach is much more sensitive to the choice of trace formula. The analysis of convergence properties in [2] succeeds because of a very delicate choice of trace formula. Even with this best choice of trace formula, however, Chen and Rokhlin must settle for solving an approximate (truncated at high frequencies) problem. Practically speaking, this is not a serious issue.

What we add in this paper is a precise characterization of the reflection coefficient as well as an exact solution to the inverse problem. We also remove the apparent need for extra smoothness assumptions, a by-product of the use of the trace formula. In our approach to layer stripping we eliminate the trace formula entirely.

There are two observations which are basic to the layer stripping approach. The first is that it is possible to define a reflection coefficient for any $x < 0$, even though the representation (14) does not hold there. we define

$$r(x,\omega) = f\left(\frac{u'(x,\omega)}{-i\omega u(x,\omega)}\right)$$

where $f(z) = \frac{1-z}{1+z}$. A brief calculation will check that $r(0,\omega) = R(\omega)$. Secondly, $r(x,\omega)$ is the unique solution to an ordinary differential equation in x with parameter ω.

$$r' = 2i\omega r + \frac{\alpha}{2}(1 - r^2) \tag{16}$$

$$r(0,\omega) = R(\omega) \tag{17}$$

$$r(-\infty,\omega) = 0 \tag{18}$$

In these terms, the forward scattering problem is to calculate (17) from (16) and (18), while the inverse scattering problem is calculate α from the overposed first order boundary value problem (16)–(17)–(18).

The primary advantage of this approach is that it puts at the disposal of the scatterer/inverse scatterer, the substantial repertoire of tools available for estimating and calculating solutions to ODE's. In addition, it exhibits very clearly the very substantial parallels between the Fourier transform and its inverse and the scattering inverse scattering pair.

Before proceeding with our discussion of the inverse problem, we take a moment to derive the classical Plancherel equality from this layer stripping point of view. We hope that this will serve as a partial justification for the previous statements and will help to guide the reader through some of the more technical computations in section 1.

If we denote the scattering transform by S

$$\alpha \overset{S}{\mapsto} R(\omega)$$

then the Born approximation is given by

$$a \overset{B}{\mapsto} \rho_0$$

where

$$\rho_0(\omega) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(0 + \varepsilon a).$$

Differentiating (16) with respect to ε gives

$$\rho' = 2i\omega\rho + \frac{a}{2} \tag{19}$$

$$\rho(-\infty, \omega) = 0 \tag{20}$$

$$\rho(0, \omega) = \rho_0(\omega) \tag{21}$$

which can be integrated directly to yield

$$\rho(x, \omega) = \int_{-\infty}^{x} e^{2i\omega(x-y)} \frac{a(y)}{2} dy \tag{22}$$

so that

$$\begin{aligned} \rho_0(\omega) &= \textstyle\int_{-\infty}^{0} e^{-2i\omega y} \frac{a(y)}{2} dy \\ &= \left(H_{y<0} \frac{a(y)}{2}\right)^{\wedge} \end{aligned} \tag{23}$$

where $H_{y<0}$ denotes the Heavyside function supported on the left half line and $^\wedge$ denotes the Fourier transform. Notice that it is convenient for us to use the definitions:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-2i\omega z} f(z) dz \tag{24}$$

$$f^{\vee}(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{2i\omega z} f(\omega) d\omega \tag{25}$$

Equation (23) exhibits the Born approximation as the Fourier transform on the half line. Equally important from our point of view, (16) is just (19) with an additional nonlinear term. Our nonlinear Plancherel equality will be proved in Section 1 by paralleling the proof of the linear Plancherel equality which we give below. Begin with (16) and multiply by $\overline{\rho}$ and take real parts to obtain

$$\overline{\rho}\rho' + \rho\overline{\rho}' = \frac{a}{2}(\rho + \overline{\rho})$$

Next integrate w.r.t. ω to obtain

$$\left(\int_{-\infty}^{\infty} |\rho(x, \omega)|^2 d\omega\right)' = \frac{a(x)}{2} \int_{-\infty}^{\infty} (\rho(x, \omega) + \overline{\rho}(x, \omega)) d\omega. \tag{26}$$

Now, according to (22),

$$\begin{aligned} \rho(x, \omega) + \overline{\rho}(x, \omega) &= \left(H_{y<0} \frac{a}{2}(y + x)\right)^{\wedge} \\ &+ \left(H_{y>0} \frac{a}{2}(x - y)\right)^{\wedge} \end{aligned}$$

The term on the right of (26) is the inverse Fourier transform, evaluated at $y = 0$, so that (26) becomes

$$\left(\int_{-\infty}^{\infty} |\rho(x,\omega)|^2 d\omega\right)' = \frac{a^2(x)}{4\pi}.$$

Integrating in x gives the Plancherel equality

$$\int_{-\infty}^{\infty} |\rho(x,\omega)|^2 d\omega = \frac{1}{4\pi}\int_{-\infty}^{x} a^2(y)dy.$$

One consequence of (23) is that the Fourier transform can be computed by solving the initial value problem (19)–(20) in the forward (upward) direction. The principle behind our inverse scattering algorithm in Section 2 is that one may invert the Fourier Transform by solving the initial value problem (19)–(21) in the backward (downward) direction.

There are several technical issues in Section 1 and Section 2 which appear even in the present context of the Born approximation. The first is that, to calculate the Fourier transform in (22) requires some justification, as $a \in L^2$ and not necessarily L^1. One way to deal with this is to let ω be complex, Im $\omega = b \geq 0$, in the upper half plane, $\mathbb{C}^+$. For $b > 0$, the integrand in (22) is L^1. Now, in the presence of the *à priori* bound provided by the Plancherel Equality, we may let $b \to 0$ and conclude that limits exist in the L^2 sense. Indeed, in Section 1 we will follow exactly this strategy, which, in the scattering context, is often referred to as the limiting absorption principle.

This approach yields an observation which will be crucial to our inverse scattering algorithm, that is, the function $\rho_0(\omega)$ in (23) belongs to the Hardy space $H^2(\mathbb{C}^+)$.

The Hardy space, $H^2(\mathbb{C}^+)$, is the space of functions holomorphic in the upper half plane with

$$\|\rho\|_{H^2} = \sup_{\beta>0}\left(\int |\rho(\omega + i\beta)|^2 d\omega\right)^{\frac{1}{2}} < \infty \tag{27}$$

Such analytic functions necessarily have boundary values at $b = 0$. It is customary to use the notation $\rho \in H^2(\mathbb{C}^+)$ to refer to either the analytic function defined for all $\omega \in \mathbb{C}^+$ or the restriction of r to the real line, Im $\omega = 0$. The confusion is minimal as the analytic function is uniquely determined from its boundary values via the Cauchy-Riemann equations. In particular, the functions in $H^2(\mathbb{C}^+)$ are exactly the Fourier Transforms of L^2-functions supported in $(-\infty, 0)$. Moreover, with $H^2(\mathbb{C}^-)$ defined similarly

$$L^2(\mathbb{R}) = H^2(\mathbb{C}^+) \oplus H^2(\mathbb{C}^-)$$

and we will let P^+ and P^- denote the orthogonal projections onto $H^2(\mathbb{C}^+)$ and $H^2(\mathbb{C}^-)$ respectively.

With these facts in mind, we investigate the downward evolution of (19)–(21). The solution is

$$\rho(x, \omega) = e^{2i\omega x}\rho_0(\omega) + \int_0^x e^{2i\omega(x-y)}\frac{a(y)}{2}dy. \tag{28}$$

Now, because $y > x$, the integral in (28) is in $H^2(\mathbb{C}^-)$, while $\rho(x, \omega)$ must be in $H^2(\mathbb{C}^+)$ because of (22), so that, applying P^+ and P^- to (28) gives

$$\rho(x, \omega) = P^+\rho(x, \omega) = P^+e^{2i\omega x}\rho_0(\omega) \tag{29}$$

$$\left(H_{0<y<x}\frac{a}{2}(y + x)\right)^{\wedge} = \int_0^x e^{2i\omega(x-y)}\frac{a(y)}{2}dy = P^-e^{2i\omega x}\rho_0(\omega). \tag{30}$$

That is, the downward evolution in (28), in the presence of the *à priori* information that $\rho(x, \omega) \in H^2(\mathbb{C}^+)$ (the characterization of the forward evolution), splits into two evolutions. The evolution of ρ is described by (29) while the evolution of $\frac{a}{2}$ is described by (30).

There is no need for an additional "trace formula" to couple ρ and a, as has been typical in other implementations of layer stripping. Apart from this, our method is similar to other implementations; we state our algorithm in terms of the Volterra integral equations (29) and (30) derived from (28) rather than in terms of the ODE (19) directly because this formulation is always the first step in proving theorems about ODE's. There are some subtlies involved in implementing the projection P^+, and a good algorithm should be designed to respect a discrete form of the energy defined below in (32). We intend to discuss such an algorithm in detail in a future paper.

This is exactly how we will solve the inverse problem in Section 2, with a few additional technicalities due to the nonlinear term in (16). Instead of H^2, we will define $H^E_{\mathbb{R}}$ to be the subset of H^2 defined by

$$H^E_{\mathbb{R}} = \left\{\rho \in H^2 \mid \ \rho(-\overline{\omega}) = \overline{\rho(\omega)}\ ;\ |\rho| \leq 1\ ;\ \sup_{\beta>0} E(\rho(\omega + i\beta)) < \infty\right\} \tag{31}$$

where

$$E(r) = -\int_{-\infty}^{\infty} \log(1 - |r|^2) d\omega \tag{32}$$

The subscript $\mathbb{R}$ is a reminder that we have included in the definition of $H^E_{\mathbb{R}}$ a symmetry condition which is equivalent to the reality of the coefficient α (i.e. $\alpha \in L^2_{\mathbb{R}}$) in (16). We shall prove

Theorem 0.2 (Forward Scattering) *There exists a unique solution to (16) and (18) such that $r(x, \cdot) \in H^E_{\mathbb{R}}$ for every $x \in (-\infty, 0]$. The map*

$$(\alpha, x) \mapsto r(x, \cdot)$$

is continuous from $L^2_{\mathbb{R}} \oplus \mathbb{R}$ into $H^E_{\mathbb{R}} \subset H^2$ (i.e. if $x_n \xrightarrow{\mathbb{R}} x$ and $\alpha_n \xrightarrow{L^2} \alpha$, then $r_n(x_n, \cdot) \xrightarrow{H^2} r(x, \cdot)$) and

$$E(r(x, \cdot)) = \frac{\pi}{4} \int_{-\infty}^{x} \alpha^2 dy \tag{33}$$

Theorem 0.3 (Inverse Scattering) *Let $R \in H^E_{\mathbb{R}}$. There exists a unique pair, $\alpha \in L^2_{\mathbb{R}}(-\infty, 0)$ and $r(x, \omega)$, continuous in x with values in $H^E_{\mathbb{R}}$, satisfying (16) and (17).*

The map $R \mapsto \alpha$ is continuous (i.e. if $R_n \xrightarrow{H^2} R$ then $\alpha_n \xrightarrow{L^2} \alpha$). In addition,

$$E(r(x, \cdot)) \to 0 \text{ as } x \to -\infty \tag{34}$$

and (33) holds.

Theorem (0.2) and Theorem (0.3) show that there is a dramatic difference between the upward and downward propagation of the ODE (16). In fact, if we are given any initial data, $r(x_0, \omega) = r_0(\omega) \in H^E_{\mathbb{R}}$, then we may choose any $\alpha(x)$, and find a unique solution to (16) for $x > x_0$ with the prescribed initial data; however, for $x < x_0$, there is a unique $\alpha(x)$ for which (16) will have a solution $r \in H^E_{\mathbb{R}}$ for any $x < x_0$. For the Born approximation, this is manifested by the splitting of (28) into (29) and (30) in the downward evolution. If we applied the projections P^+ and P^- to (28) with $x > 0$ (i.e. we tried to evolve upward from $x = 0$), we would obtain only (28); applying P^- to (28) would simply give zero on both sides of the equation.

1 Forward Scattering

We consider the equation

$$u'' + \alpha u' + \omega^2 u = 0 \tag{35}$$

where $\alpha \in L^2(-\infty, \infty)$ is supported in $(-\infty, x_0)$. There exists a unique solution to (35) satisfying

$$u \sim e^{-i\omega x} \text{ as } x \to -\infty \tag{36}$$

We do not give a proof here because this fact is well known and we will prove this fact in a slightly different form below. As $\alpha = 0$ for any $x > x_0$, u can be represented as

$$u = \frac{1}{T(\omega)} \left(e^{-i\omega(x-x_0)} + R(\omega) e^{i\omega(x-x_0)} \right) \tag{37}$$

One can easily check that, with $f(z) = \frac{1-z}{1+z}$,

$$R(\omega) = r(x_0, \omega) = f\left(\frac{u'(x_0)}{-i\omega u(x_0)} \right) \tag{38}$$

If we dispense with the condition that supp $\alpha \subset (-\infty, x_0)$, we can no longer use (37) but we may define $r(x, \omega)$ for any x via (38). It is a straightforward computation to check that

$$r' = 2i\omega r + \frac{\alpha}{2}(1 - r^2) \tag{39}$$

We shall prove the existence of a solution to (39) which satisfies

$$r(x, \omega) \xrightarrow{L^2(d\omega)} 0 \text{ as } x \to -\infty \tag{40}$$

by considering the integral equation

$$r(x, \omega) = \int_{-\infty}^{x} e^{2i\omega(x-y)} \frac{\alpha(y)}{2} (1 - r^2(y, \omega)) dy \tag{41}$$

We remark that, if we were willing to assume that $\alpha \in L^1(\mathbb{R})$, we could produce a solution to (41) by a straightforward iteration process. In order to deal with $\alpha \in L^2$, we shall work with complex $\omega \in \mathbb{C}^+$. We shall use the Hardy space norms,

$$\begin{aligned} \|r(x,\cdot)\|^2_{H^2(\mathbb{C}^+)} &= \sup_{\beta>0} \left(\int_{-\infty}^{\infty} |\rho(\omega + i\beta)|^2 d\omega \right) \\ &= \int_{-\infty}^{\infty} |\rho(\omega)|^2 d\omega \end{aligned} \tag{42}$$

where x is a fixed parameter. Implicit in the second equation is the the fact that (see [5]), if the right hand side of (42) is finite, then L^2 boundary values at $\beta = 0$ exist. We shall also use the H^∞ norm,

$$\begin{aligned} \|r(x,\cdot)\|^2_{H^\infty(\mathbb{C}^+)} &= \sup_{\beta>0} \sup_{\omega\in\mathbb{R}} |\rho(\omega + i\beta)| \\ &= \sup_{\omega\in\mathbb{R}} |\rho(\omega)| \end{aligned} \tag{43}$$

We shall begin by replacing ω in equation (41) by

$$\omega_b = \omega + ib \tag{44}$$

with $b > 0$.

We will show in propositions 1.1, 1.2, and 1.3 that (41) with ω replaced by ω_b has a unique solution, $r_b(x,\omega)$, which is in $H^2(\mathbb{C}^+) \cap B^\infty_m$ (B^∞_m is the ball of radius m in $H^\infty(\mathbb{C}^+)$). In the end, the desired solution $r(x,\omega)$ will be related to $r_b(x,\omega)$ by

$$r_b(x,\omega) = r(x, \omega + ib) \tag{45}$$

Notice that

$$\|r_b(x,\cdot)\|^2_{H^2(\mathbb{C}^+)} = \sup_{\beta>b} \int_{-\infty}^{\infty} |\rho(\omega + i\beta)|^2 d\omega \tag{46}$$

In Corollary 1.5 we will estimate $\|r_b(x,\cdot)\|^2_{H^2(\mathbb{C}^+)}$ independently of b, thus proving the existence of $r(x,\omega)$ and solving the forward scattering problem.

We will never actually show that r satisfies (41) for $b = 0$. We will show that r satisfies (39)–(40), which is not exactly equivalent because α may not be integrable.

In the following proposition, $\mathcal{C}([x_0,x_1]; H^2(\mathbb{C}^+) \cap B^\infty_m)$ denotes continuous maps from $[x_0,x_1]$ into $H^2(\mathbb{C}^+)$, whose image is contained in B^∞_m.

$$\|r\|_{\mathcal{C}([x_0,x_1];H^2(\mathbb{C}^+)\cap B^\infty_m)} = \sup_{x_0\le x\le x_1} \|r(x,\cdot)\|_{H^2(\mathbb{C}^+)} \tag{47}$$

We begin our existence proof with

Proposition 1.1 *Let* $-\infty \leq x_0 < x_1$, $b > 0$,

$$r^0 \in H^2(\mathbb{C}^+) \cap B^\infty_{\frac{m}{2}}$$

and

$$\|\alpha\|_{L^2(x_0,x_1)} < \frac{b^{\frac{1}{2}}}{2} \min(m, \frac{1}{m}) \tag{48}$$

then the map

$$\Phi\rho := r^0_b + \int_{x_0}^{x} e^{2i\omega_b(x-y)} \frac{\alpha(y)}{2}(1 - \rho^2(y,\omega))dy \tag{49}$$

is a contraction on $\mathcal{C}([x_0, x_1]; H^2(\mathbb{C}^+) \cap B^\infty_m)$ *which maps tbe disk,* D_M*, of radius* $M = \frac{\|r^0\|_{L^2} + \|\alpha\|_{L^2}}{\left(1 - \frac{\|\alpha\|_{L^2} m}{2b^{1/2}}\right)}$ *to itself.*

Proof.

Let ρ and r belong to D_M, then

$$\Phi r - \Phi\rho = \int_{x_0}^{x} e^{2i\omega_b(y-x)} \frac{\alpha}{2}(r^2 - \rho^2)dy$$

so that

$$\begin{aligned} \|\Phi r(x,\cdot) - \Phi\rho(x,\cdot)\|^2_{L^2(d\omega)} &\leq \left(\int_{x_0}^{x} |\frac{\alpha^2}{4}|dy\right)\left(\int_{x_0}^{x} e^{-4b(x-y)}dy\right) \times \\ &\quad (\sup_{\omega\in\mathbb{R}} |r(x,\omega) + \rho(x,\omega)|^2) \times \\ &\quad \|r(x,\cdot) - \rho(x,\cdot)\|^2_{L^2(d\omega)} \\ \sup_{x_0\leq x\leq x_1} \|\Phi r(x,\cdot) - \Phi\rho(x,\cdot)\|^2_{L^2(d\omega)} &\leq \|\alpha\|^2_{L^2(x_0,x_1)}(\frac{1}{4b})(2m)^2 \times \\ &\quad \sup_{x_0\leq x\leq x_1} \|r(x,\cdot) - \rho(x,\cdot)\|^2_{L^2(d\omega)} \end{aligned}$$

which shows that Φ is a contraction, in view of (48). We shall also need a similar H^∞ estimate in the case $\rho = 0$, namely,

$$\sup_{x,\omega} |\Phi r - \Phi 0| \leq \|\alpha\|_{L^2(x_0,x_1)}(\frac{1}{4b})^{\frac{1}{2}} m \sup_{x,\omega} |r| \leq \frac{m}{4}$$

To see that Φ maps the ball of radius M to itself, we note that

$$\Phi 0 = r^0 + \int_{x_0}^{x} e^{2i\omega_b(x-y)} \frac{\alpha(y)}{2}dy \tag{50}$$

and that, using the Plancherel equality on the second term in (50) gives

$$\|\Phi 0\|_{L^2(d\omega)} \leq \|r^0\|_{L^2} + \frac{\sqrt{\pi}}{2}\|\alpha\|_{L^2} \tag{51}$$

while, applying the Cauchy-Schwartz inequality to the second term yields

$$\sup_{x,\omega} |\Phi 0| \leq \frac{m}{2} + \frac{\|\alpha\|_{L^2}}{4b^{1/2}} \tag{52}$$

Therefore,

$$\begin{aligned}
\|\Phi r(x,\cdot)\|_{L^2(d\omega)} &\leq \|\Phi 0(x,\cdot)\|_{L^2(d\omega)} + \|\Phi r(x,\cdot) - \Phi 0(x,\cdot)\|_{L^2(d\omega)} \\
&\leq \|r^0\|_{L^2(d\omega)} + \|\alpha\|_{L^2} + \left(\frac{m\|\alpha\|_{L^2}}{2b^{1/2}}\right)\|r(x,\cdot)\|_{L^2(d\omega)} \\
&\leq (\|r^0\|_{L^2(d\omega)} + \|\alpha\|_{L^2})\left(1 + \frac{\theta}{1-\theta}\right) \\
&= (\|r^0\|_{L^2(d\omega)} + \|\alpha\|_{L^2})/(1-\theta)
\end{aligned}$$

where θ denotes $(m\|\alpha\|_{L^2})/2b^{1/2}$. Similarly,

$$\begin{aligned}
|\Phi r(x,\omega)| &\leq |\Phi 0(x,\omega)| + |\Phi r(x,\omega) - \Phi 0(x,\omega)| \\
&\leq \frac{m}{2} + \frac{m}{4} + \frac{m}{4} \\
&\leq m
\end{aligned}$$

■

The previous proposition implies the existence of a unique fixed point $r_b(x,\omega)$ of Φ for every $b > 0$ on some interval in x. In order to prove global existence (in x) of solutions to (41), we need an *à priori* estimate.

Proposition 1.2 *Let $b > 0$, $\rho \in \mathcal{C}([x_0, x_1]; H^2(\mathbb{C}^+) \cap B_m^\infty)$, and suppose that, for $-\infty \leq x_0 < x$, ρ satisfies*

$$\rho' = 2i\omega_b\rho + \frac{\alpha}{2}(1-\rho^2) \tag{53}$$

and

$$|\rho(x_0,\omega)| < 1 \tag{54}$$

then

$$\frac{1}{1-|\rho(x,\omega)|^2} \leq \frac{1}{1-|\rho(x_0,\omega)|^2} e^{\frac{\|\alpha\|_{L^2}}{4b^{1/2}}} \tag{55}$$

Proof. Multiplying (53) by $\overline{\rho}$ and taking real parts yields

$$\left(|\rho|^2\right)' = -4b|\rho|^2 + \frac{\alpha}{2}(\rho + \overline{\rho})(1 - |\rho|^2) \tag{56}$$

Dividing both sides by $1 - |\rho|^2$ gives

$$-\log(1 - |\rho|^2)\ ' = -4b\frac{|\rho|^2}{(1 - |\rho|^2)} + \frac{\alpha}{2}(\rho + \overline{\rho}) \tag{57}$$

so that with $f(x, \omega) := -\log(1 - |\rho(x, \omega)|^2)$,

$$f' \leq -4bf + \frac{\alpha}{2}(\rho + \overline{\rho}) \tag{58}$$

so that

$$f(x, \omega) \leq f(x_0, \omega)e^{-4b(x-x_0)} + \int_{x_0}^{x} e^{-4b(x-s)}|\alpha(s)||\rho(s, \omega)|ds \tag{59}$$

An application of the Cauchy-Schwartz inequality to the second term gives

$$f(x) \leq f(x_0) + \frac{\|\rho(x, \cdot)\|_{H^\infty}\|\alpha\|_{L^2}}{4b^{1/2}} \tag{60}$$

or

$$\frac{1}{1 - |\rho(x, \omega)|^2} \leq \frac{1}{1 - |\rho(x_0, \omega)|^2} e^{\frac{\|\rho(x,\cdot)\|_{H^\infty}\|\alpha\|_{L^2}}{4b^{1/2}}} \tag{61}$$

from which we conclude that $|\rho(x, \omega)| < 1$ and hence (61) becomes (55). ■

As a consequence of the two previous propositions we have:

Proposition 1.3 *Let $\alpha \in L^2(-\infty, x_1)$ and $b > 0$, there exists a unique solution, $r_b \in \mathcal{C}((-\infty, x_1]; H^2(\mathbb{C}^+) \cap B_m^\infty)$ solving*

$$r_b(x, w) = \int_{-\infty}^{x} e^{2i\omega_b(x-y)}\frac{\alpha(y)}{2}(1 - r_b^2(x, w))dy \tag{62}$$

and

$$\frac{1}{1 - |r_b(x, \omega)|^2} \leq e^{\frac{\|\alpha\|_{L^2}}{4b^{1/2}}} \tag{63}$$

Proof. First, apply Proposition 1.1 with $x_0 = -\infty$, $r^0 = 0$ and $m = \epsilon < 1$. This gives existence of r on some interval $(-\infty, \widetilde{x}_1)$, where $\widetilde{x}_1$ satisfies $\|\alpha\|_{L^2(-\infty,\widetilde{x}_1)} \leq \epsilon\frac{b^{1/2}}{2}$. Note, for later use, that

$$\sup_{-\infty<x<x_1} |\rho| < \epsilon \tag{64}$$

Next, apply Proposition 1.1 with $x_0 = \widetilde{x}_1$ and $r^0 = r(\widetilde{x}_1)$. Because (55) implies that $|r^0| \leq 1$, we may choose $m = 2$ and obtain existence on $L^2(-\infty, \widetilde{\widetilde{x}}_1)$ where $\widetilde{\widetilde{x}}$ satisfies $\|\alpha\|_{L^2(\widetilde{x}_1,\widetilde{\widetilde{x}})} \leq b^{1/2}$. The second step may be repeated, with $m = 2$. As $\|\alpha\|_{L^2(-\infty,x_1)} < \infty$, this eventually exhausts the interval in $\|\alpha\|_{L^2}/b^{1/2}$ iterations and proves existence.

If we note that Proposition 1.1 states that Φ is a contraction on all of $\mathcal{C}([x_0, x_1]; H^2(\mathbb{C}^+) \cap B^\infty_m)$, then uniqueness follows as well.

The estimate (63) follows from (55) on letting ϵ approach zero and recalling (64) ■

It remains to let b decrease to zero and show that $r \in \mathcal{C}([x_0, x_1]; H^2(\mathbb{C}^+) \cap B^\infty_m)$; so far we have no control of the $L^2(d\omega)$ norm of $r_b(x, \omega)$ as b approaches zero. Towards this end, we introduce an approximate identity as follows: let g be even and $0 \leq g(x) \in C^\infty_0$ with $\int g = 1$; Let $g_M(x) := Mg(Mx)$. We denote the Fourier transform and its inverse by

$$\hat{f}(x, \omega) = \int_{-\infty}^{\infty} e^{-2i\omega z} f(x, z)dz \tag{65}$$

$$f^\vee(x, z) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{2i\omega z} f(x, \omega)d\omega \tag{66}$$

where, for each fixed x, $f \in L^2$ and the integrals exist in the L^2 sense. We shall also denote the L^2 pairing by

$$\langle g, f\rangle = \int_{-\infty}^{\infty} fgd\omega \tag{67}$$

Theorem 1.4 *Let* $-\infty \leq x_0 < x < x_1$, $b \geq 0$, $\alpha \in L^2(x_0, x_1)$, $\rho \in \mathcal{C}([x_0, x_1]; H^2(\mathbb{C}^+) \cap B^\infty_m)$, $\rho_0 \in H^2(\mathbb{C}^+)$ *(if* $-\infty = x_0$*, then* $b > 0$ *and* $\rho_0 = 0$*). Let*

$$\begin{aligned} R(x, \omega) := \rho_0 e^{2i\omega_b(x-x_0)} &+ \int_{x_0}^{x} e^{2i\omega_b(x-y)}\frac{\alpha(y)}{2}dy \\ &- \int_{x_0}^{x} e^{2i\omega_b(x-y)}\frac{\alpha(y)}{2}\rho^2(y, \omega)dy \end{aligned} \tag{68}$$

then,

$$\lim_{M\to\infty}\langle \hat{g_M}, R(x,\cdot)\rangle = \frac{\pi}{4}\alpha(x) \tag{69}$$

Proof of theorem 1.4. There are three terms in (68). For the first,

$$\langle \hat{g_M}, \rho_0 e^{2i\omega_b(x-x_0)}\rangle = \pi e^{-2b(x-x_0)}\langle g_M, \rho_0^\vee(\cdot+(x-x_0))\rangle \tag{70}$$

As $M\to\infty$, the support of g_M shrinks to zero, while $\rho_0^\vee(z+(x-x_0))$ is supported below $-(x-x_0)$, so the first term approaches zero.

The third approaches zero for similar reasons, namely,

$$\begin{aligned}&\textstyle\int_{x_0}^x e^{-2b(x-y)}\langle \hat{g_M}, e^{2i\omega(x-y)}\rho^2(y,\cdot)\rangle \frac{\alpha(y)}{2}dy\\ &\qquad = \textstyle\int_{x_0}^x e^{-2b(x-y)}\pi\langle g_M, {\rho^2}^\vee(y,\cdot+(x-y))\rangle\frac{\alpha(y)}{2}dy\end{aligned} \tag{71}$$

and again the supports of g_M and ${\rho^2}^\vee(y,\cdot+(x-y))$ become disjoint as $M\to\infty$.

This leaves only the second term, we have

$$\langle \hat{g_M}, \int_{x_0}^x e^{2i(\omega+ib)(x-y)}\tfrac{\alpha(y)}{2}dy\rangle \;=\; \pi\langle g_M, e^{-2bz}\tfrac{\alpha(\cdot+x)}{2}H_{x_0-x<z<0}\rangle \tag{72}$$

$$= \pi\langle g_M, e^{-2b|z|}\tfrac{\alpha(x-|z|)}{4}H_{x_0-x<z<x-x_0}\rangle \tag{73}$$

$$\to \pi\tfrac{\alpha(x)}{4} \tag{74}$$

where $H_{a<z<b}$ denotes the indicator function of the interval (a,b) and we have used the fact that g_M is even in the second step. ■

Corollary 1.5 *The unique solution, r_b to (62) satisfies, with E defined as in (32),*

$$E(r_b(x,\cdot)) \le \frac{\pi}{4}\int_{-\infty}^x \alpha^2(y)dy \tag{75}$$

Proof. We begin with (57) with $\rho = r_b$:

$$\log(1-|r_b|^2)' = -2b\frac{|r_b|^2}{(1-|r_b|^2)} + \frac{\alpha}{2}(r_b+\overline{r}_b) \tag{76}$$

so that with $f(x,\omega) := \log(1-|r_b(x,\omega)|^2)$ and $-\infty < x_0$,

$$f(x,\omega) - f(x_0,\omega) = -2b\int_{x_0}^{x} \frac{|r_b|^2}{(1-|r_b|^2)}dy + \int_{x_0}^{x} \frac{\alpha}{2}(r_b+\overline{r}_b)dy \tag{77}$$

Now, in view of (63) of proposition 1.3, the two terms on the left as well as the first time on the right is in $L^1(d\omega)$, hence the remaining term is also L^1. We may pair each term with g_M and let M approach ∞. Using (69), we obtain

$$\int_{-\infty}^{\infty} f(x,\omega)d\omega - \int_{-\infty}^{\infty} f(x_0,\omega)d\omega = -2b\int_{-\infty}^{\infty}\int_{x_0}^{x} \frac{|r_b|^2}{(1-|r_b|^2)}d\omega + \pi\int_{x_0}^{x} \frac{\alpha^2}{4} \tag{78}$$

If we now note that the first term on the right is negative and the second term on the left tends to zero as x_0 approaches $-\infty$, we obtain (75). ■

Corollary 1.6 *Let $-\infty < x_0 < x_1$, and $\rho \in \mathcal{C}([x_0,x_1]; H^2(\mathbb{C}^+)\cap B_m^\infty)$ satisfy (53) with $b=0$, If either $\rho(x_0)$ or $\rho(x_1)$ is in $H_{\mathbb{R}}^E$, then*

$$E(\rho(x_1,\cdot)) - E(\rho(x_0,\cdot)) = \frac{\pi}{4}\int_{x_0}^{x_1} \alpha^2 dy \tag{79}$$

Proof. We begin with (57) with $b=0$, i.e.

$$-\log(1-|\rho|^2)\ ' = \frac{\alpha}{2}(\rho+\overline{\rho})dy \tag{80}$$

so that

$$-\log(1-|\rho(x_1)|^2) + \log(1-|\rho(x_0)|^2) = \int_{x_0}^{x_1} \frac{\alpha}{2}(\rho+\overline{\rho}) \tag{81}$$

Two of the three terms in (81) are locally integrable with respect to ω, so the third is also, and we may pair with g_M and send M to infinity. By hypothesis, one of the two terms on the left, say $\log(1-|\rho(x_1)|^2)$, is L^1, so that $\langle g_M, \log(1-|\rho(x_1)|^2)\rangle$, approaches $E(\rho(x_1)) < \infty$. The pairing of g_M with the right hand side has a limit which can be calculated by (69), so that $\langle g_M, \log(1-|\rho(x_0)|^2)\rangle$ must also have a limit, which can only be $E(\rho(x_0))$. This gives (79). ■

Proof of Theorem 0.2. The estimate (75) allows us to take b to zero and thus produce r satisfying

$$r(x,\omega)=\int_{x_0}^{x}e^{2i\omega(x-y)}\frac{\alpha(y)}{2}(1-r^2(y,\omega))dy \tag{82}$$

for any x_0. The same estimate (75) shows that $E(r(x_0))$ approaches zero as x_0 approaches minus infinity, establishing (16) and (18). It only remains to check the continuity of the map

$$\alpha\mapsto r$$

Towards this end, let (α,r) and (β,ρ) satisfy (16) and (18); subtracting gives

$$(r-\rho)'=2i\omega(r-\rho)+\alpha(r+\rho)(r-\rho)+(\alpha-\beta)\rho^2$$

or

$$(r-\rho)(x,\omega)-(r-\rho)(x_0,\omega)=\int_{x_0}^{x}e^{2i\omega(x-y)}e^{\int_{x_0}^{x}\alpha(r+\rho)}(\alpha-\beta)\rho^2dy \tag{83}$$

so that, recalling that $|\rho|$ and $|r|$ are less than one, and denoting the right handside of (83) by A

$$\begin{aligned}|A(x,\omega)| &\leq e^{2|x-x_0|^{1/2}\|\alpha\|_{L^2}}\|\alpha-\beta\|_{L^2}\left(\textstyle\int_{x_0}^{x}|\rho(y,\omega)|^4dy\right)^{\frac{1}{2}}\\ \|A(x,\omega)\|^2_{L^2(d\omega)} &\leq e^{4|x-x_0|^{1/2}\|\alpha\|_{L^2}}\|\alpha-\beta\|^2_{L^2}\times\\ &\quad |x-x_0|\sup_{x_0<y<x}\|\rho(y,\cdot)\|^2_{L^2(d\omega)}\\ &\leq e^{4|x-x_0|^{1/2}\|\alpha\|_{L^2}}\|\alpha-\beta\|^2_{L^2}|x-x_0|\|\beta\|^2_{L^2}\end{aligned} \tag{84}$$

where we have used $\|\rho(y,\cdot)\|^2_{L^2(d\omega)}\leq E(\rho(y,\cdot))=\frac{\pi}{4}\|\beta\|^2_{L^2(-\infty,y)}$ to obtain the last line. Therefore,

$$\begin{aligned}\|(r-\rho)(x,\cdot)\|^2_{L^2(d\omega)} &\leq \|(r-\rho)(x_0,\cdot)\|^2_{L^2(d\omega)}+\\ &\quad +e^{4|x-x_0|^{1/2}\|\alpha\|_{L^2}}\|\alpha-\beta\|^2_{L^2}|x-x_0|\|\beta\|^2_{L^2}\\ &\leq \|\alpha\|^2_{L^2(-\infty,x_0)}+\|\beta\|^2_{L^2(-\infty,x_0)}+\\ &\quad +e^{4|x-x_0|^{1/2}\|\alpha\|_{L^2}}\|\alpha-\beta\|^2_{L^2}|x-x_0|\|\beta\|^2_{L^2}\\ &\leq 2\|\alpha\|^2_{L^2(-\infty,x_0)}+\|\beta-\alpha\|^2_{L^2(-\infty,x_0)}+\\ &\quad +e^{4|x-x_0|^{1/2}\|\alpha\|_{L^2}}\|\alpha-\beta\|^2_{L^2}|x-x_0|\times\\ &\quad (\|\alpha\|^2_{L^2}+\|\beta-\alpha\|^2_{L^2})\end{aligned} \tag{85}$$

Given α and any $\epsilon > 0$, we choose x_0 so as to make the first term small and then $\delta = \|\alpha - \beta\|_{L^2}$ to make the rest small.

It is worth remarking that, if we are willing to use L^1 norms, the continuity is actually Lipschitz. That is, (85) becomes

$$\|(r-\rho)(x,\cdot)\|_{L^2(d\omega)} \leq e^{2\|\alpha\|_{L^1}}\|\alpha - \beta\|_{L^1}$$

This finishes the proof of theorem 0.2 and this section as well. ■

2 Inversion

The task of this section is to prove Theorem 0.3. That is, given $r^0 \in H^E_{\mathbb{R}}$, we will produce a unique solution (r, α) to

$$\begin{array}{rcl} r' &=& 2i\omega r + \frac{\alpha}{2}(1-r^2) \\ r(0,\omega) &=& r^0 \end{array} \tag{86}$$

We begin by replacing (86) by its equivalent integral equation:

$$r(x,\omega) = r(0,\omega)e^{2i\omega x} - \int_x^0 e^{2i\omega(x-y)}\frac{\alpha(y)}{2}(1-r^2)dy \tag{87}$$

Notice that, while (41) naturally preserved $H^2(\mathbb{C}^+)$, (87) does not, because $e^{2i\omega x}$ does not decay in $\mathbb{C}^+$ for $x < 0$. Since we insist that $r(x,\omega) \in H^2(\mathbb{C}^+)$, we must choose α to make it so. To this end, we introduce the orthogonal projections $P_{(a,b)}$ on $L^2(\mathbb{R}^1)$ by

$$P_{(a,b)}\rho(x,\omega) = (H_{(a<z<b)}\rho^\vee(x,z))^\wedge \tag{88}$$

where $H_{(a<z<b)}$ is the characteristic function of the interval (a,b).

The operator $P_{(-\infty,0)}$ projects onto $H^2(\mathbb{C}^+)$ along $H^2(\mathbb{C}^-)$ and $P_{(0,\infty)}$ onto $H^2(\mathbb{C}^-)$ along $H^2(\mathbb{C}^+)$. As we insist that

$$P_{(-\infty,0)}r = r \tag{89}$$

we may rewrite (87) as

$$\begin{array}{rcl} 0 &=& P_{(0,\infty)}(e^{2i\omega x}r^0 - \int_x^0 e^{2i\omega(x-y)}\frac{\alpha(y)}{2}(1-r^2)dy) \\ r &=& P_{(-\infty,0)}(e^{2i\omega x}r^0 - \int_x^0 e^{2i\omega(x-y)}\frac{\alpha(y)}{2}(1-r^2)dy) \end{array} \tag{90}$$

If we note that, for $\rho \in L^2(d\omega)$

$$P_{(a,b)} e^{2i\omega x} \rho = e^{2i\omega x} P_{(a+x,b+x)} \rho \tag{91}$$

then, we may rewrite (90) as

$$\begin{aligned} 0 &= e^{2i\omega x} \Big[P_{(x,0)} r^0 - \textstyle\int_x^0 e^{-2i\omega y} \frac{\alpha}{2} dy \\ &\qquad + \textstyle\int_x^0 e^{-2i\omega y} \frac{\alpha}{2} P_{(x-y,0)} r^2 dy \Big] \\ r &= e^{2i\omega x} \Big[P_{(-\infty,x)} r^0 - \textstyle\int_x^0 e^{-2i\omega y} \frac{\alpha}{2} P_{(-\infty,x-y)} r^2 dy \Big] \end{aligned} \tag{92}$$

where we have used (89) and the tautology

$$P_{(-\infty,x)} \int_x^0 e^{-2i\omega y} \frac{\alpha(y)}{2} dy = 0.$$

The first equation in (92) will be used to solve for α; we rewrite the pair of equations one more time:

$$\begin{aligned} (H_{(x<z<0)} \alpha(z))^\wedge &= 2P_{(x,0)} r^0 + \textstyle\int_x^0 e^{-2i\omega y} \alpha P_{(x-y,0)} r^2 dy \\ r &= e^{2i\omega x} \Big[P_{(-\infty,x)} r^0 - \textstyle\int_x^0 e^{-2i\omega y} \frac{\alpha}{2} P_{(-\infty,x-y)} r^2 dy \Big] \end{aligned} \tag{93}$$

This is the system we solve by iteration for x small enough.

We fix x^0 and r^0, and define

$$\Phi(\alpha, r; r^0) = \begin{pmatrix} 2 \left(P_{(x,x_0)} r^0 + \int_x^{x_0} e^{-2i\omega y} \frac{\alpha}{2} P_{(x-y,x_0)} r^2 dy \right)^\vee \\ e^{2i\omega x} \left(P_{(-\infty,x)} r^0 + \int_x^{x_0} e^{-2i\omega y} \frac{\alpha}{2} P_{(-\infty,x-y)} r^2 dy \right) \end{pmatrix} \tag{94}$$

We recall the notation from (47) that $\mathcal{C}([x_0,x_1]; H^2(\mathbb{C}^+) \cap B_m^\infty)$ denotes continuous maps from $[x_0,x_1]$ into $H^2 \cap B_m^\infty$. We have

Proposition 2.1 *Suppose that $r^0 \in H^2 \cap B_1^\infty$ and that $|x|^{1/2} \le 1/(144\|r^0\|_{H^2})$; then Φ is a contraction which maps the ball in $L^2(x_0,x_1) \oplus \mathcal{C}([x_0,x_1]; H^2(\mathbb{C}^+) \cap B_2^\infty)$, $B = \{(\alpha, r) \mid \|\alpha\|_{L^2} + \sup_{x_0<y<x_1} \|r(y,\cdot)\|_{L^2} \le 6\|r^0\|_{L^2}\}$, to itself.*

Proof. Without loss of generality we may assume that $x_1 = 0$. We begin by noting that the second component of $\Phi(\alpha, r; r_0)$ does indeed belong to H^2

for every x. To see this, recall that the right hand side of (94) is the same as the right hand side of (90), whence the property is obvious.

Once we know the image is in H^2, we need only prove estimates for ω real. We take $\omega \in \mathbb{R}$ for the remainder of the proof. We first estimate

$$\begin{pmatrix} \beta \\ \rho \end{pmatrix} = \Phi(\alpha_1, r_1; r^0) - \Phi(\alpha_2, r_2; r^0)$$

$$\widehat{\beta} = \int_x^0 e^{-2i\omega y} \left\{ \alpha_1 P_{(x-y,0)}(r_1^2 - r_2^2) + (\alpha_1 - \alpha_2) P_{(x-y,0)} r_2^2 \right\}$$

As $P_{(-\infty,x-y)}$ is an orthogonal projection on $L^2(d\omega)$, we have

$$\begin{aligned}
\|\beta\|_{L^2(x,0)} &= \|\widehat{\beta}\|_{L^2(d\omega)} \\
&\le \|\alpha_1\|_{L^2(x,0)} |x|^{1/2} \sup_{x<y<0} \|r_1^2 - r_2^2\|_{L^2} \\
&\quad + \|\alpha_1 - \alpha_2\|_{L^2(x,0)} |x|^{1/2} \sup_{x<y<0} \|r_2^2\|_{L^2(d\omega)} \\
&\le |x|^{1/2} \Big(\|\alpha_1\|_{L^2(x,0)} \cdot 4 \cdot \sup_{x<y<0} \|r_1 - r_2\|_{L^2(d\omega)} \\
&\quad + \|\alpha_1 - \alpha_2\|_{L^2(x,0)} \cdot 2 \cdot \sup_{x<y<0} \|r_2\|_{L^2(d\omega)} \Big) \\
&\le |x|^{1/2} \Big(6D_0 \cdot 4 \cdot \sup_{x<y<0} \|r_1 - r_2\|_{L^2(d\omega)} \\
&\quad + 12 D_0 \|\alpha_1 - \alpha_2\|_{L^2(x,0)} \Big) \\
&\le \tfrac{1}{6} \big(\sup_{x<y<0} \|r_1 - r_2\|_{L^2(d\omega)} + \|\alpha_1 - \alpha_2\|_{L^2(x,0)} \big)
\end{aligned} \tag{95}$$

where $D_0 = \|r^0\|_{L^2} = \|r^0\|_{H^2}$.

Similarly,

$$\rho = e^{2i\omega x} \left(\int_x^0 e^{-2i\omega y} \left\{ \alpha_1 P_{(-\infty,x-y)}(r_1^2 - r_2^2) + (\alpha_1 - \alpha_2) P_{(-\infty,x-y)} r_2^2 \right\} \right) \tag{96}$$

The $L^2(d\omega)$ estimate of ρ follows exactly as in (95); namely

$$\sup_{0<y<x} \|\rho\|_{L^2(d\omega)} \le \frac{1}{6} \sup_{0<y<x} (\|r_1 - r_2\|_{L^2(d\omega)} + \|\alpha_1 - \alpha_2\|_{L^2(x,0)}) \tag{97}$$

which establishes that Φ is a contraction.

In addition, for each fixed ω,

$$\begin{aligned}\sup_{0<y<x}|\rho(y,\omega)| \;\;\le\;\; |x|^{1/2}\Big(&\|\alpha_1\|_{L^2}\sup_{0<y<x}|P_{(-\infty,x-y)}(r_1^2-r_2^2)| \\ &+\|\alpha_1-\alpha_2\|_{L^2}\sup_{0<y<x}|P_{(\infty,x-y)}r_2^2|\Big)\end{aligned} \tag{98}$$

In order to estimate the supremum ofρ, we note that, for $f\in L^2(d\omega)$,

$$P_{(-\infty,x-y)}f=(I-P_{(x-y,0)})f$$

so that

$$\begin{aligned}\|P_{(-\infty,x-y)}f\|_{L^\infty} &\le \|f\|_{L^\infty}+\|\tfrac{1}{\pi}\textstyle\int_{x-y}^0 e^{2i\omega s}f(s)ds\|_{L^\infty} \\ &\le \|f\|_{L^\infty}+|x-y|^{1/2}\|f\|_{L^2(d\omega)}\end{aligned}$$

where we have used both the Cauchy-Schwartz inequality and the Plancherel equality in the last step. Returning to (98), we have

$$\begin{aligned}\sup_{y,\omega}|\;\;&\rho(y,\omega)|\le|x|^{1/2}\Big(\|\alpha_1\|_{L^2(x,0)}\times \\ &\Big[\sup_{y,\omega}|r_1^2-r_2^2|+|x|^{1/2}\sup_{0<y<x}\|r_1^2-r_2^2\|_{L^2(d\omega)})\Big] \\ &+\|\alpha_1-\alpha_2\|_{L^2(x,0)}\Big[\sup_{y,\omega}|r_2^2|+|x|^{1/2}\sup_{0<y<x}\|r_2^2\|_{L^2(d\omega)})\Big]\Big)\end{aligned} \tag{99}$$

In addition, we note that

$$\begin{pmatrix}\alpha^0\\ \rho^0\end{pmatrix}=\Phi(0,0,r^0)=\begin{pmatrix}2(P_{(x,0)}r^0)^\vee\\ e^{2i\omega x}P_{(-\infty,x)}r^0\end{pmatrix} \tag{100}$$

and therefore

$$\begin{aligned}\|\alpha^0\|_{L^2}+\sup_{0<y<x}\|\rho^0\|_{L^2(d\omega)} &\le 3\|r^0\|_{L^2(d\omega)} \\ \sup_{y,\omega}|\rho^0(y,\omega)| &\le \|r^0\|_{L^\infty}+|x|^{1/2}\|r^0\|_{L^2(d\omega)}\end{aligned} \tag{101}$$

Denote $\Phi(\alpha,r;r^0)$ by $\binom{\widetilde{\alpha}}{\widetilde{r}}$, then

$$\begin{pmatrix}\widetilde{\alpha}\\ \widetilde{r}\end{pmatrix}=\Phi(0,0;r^0)+\Phi(\alpha,r;r^0)-\Phi(0,0;r^0) \tag{102}$$

so that(101), (95), and (97) yield

$$\|\tilde{\alpha}\|_{L^2(x,0)} + \sup_{x<y<0} \|\tilde{r}\|_{L^2(d\omega)} \leq 4D_0 \tag{103}$$

while it follows from (101) and (99) applied to (102) that

$$\sup_{y,\omega} |\tilde{r}| \leq 2 \tag{104}$$

This completes the proof of the proposition. ■

We are now prepared to prove Theorem 0.3.

Proof of Theorem 0.3. We first apply the previous proposition to prove existence of a solution to (86) with $r^0 = R$ on some interval $(0, x_1)$. An application of Corollary 1.6 with $b = 0$ implies that

$$E(r^0) = E(r(x_1, \cdot)) + \|\alpha\|_{L^2(x_1,0)}$$

so that both terms on the right are bounded independently of x_1 and $r(x_1, \omega) \in H^2 \cap B_1^\infty$. Hence we may repeat the first step indefinitely to prove global existence.

Uniqueness follows from the fact that, any solution to (93) must, for $x - x_0$ small enough, belong to the ball on which Φ, defined in (94), is a contraction.

To establish (34), we shall show that (r_b, α) satisfy (62) and obtain (34) as a consequence of (33).

We know that, for $N = 1, 2, 3, \ldots$, and $b > 0$

$$r_b(x, \omega) = r_b(-N, \omega)e^{2i\omega_b(x+N)} + \int_{-N}^{x} e^{2i\omega_b(x-y)} \frac{\alpha}{2}(1 - r_b^2)dy$$

Now, since $\|r_b(x, \cdot)\|_{L^2(d\omega)} \leq E(r_b(x, \cdot) \leq E(r^0)$ and $|r_b(x, \omega)|$ is bounded above by one, we may fix x and let $N \to \infty$. We obtain

$$r_b(x, \omega) = 0 + \int_{-\infty}^{x} e^{2i\omega_b(x-y)} \frac{\alpha}{2}(1 - r_b^2)dy$$

which shows that r is indeed the unique solution to the forward scattering problem, and hence satisfies (33).

The continuity of the map $r^0 \mapsto \alpha$ for x^1 finite follows from the continity of $\Phi(\alpha, r; r^0)$ with respect to its three arguments — the continuity with respect to r^0 follows from noting that Φ is affine in r^0 and using (101). In fact, for

x^1 finite, it even follows that Φ, and hence its unique fixed point, is Lipschitz continuous with respect to r^0.

Since $\|\alpha\|_{L^2(-\infty,x_1)} \to 0$ as $x_1 \to -\infty$, we may use the same $\varepsilon-\delta$ argument as we did at the end of Section 1 to establish the continuity of the inverse mapping. Note that, as in the case of the forward mapping, there is no uniform estimate if we only assume that $R(\omega)$ has finite energy.

■

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