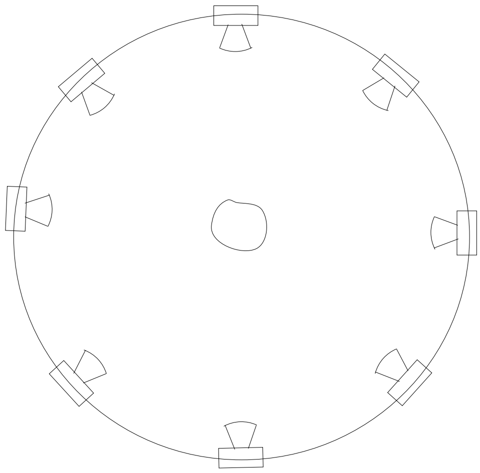
Remote Sensing Experiment



Sensors in the Far Field
many diameters/wavelengths
away
asymptotic formulas hold

Passive

The source radiates

$$u_{tt} - \Delta u = F(x,t)$$

$$u(x,0) = 0$$
$$u_t(x,0) = 0$$

Active

Incoming wave illuminates, producing an induced source

$$u_{tt} - \Delta u = (1 - n^2(x))u_{tt}$$

$$u(x,0) = g(\theta)\delta(t - |x| - r_0)$$
$$u_t(x,0) = g(\theta)\delta'(t - |x| - r_0)$$

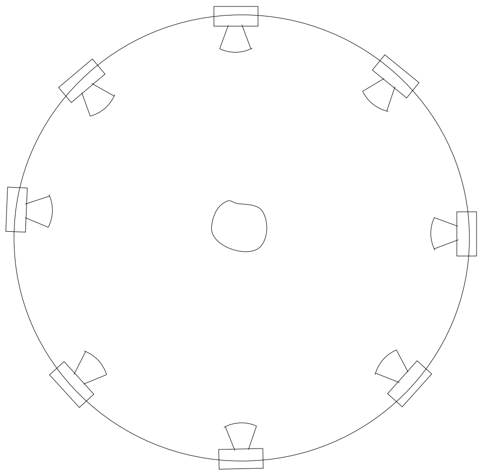
Forward Scattering — compute the far field

Inverse Scattering — infer properties of the source

- Where is the source ?
- What is the source ?

Inverse Source Problem

$$u_{tt} - \Delta u = F(x, t)$$



We measure only the field u radiated by the source. This is why we start with zero intial conditions.

$$u(x, 0) = 0$$
$$u_t(x, 0) = 0$$

Time Harmonic Model

$$
\begin{aligned}
F(x,t) &= \mathrm{e}^{i\omega t} F(x) \\
u(x,t) &= \mathrm{e}^{i\omega t} u(x)
\end{aligned}
$$

Wave equation becomes Helmholtz equation ($k = n_0 \omega$)

$$\left(\Delta + k^2\right) u = F(x,k)$$

What replaces the initial conditons?
We are already in the steady state.

Analyticity (Hardy Space Properties) in k in the lower half plane characterizes the solution u which is zero in the past.

Paley-Wiener Theorem – time

$$\begin{aligned} u(x,k) &= \int_{-\infty}^{\infty} \mathrm{e}^{ikt} u(x,t)dt \\ &= \int_{0}^{\infty} \mathrm{e}^{-ikt} u(x,t)dt \\ u_\eta(x,k) &= \int_{0}^{\infty} \mathrm{e}^{-i(k+i\eta)t} u(x,t)dt \\ ||u_\eta(x,k)||_{L^2(dk)} &= ||\mathrm{e}^{\eta t} u(x,t)||_{L^2(dt)} \end{aligned}$$

- u is holomorphic as a function of $k + i\eta \in C^-$
- $u_\eta \to 0$ as $\eta \to -\infty$.
- $u_\eta \to u$ as $\eta \to 0$.

Conversely, by deforming the contour of integration

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \mathrm{e}^{ikt} u(x,k)dk \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{i(k+i\eta)t} u(x,k+i\eta)dk \\ &\leq \mathrm{e}^{-\eta t}||u|| \\ &\to 0 \quad \text{for } t < 0 \text{ and } \eta \to \infty \end{aligned}$$

Limiting Absorption Principle

$$\begin{aligned} \left(\Delta + k^2\right) u &= F(x) \\ \left(k^2 - |\xi|^2\right) \widehat{u} &= \widehat{F}(\xi) \\ \widehat{u}(\xi, k) &= \frac{\widehat{F}(\xi)}{k^2 - |\xi|^2} \end{aligned}$$

The quotient is ambiguous because the singularity is not integrable. However, because u is zero in the past,

$$\begin{aligned} \widehat{u}(\xi, k) &= \lim_{\varepsilon \downarrow 0} \frac{\widehat{F}(\xi)}{(k - i\varepsilon)^2 - |\xi|^2} \\ \widehat{u}(\xi, k) &:= \frac{\widehat{F}(\xi)}{(k - i0)^2 - |\xi|^2} \end{aligned}$$

which is equivalent to (and more frequently written as)

$$= \frac{\widehat{F}(\xi)}{k^2 - |\xi|^2 - i0}$$

Three Scattering Subspaces

Outgoing Functions

$$\left(\frac{\widehat{F}(\xi)}{k^2 - |\xi|^2 - i0} \right)^\vee$$

Incoming Functions

$$\left(\frac{\widehat{F}(\xi)}{k^2 - |\xi|^2 + i0} \right)^\vee$$

Free (Herglotz Wave) Functions

$$\left(\frac{\widehat{F}(\xi)}{k^2 - |\xi|^2 - i0} - \frac{\widehat{F}(\xi)}{k^2 - |\xi|^2 + i0} \right)^\vee = \left(2i\widehat{F}(\xi) dS_{|\xi|2=k^2} \right)^\vee$$

Three Special Functions

Outgoing Hankel Functions

$$\left(\frac{\left(\frac{|\xi|}{ik}\right)^n \mathrm{e}^{in\theta}}{k^2 - |\xi|^2 - i0} \right)^\vee = H_n^-(kr)\mathrm{e}^{in\phi} \sim \frac{\mathrm{e}^{-ikr}}{r}\,\mathrm{e}^{in\phi}$$

Incoming Hankel Functions

$$\left(\frac{\left(\frac{|\xi|}{ik}\right)^n \mathrm{e}^{in\theta}}{k^2 - |\xi|^2 + i0} \right)^\vee = H_n^+(kr)\mathrm{e}^{in\phi} \sim \frac{\mathrm{e}^{ikr}}{r}\,\mathrm{e}^{in\phi}$$

Free Bessel Functions

$$\begin{aligned} \left((-i)^n \mathrm{e}^{in\theta} dS_{|\xi|^2=k^2}\right)^\vee &= \mathrm{e}^{in\phi} J_n(kr) \\ &= \frac{\mathrm{e}^{in\phi}}{2}\left(H_n^+(kr) + H_n^-(kr)\right) \end{aligned}$$

The Paley-Wiener Theorem — space

$$\widehat{u}(\xi + i\eta, k) = \int_{B_R} \mathrm{e}^{-i(\xi+i\eta)\cdot x} u(x,k)dx \leq \int_{B_R} \mathrm{e}^{|\eta||x|}|u|dx$$

$\widehat{u}(\xi)$ extends to be holomorphic for all $\xi + i\eta \in C^2$ and

$$|\widehat{u}| \leq C\mathrm{e}^{R|\eta|}$$

$$\Longleftrightarrow$$

$$\mathrm{supp}\,(u) \subset B_R$$

$$\widehat{u} = \frac{\widehat{F}(\xi)}{k^2 - |\xi|^2 - i0}$$

If $\widehat{F}(k\Theta)$ vanishes, then the quotient is holomorphic and has the same exponential growth properties as $\widehat{F}$.

If $\widehat{F}(k\Theta)$ vanishes,

$$\mathrm{supp}\,(u) \subset \mathrm{supp}\,(F)$$

and F has zero far field.

Hankel Function Expansion

$$\widehat{u} = \frac{\widehat{F}(\xi)}{k^2 - |\xi|^2 - i0}$$

$$\widehat{F}(k,\theta) = \sum f_n \mathrm{e}^{in\theta}$$

$$\widehat{F}_H(\xi) = \sum f_n \mathrm{e}^{in\theta} \left(\frac{|\xi|}{k}\right)^n$$

$$\widehat{F}_H - \widehat{F} \text{ vanishes on } |\xi|^2 = k^2$$

so $u - u_H$ vanishes outside a ball containing $\operatorname{supp}(F)$.

Outside that ball,

$$\begin{aligned} u &= u_H \\ &= \sum f_n \left(\frac{\left(\frac{|\xi|}{k}\right)^n \mathrm{e}^{in\theta}}{k^2 - |\xi|^2 - i0}\right)^\vee \\ &= \sum f_n \mathrm{e}^{in\theta} H_n^-(kr) \\ &\sim \left(\sum f_n \mathrm{e}^{in\theta}\right) \frac{\mathrm{e}^{-ikr}}{ikr} \\ &\sim \widehat{F}(k,\theta) \frac{\mathrm{e}^{-ikr}}{ikr} \end{aligned}$$

Descriptions of the Far Field

Source

$$\widehat{F}(k,\theta) = \sum f_n \mathrm{e}^{in\theta}$$

The very Far Field

$$u^\infty = \frac{\mathrm{e}^{-ikr}}{2ikr}\widehat{F}(k,\theta) + ...$$

The not so Far Field – outside a ball containing supp F

$$u = \sum f_n \mathrm{e}^{in\theta} H_n^+(kr)$$

Near Field — for any Ω containing the support F

$$\begin{aligned} f_n &= \int_{\partial\Omega} \{u, \mathrm{e}^{in\phi} J_n(kr)\} \\ &= \int_{\Omega} F J_n(kr) \mathrm{e}^{in\phi} dV \end{aligned}$$

$$\{u,v\} := \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) dS$$

The Wronskian

$$\{u, v\} := \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) dS \qquad [v\nabla u - u\nabla v]$$

$$\int_{\partial\Omega} \{u, v\} = \int_{\Omega} \left(v\left(\Delta + k^2\right) u - u\left(\Delta + k^2\right) v\right) dV$$

For an outgoing u and a free solution v, i.e.

$$\left(\Delta + k^2\right) u = F(x)$$

$$\left(\Delta + k^2\right) v_f = 0$$

$$\int_{\partial\Omega} \{u, v_f\} = \int_{\Omega} F v_f dV$$

If we choose v_f to be a $\mathrm{e}^{ik\Theta\cdot x}$,

$$\int_{\partial\Omega} \left\{u, \mathrm{e}^{ik\Theta\cdot x}\right\} = \widehat{F}(k\Theta)$$

Wronskian continued

If we choose v_f to be a $J_n(r)\mathrm{e}^{in\phi}$,

$$\begin{aligned}
\int_{\partial\Omega} \left\{u, J_n(kr)\mathrm{e}^{in\phi}\right\} &= \int_{\Omega} F J_n(kr)\mathrm{e}^{in\phi} dV \\
\lim_{R\to\infty} \int_{\partial B_R} \left\{u, J_n(kr)\mathrm{e}^{in\phi}\right\} &= \\
\sum f_m \int_0^{2\pi} \left\{H_m(kr)\mathrm{e}^{im\phi}, J_n(kr)\mathrm{e}^{in\phi}\right\} d\phi &= \\
f_n \left\{H_n(kr), J_n(kr)\right\} &= \\
2ikf_n &=
\end{aligned}$$

$$\begin{aligned}
f_n &= \frac{1}{2ik} \int_{\Omega} F J_n(kr)\mathrm{e}^{in\phi} dV \\
|f_n| &\leq \frac{1}{2k} ||F||_{L^2} ||J_n(kr)||_{L^2(\mathrm{supp}\,(F))}
\end{aligned}$$

The Far Field and the Support of F

If supp $(F) \subset B_R$,

$$|f_n| \leq \frac{1}{2k}||F||_{L^2}||J_n(kr)||_{L^2(B_R)}$$

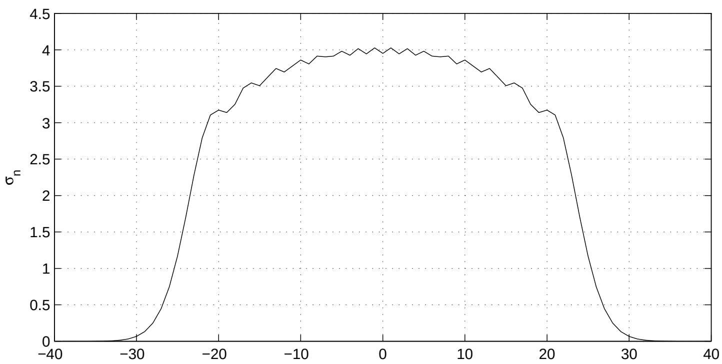
Translation Formula

$$F_c = F(x - c)$$
$$\widehat{F}_c(k\Theta) = \mathrm{e}^{ik\Theta\cdot c}\widehat{F}(k\Theta)$$
$$f_n^c = \sum J_{n-m}(k|c|)\mathrm{e}^{-in\theta_c}$$

If supp $(F) \subset B_R(c)$,

$$|f_n^c| \leq \frac{1}{2k}||F||_{L^2}||J_n(kr)||_{L^2(B_R)}$$

$||J_n(r)||_{L^2(B_{25})}$



The Circular Paley-Wiener Theorem

$$\widehat{F}(k\Theta) = \Sigma f_n \mathrm{e}^{in\theta}$$

There exists F supported in B_R

$$\Longleftrightarrow$$

$$\left\{ \frac{f_n}{\left(\int_0^R J_n^2(ks)ds \right)^{\frac{1}{2}}} \right\} \quad \epsilon \quad l^2$$

Translation Formula

$$F_c = F(x - c)$$

$$\widehat{F}_c(k\Theta) = \mathrm{e}^{ik\Theta \cdot c} \widehat{F}(k\Theta)$$

$$f_n^c = \sum J_{n-m}(k|c|)\mathrm{e}^{-in\theta_c}$$

There exists F supported in $\bigcap B_{R_c}(c)$

$$\Longleftrightarrow$$

$$\left\{ \frac{f_n^c}{\left(\int_0^R J_n^2(ks)ds \right)^{\frac{1}{2}}} \right\} \quad \epsilon \quad l^2 \quad \forall \quad c$$

$$
\begin{aligned}
f_n &= \int F(x)\mathrm{e}^{ix\cdot\xi}\mathrm{e}^{-in\theta}dxd\theta \\
&= \int F(r\Phi)\mathrm{e}^{irk\cos(\theta-\phi)}\mathrm{e}^{-in\theta}rdrd\phi d\theta \\
&= \int_0^{2\pi} d\phi \int_0^R rdr\mathrm{e}^{in\phi}f(r\Phi)\int_0^{2\pi}\mathrm{e}^{irk\cos(\theta)}\mathrm{e}^{-in\theta}d\theta
\end{aligned}
$$

We make θ complex

$$
\begin{aligned}
&= \dots \int_0^{2\pi}\mathrm{e}^{irk\cos(\theta+i\psi)}\mathrm{e}^{-in(\theta+i\psi)} \\
&\leq \mathrm{e}^{Rk\sin(\theta)\sinh(\psi)}\mathrm{e}^{-n\psi} \sim \mathrm{e}^{\frac{Rk}{2}\mathrm{e}^{\psi}-n\psi}
\end{aligned}
$$

Now choose $\psi = \log\frac{2n}{RK}$ to optimize

$$
\leq \left(\frac{eRk}{2n}\right)^n \frac{1}{\sqrt{n}} \sim J_n(kR)
$$

Conversely, we may extend

$$
\widehat{F}(\theta) = \Sigma f_n \mathrm{e}^{in\theta}
$$

from the circle to the entire plane by

$$
\widehat{F}_E(\rho,\theta) = \Sigma f_n \mathrm{e}^{in\theta}\frac{J_n(\rho R)}{J_n(kR)}
$$

and check that the sum converges and satisfies the appropriate Paley-Wiener estimates.

The Simplest Example

$$\begin{aligned} F &= \delta_0(x) \\ u &= H_0(kr) \\ \widehat{F}(k,\theta) &= 1 \\ f_0 &= 1 \end{aligned}$$

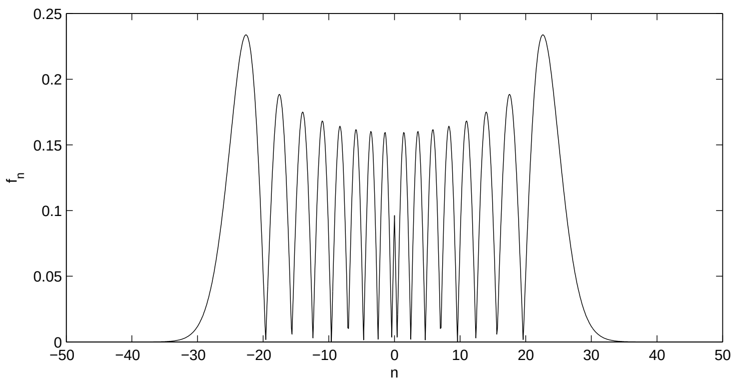
$$\begin{aligned} F &= \delta_c(x) \\ u &= \sum J_n(k|c|)H_n(kr)\mathrm{e}^{in(\theta-\theta_c)} \\ \widehat{F}(k,\theta) &= \mathrm{e}^{ik|c|\cos(\theta-\theta_c)} \\ f_n &= J_n(k|c|)\mathrm{e}^{in\theta_c} \end{aligned}$$

Translation Formula

$$F_c = F(x-c)$$

$$\widehat{F}_c(k\Theta) = \mathrm{e}^{ik\Theta\cdot c}\widehat{F}(k\Theta)$$

$$f_n^c = \sum J_{n-m}(k|c|)\mathrm{e}^{-in\theta_c}$$



$|J_n(?)|$

$$\left(r\frac{d}{dr}\right)^2 J_n - \left(r^2 - n^2\right) J_n = 0$$

Some Asymptotic Formulas

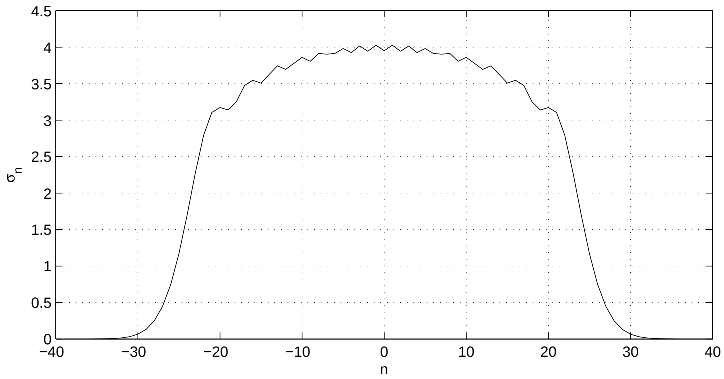
For $n > R$

$$J_n(R) \; \sim \; \frac{1}{\sqrt{n}} \left(\frac{\mathrm{e}R}{2n} \right)^n$$

$$||J_n||_{L^2(B_R)} \; \sim \; J_{n+\frac{1}{2}}(R) \sim 0$$

For $n < R$

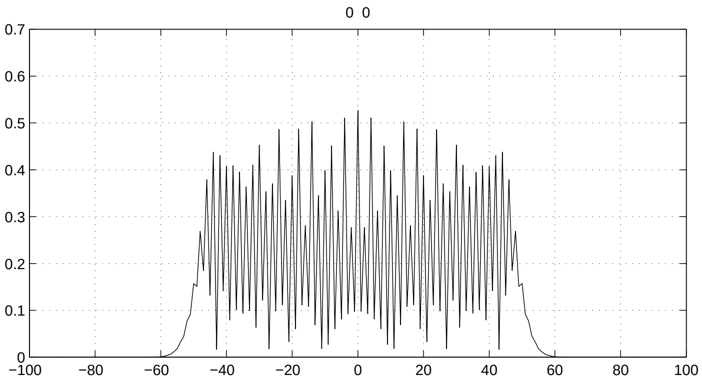
$$||J_n|| \sim (R^2 - n^2)^{\frac{1}{4}}$$



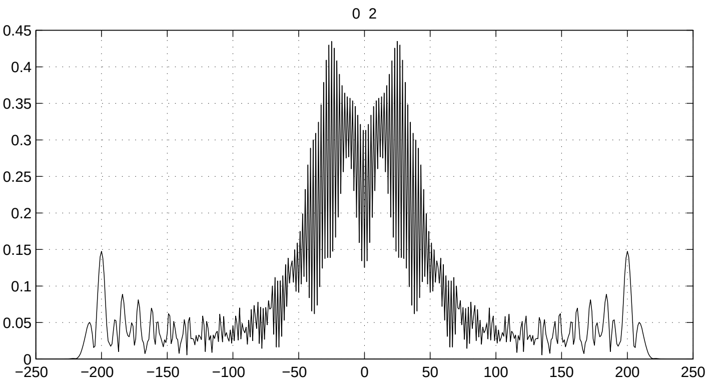
$$J_n(R) \sim \frac{\cos\left(\sqrt{R^2 - n^2} - n\cos^{-1}\frac{n}{r} - \frac{\pi}{4}\right)}{(R^2 - n^2)^{\frac{1}{4}}}$$

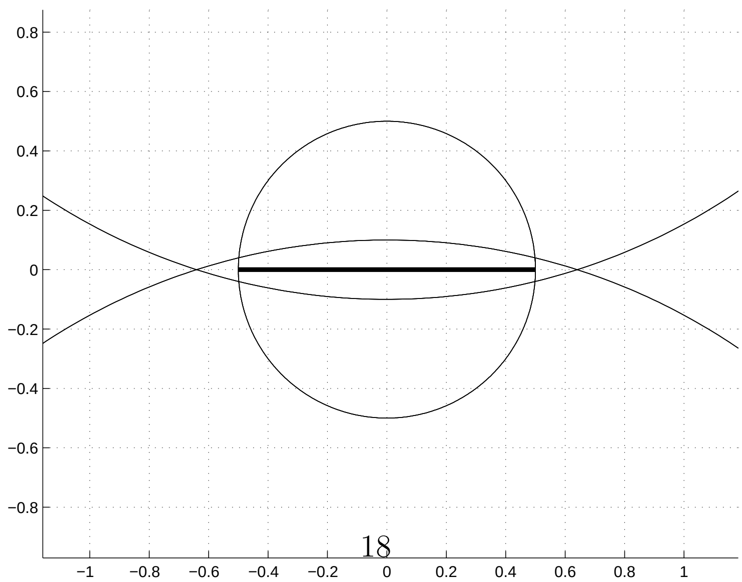
A Line Source (k = 100)

f_n



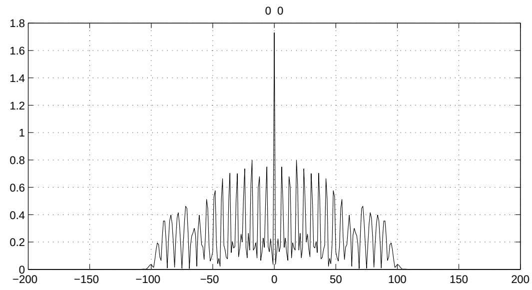
f_n^c



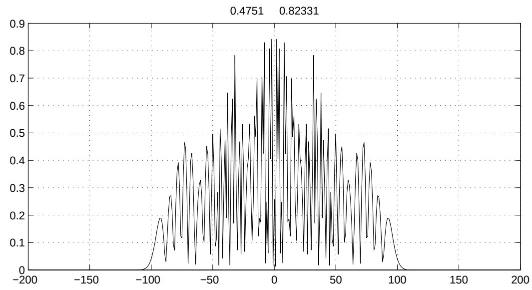


A Triangular Source (k = 100)

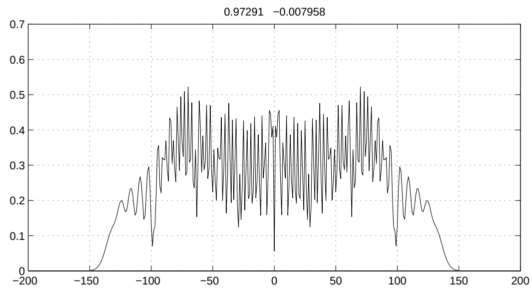
f_n

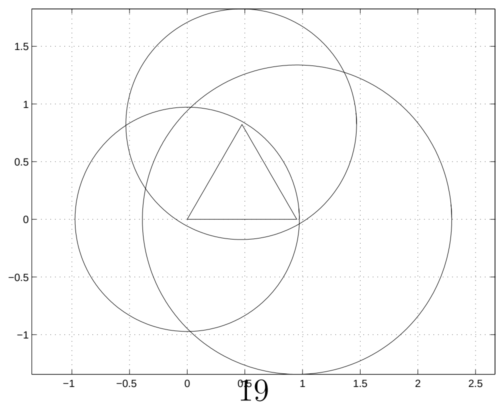


f_n^c



f_n^c





Theory and Practice

Theory:

- Only the large n asymptotics of the Fourier coefficients matter.

Practice:

- Only the the Fourier coefficients with $n < kR$ are appreciably nonzero. We can use the "transition to evanescence" to find the "scattering support".

Theory or Practice ???

- The Far field $\alpha(\theta) = \widehat{F}(k, \theta)$ is an analytic function of θ. Therefore, the restriction of α to any open interval (i.e limited aperture) completely determines α.

Problem with Circular Paley Wiener Theorem

"There exists an F" isn't good enough. The source that the theorem promises might have nothing to do with the source that I am looking for.

Two More Examples

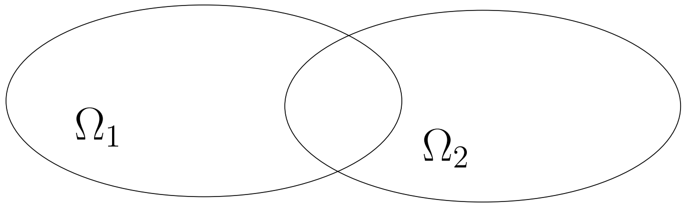
supp (Φ) compact

$$
\begin{aligned}
F &= \left(\Delta + k^2\right)\Phi \\
u &= \Phi \\
f_n &= 0
\end{aligned}
$$

$$
\begin{aligned}
F &= \Phi(r) \\
u &= C H_0(kr) \\
f_0 &= C \\
f_n &= 0
\end{aligned}
$$

Resolution

Suppose $\text{supp}\ F_1 \subset \Omega_1$ $\text{supp}\ F_2 \subset \Omega_2$ *and that* F_1 *and* F_2 *produce the same far field, and that* $\mathbb{R}^n \setminus (\Omega_1 \bigcup \Omega_2)$ *is connected and contains a neighborhood of* ∞. *then there exists an* F_3, $\operatorname{supp} F_3 \subset N_\varepsilon(\Omega_1 \cap \Omega_2)$, *which also produces that far field.*

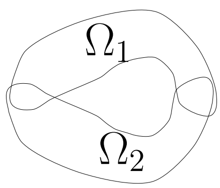


$$v = \begin{cases} \phi u_1, & x \in \mathbb{R}^n \backslash \Omega_1 \\ \phi u_2, & x \in \mathbb{R}^n \backslash \Omega_2 \\ 0, & x \in \Omega_1 \cap \Omega_2 \end{cases}$$

is a well-defined C^∞ function and $v = u_1 = u_2$ outside a compact set so that

$$f_3 = (\Delta + k^2)v$$

must also have far field α.



The Convex Scattering Support

If we take the Ω's to be convex, then the small print is always satisfied. Moreover, since the intersections of convex sets are convex, we can intersect infinitely many sources and assert that there is genuinely a “smallest convex set” which supports a far field.

$$\mathrm{cS}_k\mathrm{supp}\alpha = \bigcap_{\widehat{F}(k\Theta)=\alpha(\theta)} \mathrm{ch}(\mathrm{supp}\, F).$$

- Any F with far field α must contain $\mathrm{cS}_k\mathrm{supp}\ \alpha$ in its convex support.
- There is a source,F, supported on any neighborhood of $\mathrm{cS}_k\mathrm{supp}\alpha$, that produces the far field α. In particular, $\mathrm{cS}_k\mathrm{supp}\alpha$ can't be empty unless α is zero.

There can't be a “biggest convex set” because of the previous examples. i.e. we can always add $\left(\Delta + k^2\right)\Phi$ to any F to increase the support without changing the far field.

Special Cases

The more about the source, we can say more about the relation between the scattering support and the true support. We are only scratching the surface in understanding the relationship between the two.

- If F has a convex corner, that corner must belong to the scattering support. In particular, convex polygons have true support and convex scattering support equal.
- The scattering support of the characteristic function of an ellipse is the line between its foci (almost).
- All these criterion apply directly to the (active imaging) inverse scattering problem with a single incident wave.

  However, if we know the source must be an induced source, this is not a sharp criterion. In some cases we can show that a far field could have been produced by a source with small support, but an induced source must have had larger support.

Laplace's Equation

Outgoing solutions	Free solutions
$\Delta u = F$	$\Delta v_f = 0$
$u = \sum b_n \frac{e^{in\theta}}{r^{\|n\|}}$	$v_f = \sum a_n r^{\|n\|} e^{in\theta}$

$$-|n| a_n b_n = \int_{\partial B_R} \{u, v_f\} = \int_{B_R} F v_f$$

$$\begin{aligned} |b_n| &= \frac{1}{2|n|} \int F r^{|n|} e^{in\theta} \\ &\leq \frac{1}{2|n|} \, ||F||_{L^2} \left(\int_{\text{supp } F} r^{2|n|+1} dr \right)^{\frac{1}{2}} \\ &\leq \frac{1}{4|n||n+1|} \, ||F||_{L^2} \; R^{|n|+1} \end{aligned}$$

---

b_n's decay (or grow) like $R^{|n|+1}$ iff there exists a source supported inB_R.

Multi–polish translation formula

$$b_m^c = \sum \binom{m+k}{k} |c|^k b_{m+k} e^{ik\theta_c}$$

Maxwell's Equations

Outgoing Solutions	Free Solutions
$$\begin{aligned} dE - ik * H &= 0 \\ dH + ik * E &= j \end{aligned}$$	$$\begin{aligned} dE_f - ik * H_f &= 0 \\ dH_f + ik * E_f &= 0 \end{aligned}$$

$$\left\{ \begin{pmatrix} E \\ H \end{pmatrix}, \begin{pmatrix} E_f \\ H_f \end{pmatrix} \right\} = E \wedge H_f - E_f \wedge H$$

$$d \left\{ \begin{pmatrix} E \\ H \end{pmatrix}, \begin{pmatrix} E_f \\ H_f \end{pmatrix} \right\} = E_f \wedge j$$

$$\int_{\partial B_R} \left\{ \begin{pmatrix} E \\ H \end{pmatrix}, \begin{pmatrix} E_f \\ H_f \end{pmatrix} \right\} = \int_{B_R} E_f \wedge j$$

$$|a_{nm}| \ , \ |b_{nm}| \ \leq \ ||j||_{L^2} ||J_n(kr)||_{L^2(\text{supp}\,(j)}$$

$$\begin{aligned} E_f &= *dY_{nm} r J_n(kr) \\ E_f &= (dY_{nm}(rJ_n(kr))' + rJ_n(kr)n(n+1)Y_{nm}dr) \end{aligned}$$

$$\begin{aligned} E = & \sum a_{nm} * dY_{nm} r H_n^+(kr) + \\ & \sum b_{nm} \left(dY_{nm}(rH_n^+(kr))' + rH_n^+(kr)n(n+1)Y_{nm}dr \right) \end{aligned}$$

Inhomogeneous Media

The notion of scattering support can be extended to inhomogeneous media (i.e. non-constant coefficient equations).

$$\begin{aligned} Lu &= F \\ Lv_f &= 0 \end{aligned}$$

The Far Field Operator maps sources in Ω to far fields :

$$F \overset{G_\Omega}{\mapsto} u^\infty$$

is compact and has a singular value decompositon.

$$G_\Omega = \sum \sigma_n \Phi_n \otimes \Psi_n$$

G_Ω has a big kernel;

$$\ker(G_\Omega) = \left\{ F \middle| F = L\Phi \quad \operatorname{supp} \Phi \subset \Omega \right\}$$

The orthognal complement to that kernel is spanned by the restrictions to Ω of the free solutions.

$$\ker(G_\Omega)^\perp = \left\{ \chi_\Omega v_f \middle| Lv_f = 0 \right\}$$

A far field belongs to the range of G_Ω iff

$$\left\{ \frac{(u^\infty, \Phi_n)}{\sigma_n} \right\} \in l^2$$

In the Helmholtz case, with Ω equal to B_R,

$$\begin{aligned}
\Phi_n &= \mathrm{e}^{in\theta} \\
\Psi_n &= J_n(kr)\mathrm{e}^{in\phi} \\
\sigma_n^2 &= \int_0^R J_n^2(kr) r dr
\end{aligned}$$

and this is exactly the condition of the circular PW theorem.

Notice that the convexity of Ω isn't necessary to characterize the range of G_Ω. It is , however, necessary to conclude that:

$$Range(G_{\Omega_1}) \cap Range(G_{\Omega_2}) \subset Range(N_\varepsilon\left(G_{\Omega_1 \cap \Omega_2}\right))$$

which allows us to conclude the existence of a smallest convex set which supports the far field.

??? Do the $||\Psi_n||_{L^2(\Omega}$'s begin to decay at a specific threshold like the $||J_n(kr)||_{L^2(B_R)}$'s ???

Inverse scattering

$$\left(\Delta + k^2\right) u = k^2 q(x) u$$
$$u = u_f + u_o$$

$$\begin{aligned}\int_{B_R} q u v_f &= \int_{\partial B_R} \{u, v_f\} = \int_{\partial B_R} \{u_o, v_f\} \\ &= \lim_{R\to\infty} \int_{\partial B_R} \{u_o, v_f\} = A(\Theta_1, \Theta_2)\end{aligned}$$

If we choose

$$\begin{aligned} u_f &= \mathrm{e}^{ik\Theta_1 \cdot x} \\ v_f &= \mathrm{e}^{ik\Theta_2 \cdot x}\end{aligned}$$

If instead we choose

$$\begin{aligned} u &= J_n(kr)\mathrm{e}^{in\phi} + \ldots \\ v_f &= J_m(kr)\mathrm{e}^{im\phi}\end{aligned}$$

It appears that the Fourier coeficients of the scattering amplitude satisfy

$$A_{nm} \leq K||J_m(kr)||_{L^2(\mathrm{supp}\,(q))}||J_n(kr)||_{L^2(\mathrm{supp}\,(q))}$$

which gives a stronger criterion than simply applying the previous results.

Credits

Joint work with Steve Kusiak

Joint and Related work with Roland Potthast

Motivated by the Linear Sampling method of Colton Kirsch and the the Factorization Method of Kirsch