Inverse Problems **22** (2006) 1273–1288 doi:10.1088/0266-5611/22/4/010

Notions of support for far fields

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Received 14 March 2006, in final form 23 May 2006
Published 20 June 2006
Online at stacks.iop.org/IP/22/1273

Abstract
In practical remote sensing, faraway sources radiate fields that, within measurement precision, are nearly those radiated by point sources. Algorithms like MUSIC (Devaney *J. Acoust. Soc. Am.* at press, Kirsch 2002 *Inverse Problems* **18** 1025–40) correctly identify their number, their locations and their strengths based on observations of the near or far fields they radiate. Asymptotic perturbation formulae (Ammari *et al* 2005 *SIAM J. Appl. Math.* **65** 2107–27, Brühl *et al* 2003 *Numer. Math.* **93** 635–54) have been used to successfully locate small sparse inclusions based on remote measurements. The main motivation for this paper is to *locate* sources which are supported on sets that are larger and less sparse. Although the far field of a solution to the inhomogeneous Helmholtz equation does not determine the source, or its support, uniquely, we will show how to associate with any far field a unique *union of well-separated-convex sets* (UWSC sets) that is both big enough to support a source that can radiate that far field, and small enough that it must be contained in the UWSC-support of any source that radiates the same far field. This means that it makes theoretical sense to look for not only the number and the locations, but also the convex geometry of sources based on the far field they radiate. The only requirement is that sources be well separated—the diameter of each convex component is strictly smaller than the distance to the other components. We also give examples to illustrate the extent to which both the convexity and well-separated properties in UWSC are necessary, i.e. we will exhibit far fields with which it is not possible to associate a unique smallest *compact set* or, in $\mathbb{R}^2$, a unique smallest *disjoint union of convex sets*.

1. Introduction

The Helmholtz equation is a model for an electro-magnetic or acoustical field radiated by a time harmonic source:

$$(\Delta + k^2)u = f, \qquad x \in \mathbb{R}^n, \quad n \geqslant 2. \tag{1}$$

The *far field* radiated by the source f means the large x asymptotics of the unique *outgoing* solution to (1). As long as the source decays fast enough (e.g. $f \in L^2_\delta, \delta > \frac{1}{2}$),[1] there is a radiated field, u, an $L^2_{-\delta}$ solution to (1). Every $L^2_{-\delta}$ solution has asymptotics of the form

$$u \sim \frac{\mathrm{e}^{\mathrm{i}kr}}{r^{\frac{n-1}{2}}}\alpha(\Theta) + \frac{\mathrm{e}^{-\mathrm{i}kr}}{r^{\frac{n-1}{2}}}\beta(\Theta) \tag{2}$$

where $r = |x|$ and $\Theta = \frac{x}{|x|} \in S^{n-1}$. These asymptotics are called the far field of u. Individually, α and β are called the outgoing and incoming far fields[2]. The unique solution with $\beta = 0$ is the outgoing field radiated by f. A short calculation using the Fourier transform and the limiting absorption principle [15] shows that the unique outgoing solution to (1), which we will subsequently refer to as *the field radiated by* f, is given by

$$\widehat{u}_+ = \lim_{\epsilon \downarrow 0} \frac{\widehat{f}}{k^2 - \xi \cdot \xi + \mathrm{i}\epsilon} \tag{3}$$

and that the outgoing far field is exactly the Fourier transform of f, restricted to the sphere of radius k. That is,

$$\alpha(\Theta) = \widehat{f}(k\Theta). \tag{4}$$

In the inverse source problem, we attempt to deduce the location or properties of the source based on observations of the outgoing far field radiated by the source. A basic feature of the inverse source problem is non-uniqueness. We pointed out in (4) that the outgoing far field is the restricted Fourier transform. Thus the far field map

$$f \overset{\mathcal{F}}{\mapsto} \widehat{f}|_{|\xi|=k} \tag{5}$$

has a large kernel. In fact,

$$\ker(\mathcal{F}) = \overline{\left\{g | g = (\Delta + k^2)\phi, \phi \in C_0^\infty(\mathbb{R}^n)\right\}} \tag{6}$$

where the closure may be in the $L^2_{-\delta}$ topology or the topology of tempered distributions.

It is a consequence of Rellich's lemma and unique continuation [4] that if the far field vanishes, then the field radiated by the source is identically zero outside the support of the source f. For this reason, such an f is called a *non-radiating source*. These sources are invisible at wavenumber k.

Work on the inverse source problem must account for this non-uniqueness. One alternative is to restrict attention to a special class of sources for which uniqueness holds. In [8], it is shown that sources which belong to each of the following classes can be uniquely identified (among other sources in that same class) on the basis of observations of the outgoing far field that they radiate.

- f is the indicator (Heavyside) function of a bounded convex set.
- f is the indicator function of a bounded set which is star shaped about a fixed point p.
- f is a finite sum of point sources, i.e. $f = \sum_{i=1}^N a_i \delta_{p_i}$.

Once we make such a restriction, it is then (at least theoretically) possible to compute the source from its radiated far field. Note that, although the inverse source problem is linear, two of the three classes listed above are not linear. One reason for this is that the properties of physical interest often do not depend linearly on the source. The property that we will focus on this paper is location, the support of the source. If the source belongs to one of the classes

[1] We will give the definition in (7).

[2] This is because the first term in (2) is approximately the Fourier transform in time of $\delta(|x| - t)$ and the second is approximately the Fourier transform of $\delta(|x| + t)$.

above, it makes sense to compute the support of the source from the far field. The rub is that the special form of the source can be too restrictive for many applications.

Our approach will be to make no assumptions about the source, f. However, we will compute only a lower bound on the support of f. Because of the large kernel of the far field map (6), f and $f + (\Delta + k^2)\phi$ radiate the same far field for any $\phi \in C_0^\infty$. Since we can choose the support of ϕ arbitrarily, no upper bound for the support of the source is possible.

Roughly speaking, we will define the support of a far field to be the *smallest set* that supports a source which radiates that far field. The support of a far field must be simultaneously big enough to radiate the field (in particular, the support of a nonzero far field should not be empty) and small enough that it must be contained in the support of any source which radiates that far field. It tells us something that all sources that radiate that field must have in common.

We will develop a systematic method to construct equivalent sources (sources which radiate the same far field) in section 3. As a consequence of the examples we construct there, we will see that, in general, there can be no smallest compact set that supports (a source that radiates) a far field. We will need to restrict the class of sets to what we call *far-field support classes*.

In [12], we defined the convex scattering support, the smallest convex set that supports a source that radiates a far field. In subsequent work [13, 14], we elaborated on its properties, showed how to compute it, and applied the notion to the inverse scattering problem in addition to the inverse source problem. This allowed us to associate a convex set with a far field. This convex set must be a subset of the convex hull of the support of any source that radiates that far field. Thus, in spite of the non-uniqueness in the problem, we could make a practically useful statement about location. However, many important aspects of the support of the source cannot be deduced from that single convex set. In particular, we cannot distinguish a far field radiated by many separate sources scattered throughout a convex set from one large distributed source that covers the entire convex set.

In this paper, we show that is is possible to define a more precise notion of far field support that allows us to distinguish sources with several components. We will show how to associate with any far field a smallest *union of well-separated-convex sets* (UWSC sets) that supports a source that can radiate it. Well-separated means that the diameter of each convex component is strictly smaller than the distance to the other components.

In section 3, we show how to systematically construct examples to show why these restricted classes of sets are necessary, i.e. we will exhibit far fields with which it is not possible to associate a smallest connected set, or a *disjoint union of convex sets*.

2. Far field support

In this section and the next, we will use the letter P to represent the Helmholtz operator, $(\Delta + k^2)$.

2.1. Sources, fields and far fields

In this section we list the technical details that one needs to carefully define the far field. We will work in the Hilbert spaces that are the completions of $C_0^\infty(\mathbb{R}^n)$ in the norms

$$\|f\|_{L^2_\delta} = \|(1+|x|^2)^{\frac{\delta}{2}} f\|_{L^2}, \qquad \|f\|_{H^s_\delta} = \|(1+|x|^2)^{\frac{\delta}{2}}(-\Delta+1)^{\frac{s}{2}} f\|_{L^2}. \quad (7)$$

We will make use of both the unique continuation principle and Rellich's lemma, here, and in the rest of the paper [4, 9, 10].

Theorem 1 (unique continuation principle). *If $Pu = 0$ on a connected set Ω and $u \equiv 0$ on an open subset of Ω, then $u \equiv 0$ on all of Ω.*

Theorem 2 (Rellich's lemma). *Let B_R denote the ball of radius R. Any $L^2_{-\delta}$ solution to $Pu = 0$ in $\mathbb{R}^n \backslash B_R$ with vanishing far field is identically zero in $\mathbb{R}^n \backslash B_R$.*

We will only consider sources that are compactly supported, and our results would be unchanged if we insisted that all of our sources be smooth. However, in order to motivate our definitions, it is useful to consider sources which are compactly supported distributions. In section 3, we will make extensive use of sources that radiate single and double layer potentials. It is not hard to check that the definition in (3) defines a unique outgoing field for any compactly supported tempered distribution. Every such distribution is in $H^s_\delta(\mathbb{R}^n)$ for some $s \in \mathbb{R}$ and any $\delta \in \mathbb{R}$. We need to choose $\delta > \frac{1}{2}$ in the following theorem, which is due to [1]. The statement below can be found in [13].

Theorem 3. *If $f \in H^s_\delta(\mathbb{R}^n)$ then formula (3) gives the unique outgoing solution to $Pu = f$ which is in $H^{s+2}_{-\delta}(\mathbb{R}^n)$. Furthermore, modulo functions in $H^{s+2}_{-\delta+1}(\mathbb{R}^n)$, which decay more rapidly at infinity,*

$$u \sim \frac{\mathrm{e}^{\mathrm{i}kr}}{r^{\frac{n-1}{2}}}\alpha(\Theta)$$

and $\alpha = \widehat{f}(k\Theta)$, the Fourier transform of the source, restricted to the sphere of radius k.

We will continue to call f the source, and u the field radiated by f. We refer to α both as the far field of u and as the far field radiated by f.

2.2. Sets

Although many sources, with many different supports, can radiate the same far field, we intend to associate a unique *smallest* set with each far field. We will insist that this set *carry* the far field, in the sense defined below.

Definition 4. *A compact set M carries a far field, if every open neighbourhood of M supports a source that radiates that far field.*

To see why we make this definition, let H^+_n denote the outgoing Hankel function of order n, δ the dirac delta supported at the origin, and $\overline{\partial}$ the Cauchy–Riemann operator. Define u_n on $\mathbb{R}^2$, by

$$u_n := H^+_n(r)\,\mathrm{e}^{\mathrm{i}n\theta}$$

and note that

$$u_n \sim \frac{\mathrm{e}^{\mathrm{i}kr}}{r^{\frac{1}{2}}}\,\mathrm{e}^{\mathrm{i}n\theta}$$

and solve

$$(\Delta + k^2)u_n = (\overline{\partial})^n\delta.$$

Thus each of the far fields

$$\alpha_n = \mathrm{e}^{\mathrm{i}n\theta}$$

is radiated by sources supported at the origin. The outgoing far field

$$\alpha = \sum \frac{\mathrm{e}^{\mathrm{i}n\theta}}{(n!)^2}$$

cannot be radiated by a source that is a distribution supported at the origin, because any distribution supported at the origin must necessarily be a finite sum of derivatives of the δ, which would imply that α had a finite Fourier series. However, because the series

$$u = \sum H_n^+(r) \frac{\mathrm{e}^{\mathrm{i}n\theta}}{(n!)^2}$$

converges for all nonzero r, we may choose any C_0^∞ function ϕ which does not contain the origin in its support and see that

$$f := P(\phi u)$$

radiates α. Thus the origin is α's smallest carrier, although the origin does not support a source that radiates α.

Although a carrier of a far field may not support a distribution that radiates that field, it is a consequence of the unique continuation principle that the field u is well defined outside any carrier M whose complement has no bounded connected component. The complement of a compact set is open, and therefore a union of open components. One of these components is unbounded, and the rest are bounded. The far field uniquely continues to the unbounded component, but not to the bounded components. We will often use the condition that $\mathbb{R}^n \backslash M$ has no bounded components, or equivalently that $\mathbb{R}^n \backslash M$ is connected. A simpler statement might be to say that *M has no holes*.

Lemma 5. *Let M be compact and $\mathbb{R}^n \backslash M$ have no bounded connected component. Then M carries a far field $\alpha \iff$ there exists a unique smooth field u satisfying*

$$Pu = 0 \quad in\ \mathbb{R}^n \backslash M$$

which has far field α.

Proof. Suppose first that there exists such a u. For any $\epsilon > 0$, let $N_\epsilon(M)$ denote the ϵ-neighbourhood of M and ϕ_ϵ a C^∞ function satisfying

$$\phi_\epsilon = \begin{cases} 1 & \text{on } \mathbb{R}^n \backslash N_\epsilon(M) \\ 0 & M \end{cases}$$

then $f_\epsilon := P(\phi_\epsilon u)$ is a source supported in $N_\epsilon(M)$ that radiates α, proving that M carries α.

If M carries α, for every $\epsilon > 0$ there exists f_ϵ supported in $N_\epsilon(M)$ radiating α. Let u_ϵ denote the fields

$$Pu_\epsilon = f_\epsilon$$

and $(\mathbb{R}^n \backslash N_\epsilon(M))^\infty$ the unbounded component of $\mathbb{R}^n \backslash N_\epsilon(M)$. As ϵ decreases, $(\mathbb{R}^n \backslash N_\epsilon(M))^\infty$ increases. We claim that it increases to $(\mathbb{R}^n \backslash M)^\infty$, which we have assumed to be equal to $(\mathbb{R}^n \backslash M)$.

To show this, let $x \in (\mathbb{R}^n \backslash M)^\infty$, and let x_* denote a fixed point in some $(\mathbb{R}^n \backslash N_{\epsilon *}(M))^\infty$. Now x can be connected to x_* by a path that avoids M. Because M is compact, this path avoids $N_\delta(M)$ for some $\delta > 0$, so $x \in (\mathbb{R}^n \backslash N_\delta(M))^\infty$.

We can now define u by the formula

$$u(x) = u_\epsilon(x) \qquad \text{for} \quad x \in (\mathbb{R}^n \backslash N_\epsilon(M))^\infty.$$

The definition of u makes sense because the unique continuation principle guarantees that all the u_ϵ agree on their common domains of definition. Because each $Pu_\epsilon = 0$ on $\mathbb{R}^n \backslash N_\epsilon(M)$, it follows that

$$Pu = 0 \quad \text{on } \mathbb{R}^n \backslash M$$

and the proof is complete. □

We will construct our smallest carriers by intersecting other carriers. Lemma 6 will play a crucial role.

Lemma 6. *Let M_1 and M_2 be compact and carry the far field α. Suppose that neither $\mathbb{R}^n \backslash M_1$, $\mathbb{R}^n \backslash M_2$, nor $\mathbb{R}^n \backslash (M_1 \cup M_2)$ have bounded components. Then $M_1 \cap M_2$ carries α.*

Proof. Let u_1 and u_2 denote the unique fields with far field α described in lemma 5, which are defined on $\mathbb{R}^n \backslash M_1$ and $\mathbb{R}^n \backslash M_2$ respectively. For any $\epsilon > 0$, let $\phi_\epsilon \in C^\infty$ satisfy

$$\phi_\epsilon = \begin{cases} 1 & \text{on } \mathbb{R}^n \backslash N_\epsilon(M_1 \cap M_2) \\ 0 & M_1 \cap M_2. \end{cases}$$

The unique continuation principle allows us to define

$$v_\epsilon = \begin{cases} \phi_\epsilon u_1 & \text{on } \mathbb{R}^n \backslash M_1 \\ \phi_\epsilon u_2 & \text{on } \mathbb{R}^n \backslash M_2 \\ 0 & M_1 \cap M_2. \end{cases}$$

Because ϕ_ϵ is identically one outside some compact set, v_ϵ has far field α and therefore $f_\epsilon = P v_\epsilon$ radiates α. □

Motivated by the lemma, we make the definition below:

Definition 7. *A far-field support class is a collection, $\mathcal{M}$, of compact sets that satisfy*

1. *$\mathcal{M}$ is closed under intersection.*
2. *If $M \in \mathcal{M}$, $\mathbb{R}^n \backslash M$ has no bounded component.*
3. *If M_1 and $M_2 \in \mathcal{M}$, then $\mathbb{R}^n \backslash (M_1 \cup M_2)$ has no bounded component.*

Property 2 is actually a special case of property 3 with M_2 equal to the empty set, but it is convenient to state it and to verify it separately.

Definition 8. *The $\mathcal{M}$-support of a far field α is*

$$M_\alpha = \bigcap_{\substack{M \in \mathcal{M} \\ M \text{ carries } \alpha.}} M$$

Theorem 9. *M_α is the smallest set in $\mathcal{M}$ that carries α.*

Proof. M_α's definition guarantees that it is a subset of any $M \in \mathcal{M}$ that carries α. We only need to show that M_α itself carries α. It suffices to show that $N_\epsilon(M)_\alpha$ carries α for every $\epsilon > 0$, and this will follow from repeated application of lemma 6 as soon as we show that

$$N_\epsilon(M_\alpha) \supset \bigcap_{j=0}^{N} M_j \tag{8}$$

for some finite N. To see this, fix one (compact) $M_0 \in \mathcal{M}$ that carries α and note that

$$\begin{aligned} M_0 \backslash M_\alpha &= M_0 \backslash \Big(\bigcap_{\substack{M \in \mathcal{M} \\ M \text{ carries } \alpha}} M \Big) \\ &= \bigcup_{\substack{M \in \mathcal{M} \\ M \text{ carries } \alpha}} (M_0 \backslash M) \end{aligned}$$

so that

$$M_0 \backslash N_\epsilon(M_\alpha) \subset \bigcup_{\substack{M \in \mathcal{M} \\ M \text{ carries } \alpha.}} (M_0 \backslash M) \tag{9}$$

The set on the left-hand side of (9) is compact and covered by the open sets on the right-hand side, so a finite sub-cover $\{M_j\}_{j=1}^N$ suffices, i.e.

$$M_0 \backslash N_\epsilon(M_\alpha) \subset \bigcup_{j=1}^{N} (M_0 \backslash M_j)$$

and therefore

$$N_\epsilon(M_\alpha) \cap M_0 \supset \left(\bigcap_{j=1}^{N} M_j \right) \cap M_0$$

$$N_\epsilon(M_\alpha) \supset \bigcap_{j=0}^{N} M_j$$

which establishes (8) and completes the proof. □

2.3. Far field support classes

We recall two definitions and make a new one.

Definition 10. *A set S is p-star shaped if the line segment connecting p and any $s \in S$ belongs to S.*

Definition 11. *A set K is convex if it is p-star shaped for every $p \in K$.*

Definition 12. *A set W is a union of well-separated convex sets (UWSCS) if*

1. *W is the disjoint union of closed convex sets.*
2. *The distance between any connected component A of W and $W \backslash A$ is strictly greater than the diameter of A.*

Lemma 13. *A compact UWSCS has only finitely many connected components.*

Proof. Suppose W contains infinitely many connected (closed convex) components, choose $x_i \in B_i$, they converge to some x_* because W is compact. The distance between the connected component of W containing x_* and the rest of W is zero, so that component cannot be well separated. □

Theorem 14. *The following classes of subsets of $\mathbb{R}^n$ are far field support classes. That is, they satisfy the requirements in definition 7.*

1. *compact convex sets;*
2. *compact p-star shaped sets;*
3. *compact unions of well-separated convex sets.*

Proof. We shall prove each of the required properties 1 through 3 for each of the three classes.

(a) *$\mathcal{M}$ is closed under intersection.* That the intersection of compact sets is compact, the intersection of convex sets is convex, and the intersection of p-star shaped sets is p-star shaped is immediate. That the intersection of UWSC sets is UWSC requires a short proof. Let W_α be UWSC sets and

$$W = \cap W_\alpha.$$

Let $p \in W$ and let B^p denote the maximal connected component of W containing p and B_α^p the corresponding components of the W_α. Now B_p must be a subset of B_α^p for every α—otherwise it would be disconnected. Each B_α^p is compact and convex, so the intersection of the B_α^p is compact and convex, hence connected. Thus we conclude that

$$B_p = \bigcap_\alpha B_p^\alpha$$

and hence is both compact and convex. Now intersection decreases diameters, i.e.

$$\mathrm{diam}(B_p) \leqslant \mathrm{diam}\big(B_p^\alpha\big)$$

and increases the distances between components. To see the latter, let $p \in B_p$ and $q \in B_q$. If B_p and B_q are different components, then B_p^α and B_q^α are different for at least one α, so

$$\begin{aligned}\mathrm{dist}(p,q) &> \min\big(\mathrm{diam}\big(B_p^\alpha\big), \mathrm{diam}\big(B_q^\alpha\big)\big)\\ &\geqslant \min(\mathrm{diam}(B_p), \mathrm{diam}(B_q))\end{aligned}$$

which shows that B_p and B_q are well separated, completing the proof that W is UWSCS.

(b) $\mathbb{R}^n\backslash M$ *is connected*. The complement of a convex or a p-star shaped set is homeomorphic to the complement of a point, which is certainly connected. The homeomorphism is just a stretching of the radial coordinate of spherical coordinates based at p. Similarly, the complement of a disjoint union of finitely many compact convex sets is homeomorphic to the complement of finitely many points, so it is also connected.

(c) $\mathbb{R}^n\backslash M_1 \cup M_2$ *is connected*. The union of p-star shaped sets is again p-star shaped, and the union of two convex sets is either disjoint or star shaped about any point in the intersection, so property (c) for the convex and p-star shaped classes follows from what we have already proved in (b).

For the class of compact UWSC sets, we first observe that the connected components of $M_1 \cup M_2$ are unions of the finitely many (compact convex) connected components of M_1 and M_2. That is, if C denotes one connected component of $M_1 \cup M_2$, then

$$C = \left(\bigcup_{j=1}^{N_1} M_1^j\right) \cup \left(\bigcup_{k=1}^{N_2} M_2^k\right). \tag{10}$$

From among these components of M_1 and M_2, choose one with largest diameter d, say M_1^1. Now the distance between M_1^1 and any other M_1^j is strictly greater than d, and the diameters of each of the M_2^k is less than or equal to d, so M_1^1 can be the only M_1 component contained in C. Thus (10) actually reads

$$C = M_1^1 \bigcup_{k=1}^{N_2} M_2^k.$$

Recall that the M_2^k are disjoint. Each of the intersections $M_1^1 \cap M_2^k$ is nonempty, convex and disjoint. We illustrate this in figure 1.

Because each M_2^k is star shaped about any point p_k in $M_1^1 \cap M_2^k$ it is possible to construct a homeomorphism from $\mathbb{R}^n\backslash C$ to $\mathbb{R}^n\big\backslash\big(C\backslash M_2^k\big)$ by just deforming the radial coordinate in spherical coordinates based at p_k. If we do this for each k we construct a homeomorphism to $\mathbb{R}^n\backslash M_1^1$. In addition, we can construct this homeomorphism so that it is the identity outside an arbitrary open neighbourhood of C. Therefore, we can find a homeomorphism, h,

$$\mathbb{R}^n\backslash(M_1 \cup M_2) \stackrel{h}{\longmapsto} \mathbb{R}^n\backslash \cup C_l$$

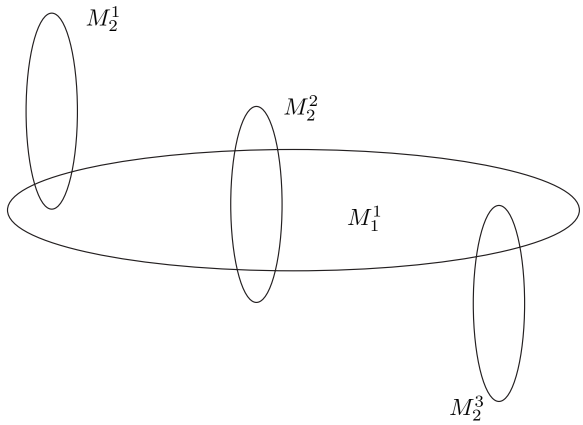

Figure 1. A component in the union of two UWSCS.

where the C_l are disjoint convex and compact (each C_l is either an M_1^k or an M_2^k). We showed in the proof of property (b) that $\mathbb{R}^n \setminus \cup C_l$ is connected, so we may conclude that $\mathbb{R}^n \setminus (M_1 \cup M_2)$ is connected as well.

This completes the proof of the theorem. □

3. Carriers, minimal carriers and supports

The more restrictive a class of sets we consider, the less we learn about the source. The class of UWSC sets is bigger than the class of convex sets, so we learn more about a source by finding the UWSC-support of its far field than by finding the convex-support of its far field. We would learn still more if we could work in the class of *closed bounded sets*, the class *closed bounded sets with connected complements*, or even the class of *compact unions of disjoint convex sets*.

In this section, we will illustrate that this is not possible by constructing far fields which are carried by two different sets in these classes but not by their intersection—for the last class we only prove this in dimension two. To accomplish this, we need a way to show that some far fields cannot be carried by sets which are too small. We introduce the notion of a minimal carrier[3] for this purpose.

Definition 15. *A set Γ is a minimal carrier for a far field α if it carries α and no proper subset of Γ carries α. We say that $\Gamma \in \mathcal{M}$ is an $\mathcal{M}$-minimal carrier if no proper subset of Γ in the class $\mathcal{M}$ also carries α.*

A restatement of definition 8 is

Definition 16. *The $\mathcal{M}$-support of a far field is the unique $\mathcal{M}$-minimal carrier of that far field.*

The following theorem shows that minimal carriers must be small. Roughly speaking, these are the sets that support the distributional sources which arise in the theory of single and double layer potentials.

[3] Both the notion of carrier and minimal carrier appear in the study of bounded linear functionals on spaces of holomorphic functions of several complex variables [7]. It turns out that it is impossible to define even the convex support of such a functional. Although we have not done so, it is possible to pursue that analogy here. One can identify every far field with a unique linear functional on the space of free solutions.

Theorem 17. *Suppose that a far field is carried by a compact set Ω, then it is also carried by $\partial\Omega$. In fact, it is carried by the subset of $\partial\Omega$ that is the boundary of the unbounded component of $\mathbb{R}^n\backslash\Omega$.*

Proof. Without loss of generality, we can assume that $\mathbb{R}^n\backslash\Omega$ is connected, since we can replace Ω with the complement of the unbounded component of its complement (i.e. fill in the bounded holes). Let u be the free field on $\mathbb{R}^n\backslash\Omega$ guaranteed by lemma 5, let α be its far field, and let $\epsilon > 0$. Because u is in $L^1_{\text{loc}}(\mathbb{R}^n\backslash\Omega)$, we may define the distribution ω_ϵ by

$$\langle\omega_\epsilon, \phi\rangle = \int_{\mathbb{R}^n\backslash N_\epsilon(\Omega)} (P\phi)u. \tag{11}$$

Because $Pu = 0$ on $\mathbb{R}^n\backslash N_\epsilon(\Omega)$ we see that ω_ϵ is supported in $\partial N_\epsilon(\Omega)$ and that the L^1_{loc} field

$$v_\epsilon := \begin{cases} 0 & x \in N_\epsilon(\Omega) \\ u & x \in \mathbb{R}^n\backslash N_\epsilon(\Omega) \end{cases}$$

satisfies

$$Pv_\epsilon = \omega_\epsilon$$

and has far field α. Thus we have produced a source supported in $\partial N_\epsilon(\Omega)$, which is a subset of $N_{2\epsilon}(\partial\Omega)$, proving that $\partial\Omega$ carries α. □

If $\partial\Omega$ and the field u are smooth, then we may integrate by parts in (11) to see that

$$\langle\omega_\epsilon, \phi\rangle = \int_{\partial N_\epsilon(\Omega)} \left(\phi\frac{\partial u}{\partial\nu} - u\frac{\partial\phi}{\partial\nu}\right) \mathrm{d}S$$

and let $\epsilon \to 0$ to obtain the source of the classical single and double layer potentials on $\partial\Omega$ which radiates our far field.

$$\langle\omega, \phi\rangle = \int_{\partial\Omega} \left(\phi\frac{\partial u}{\partial\nu} - u\frac{\partial\phi}{\partial\nu}\right) \mathrm{d}S. \tag{12}$$

Single and double layer potentials are fields radiated by distributional sources. We can construct these sources from the Cauchy data of any function restricted to a surface, as long as both the function and the surface are smooth enough. If S_1 is a C^1 surface with boundary and $u \in H^{\frac{3}{2}+}(\mathbb{R}^n)$ (i.e. $H^s(\mathbb{R}^n)$ for some $s > \frac{3}{2}$), then $\mathcal{C}_{S_1}u$ is an $H^{\frac{-3}{2}-}(\mathbb{R}^n)$ distribution defined by

$$\langle\mathcal{C}_{S_1}u, \phi\rangle = \int_{S_1} \left(\phi\frac{\partial u}{\partial\nu} - u\frac{\partial\phi}{\partial\nu}\right) \mathrm{d}S. \tag{13}$$

When S is the boundary of a bounded open set, and u is a free field in that set, we need not require as much smoothness. In the previous proof, even if Ω were not smooth, we could have replaced $N_\epsilon(\Omega)$ with a sequence (not a continuously parameterized family) of sets B_ϵ with smooth boundary that decrease to Ω. In this case, the distributions ω_ϵ could have been represented as sources of single and double layer potentials supported on ∂B_ϵ

$$\langle\omega_\epsilon, \phi\rangle = \int_{\partial B_\epsilon} \left(\phi\frac{\partial u}{\partial\nu} - u\frac{\partial\phi}{\partial\nu}\right) \mathrm{d}S.$$

In general, we cannot send ϵ to zero to exhibit a source supported on $\partial\Omega$ that radiates this far field. However, if the free field u is in $L^1(\mathbb{R}^n\backslash\Omega)$, then we can let $\epsilon \to 0$ in (11) and produce a distribution ω supported on $\partial\Omega$ that radiates the far field. Only if both $\partial\Omega$ and u are smoother, can ω be represented as the source of a classical multiple layer potential as in (12). With this in mind, we make the following definition.

Definition 18. *Suppose that Ω is a bounded open set and that $u \in L^1(\Omega)$ is free (i.e. $Pu = 0$ in Ω), then we define $\mathcal{C}_{\partial\Omega}u$, the Cauchy data of u on $\partial\Omega$, to be the distribution*

$$\langle \mathcal{C}_{\partial\Omega}u, \phi\rangle = \int_\Omega (P\phi)u.$$

The distribution $\mathcal{C}_{\partial\Omega}u$ clearly vanishes on any ϕ supported away from $\partial\Omega$. If $\partial\Omega$ and u are smooth, then $\mathcal{C}_{\partial\Omega}u$ is exactly the multiple layer source with densities equal to the restrictions of u and $\frac{\partial u}{\partial \nu}$ to $\partial\Omega$. Even in the absence of additional smoothness, if we choose $B_n \subset \Omega$ with smooth boundary increasing to Ω as $n \to \infty$,

$$\langle \mathcal{C}_{\partial\Omega}u, \phi\rangle = \lim_{n\to\infty} \int_{\partial B_n} \left(\phi \frac{\partial u}{\partial \nu} - u \frac{\partial \phi}{\partial \nu}\right) \mathrm{d}S.$$

This natural extension of the definition of Cauchy data allows us to make a precise statement about non-radiating sources.

Theorem 19. *Suppose that $\omega \in H^{-2+}(\mathbb{R}^n)$, and that* $\operatorname{supp}\omega$ *is compact and has measure zero, then*

$$\omega \text{ is a non-radiating source}$$
$$\Longleftrightarrow$$
$$\operatorname{supp}(\omega) \text{ is the boundary of a bounded open set } B$$
$$\text{and}$$
$$\omega = \mathcal{C}_{\partial B}u \text{ with } u \text{ free } (Pu = 0) \text{ in } B.$$

Proof. Because the support of ω is a compact set,

$$\mathbb{R}^n \backslash \operatorname{supp}\omega = B \cup U$$

with B representing the union of the open-bounded components of $\mathbb{R}^n\backslash\operatorname{supp}\omega$ and U representing the unbounded component. Because $\omega \in H^{-2+}(\mathbb{R}^n)$, $u \in H^{0+}(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}$, so that

$$\langle \omega, \phi\rangle = \int_{\mathbb{R}^n} (P\phi)u = \int_B (P\phi)u + \int_U (P\phi)u.$$

If ω is non-radiating, u is identically zero in U, so the second term is zero, and the first term is exactly $\mathcal{C}_{\partial B}u$.

Conversely, if $\omega = \mathcal{C}_{\partial B}u$ with $Pu = 0$ in B, then

$$w = \begin{cases} u & x \in B \\ 0 & x \in \mathbb{R}^n\backslash B \end{cases}$$

satisfies

$$Pw = \omega$$

and has zero far field, proving that ω is non-radiating. □

The previous theorem told us that the compact measure zero supports of non-radiating sources were boundaries of bounded open sets. The next theorem tells us that the compact measure zero supports that do not contain such a boundary are minimal carriers.

Theorem 20. *Suppose that $\omega \in H^{-2+}(\mathbb{R}^n)$, $\operatorname{supp}\omega$ is compact and has measure zero, and $\mathbb{R}^n\backslash\operatorname{supp}\omega$ has no bounded component. Then $\operatorname{supp}\omega$ is a minimal carrier for the far field radiated by ω and ω is the only distribution supported on this set that radiates that far field.*

Proof. Suppose that a compact subset of $\operatorname{supp}\omega$, denoted by S_1, carries the same far field. $\mathbb{R}^n\backslash S_1$ is connected, so, according to lemma 5, there is a field u_1 which is free ($Pu_1 = 0$) on $\mathbb{R}^n\backslash S_1$. Unique continuation guarantees that this field agrees with the field $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ radiated by ω on $\mathbb{R}^n\backslash\operatorname{supp}\omega$. Thus

$$\langle\omega, \phi\rangle = \int_{\mathbb{R}^n}(P\phi)u = \int_{\mathbb{R}^n}(P\phi)u_1 \tag{14}$$

$$= \int_{\mathbb{R}^n\backslash S_1}(P\phi)u_1 \tag{15}$$

which shows that $\operatorname{supp}\omega$ is contained in S_1. But we began with the hypothesis that $S_1 \subset \operatorname{supp}\omega$, so the two sets must be equal. Finally, note that (15) gives a formula for ω that depends only on the unique continuation of the far field, proving that ω is unique. □

Corollary 21. *Let v be free in $\mathbb{R}^n$ and let S_1 and S_2 be two smooth surfaces with boundary that satisfy*

1. *$\mathbb{R}^n\backslash S_1$ and $\mathbb{R}^n\backslash S_2$ are connected;*
2. *$S_1 \cup S_2$ is the boundary of a bounded open set;*

then $\mathcal{C}_{S_1}v$ and $-\mathcal{C}_{S_2}v$ radiate the same far field and both S_1 and S_2 are minimal carriers for that far field.

Proof. Because v, S_1 and S_2 are smooth, formula (13) shows us that the distributions $\mathcal{C}_{S_1}v$ and $-\mathcal{C}_{S_2}v$ are in $H^{-3/2}(\mathbb{R}^n)$, so we may apply theorem 20 to conclude that each surface is a minimal carrier and theorem 19 to conclude that

$$\mathcal{C}_{S_1}v + \mathcal{C}_{S_2}v = \mathcal{C}_{S_1\cup S_2}v$$

is non-radiating, and therefore that $\mathcal{C}_{S_1}v$ and $-\mathcal{C}_{S_2}v$ radiate the same far field. □

Corollary 22. *The class of compact sets (with connected complements) is not a support class for far fields. In $\mathbb{R}^2$, compact unions of disjoint convex sets are not a support class.*

Proof. Recall that within a support class $\mathcal{M}$, there can be at most one minimal carrier (that carrier is the $\mathcal{M}$-support of the far field) for any far field. According to corollary 21, a support class cannot contain two distinct sets of measure zero whose union is a boundary. If it does, then the Cauchy data of any free solution (e.g. $v = \mathrm{e}^{\mathrm{i}k\theta\cdot x}$) on those sets provide distinct minimal carriers of the same field.

The class of compact sets (with connected complements) certainly contains two hemispheres. In $\mathbb{R}^2$, we may choose S_1 to be the two vertical sides of a rectangle and S_2 to be the two horizontal sides of the same rectangle. Both are compact disjoint unions of convex sets, and their union is the boundary of the interior of the rectangle. □

It is worth remarking that at most one of S_1 and S_2, in the previous proof, is a well-separated union of convex sets. If the rectangle is not a square, then the two shorter sides are well separated but the longer ones are not. If the rectangle is a square, then neither S_1 nor S_2 is well separated.

We emphasize that we have only shown that disjoint unions of convex sets cannot provide an adequate notion of far field support in $\mathbb{R}^2$. It appears that, in $\mathbb{R}^3$, it is not possible to find a boundary that is the union of two disjoint unions of convex sets.

4. Generalizations

The analysis and the conclusions in section 2 do not depend on the wave nature of the solutions to the Helmholtz equation, or the details of the behaviour or the far field as described in (2). We could replace the Helmholtz equation by any differential equation or system for which the unique continuation principle holds, and the far field by a set of data which uniquely determines a solution to the free equation ($P(x, D)u = 0$) on an open set. We will list a few examples below. In each case the first line is the partial differential equation and the second the measured data.

1. The gravitational far field inverse source problem:

$$\Delta u = f \quad \text{in } \mathbb{R}^3$$
$$u \sim \sum_{n=-\infty}^{\infty} a_n r^{|n|} Y_n(\Theta)$$

where the Y_n are spherical harmonics.

2. A near field inverse source problem in a bounded domain $\mathcal{D}$, with smooth boundary:

$$(\Delta + k^2)u = f \quad \text{in } \mathcal{D}$$
$$u|_{\mathcal{D}} = \alpha \qquad \left.\frac{\partial u}{\partial \nu}\right|_{\mathcal{D}} = \beta. \tag{16}$$

3. The inhomogeneous Cauchy–Riemann equations in $\mathbb{R}^2$:

$$\bar{\partial} u = f \qquad u \sim \sum_{n=1}^{\infty} \frac{a_n}{z^n}.$$

In each case, it is possible to associate a smallest convex set, a smallest UWSC set, and a smallest set that is star shaped about p with the measured data. In each case the data are somewhat different. The correct Hilbert spaces describing the sources f, and the fields u are different. These differences would undoubtedly be important in any reconstruction, but in the issue we treat here—the fact that certain notions of support of the measured data make sense and others do not—these differences are not relevant.

In the following paragraphs, we indicate how to modify the previous definitions, which were specific to the Helmholtz equation, in a way that the concepts in section 2 will apply to other equations defined on an open domain D. We have structured the proofs in section 2 in such a way, that, together with the definitions below, they would suffice to prove an abstract theorem analogous to theorem 14. Our goal, however, is not to prove a general theorem, but simply to convince the reader that it is possible to apply these concepts in other inverse problems, should the need arise.

We start with an open domain D on which our sources and fields will be defined (e.g. $\mathbb{R}^n$ if our data are asymptotic far fields or a bounded domain if our data are to be Cauchy data on ∂D). We let $P(x, D)$ denote a differential operator for which the unique continuation principle holds (e.g. a uniformly elliptic operator). Our sources f will be compactly supported distributions in D. We say a field u is radiated by a source f if

$$Pu = f \quad \text{in } D. \tag{17}$$

If D is $\mathbb{R}^n$, then we say that two solutions to (17) have the same far field if they agree outside some ball. Rellich's lemma tells us that this is the case for two solutions to the Helmholtz equation with the same asymptotics. Because $Pu = 0$ outside the compact support of f, two fields which have the same Cauchy data on the boundary of a bounded domain D will agree on some open neighbourhood of ∂D. To accommodate both situations, we define our far field as the germ of a field.

Definition 23. *We say that two fields (solutions to (17)) have the same far field if they are identical outside some compact subset of D.*

Definition 24. *Two sources radiate the same far field if there exist fields u_1 and u_2 satisfying*

$$Pu_1 = f_1 \qquad Pu_2 = f_2$$

which have the same far field.

Definition 24 should be an equivalence relation; two compactly supported distributions are equivalent if there is a *far field* that both radiate. In particular, f should be equivalent to itself. We therefore require the existence of a right fundamental solution in D, that is the existence of an operator E, such that, for every source f

$$PEf = f.$$

However, we do not require in (17), or in definition 24, that u be the unique solution to a particular boundary value problem, or that it satisfies a radiation condition. This definition has the advantage that it simultaneously applies to far field data and the near-field data described in item 2, where we do not have (or need) a natural notion of outgoing solution.

For the far-field inverse source problem for the Helmholtz equation, we did require that u be the unique outgoing solution. We did this because the outgoing field is the physically relevant solution, the one that is genuinely radiated by a time harmonic electro-magnetic or acoustic source. However, our mathematical analysis would have been identical if we had not. Every $L^2_{-\delta}$ solution to (1) has asymptotics (2), and can be uniquely represented as the sum of an outgoing wave and a free field (also called an incident wave). A free field solves

$$(\Delta + k^2)u = 0$$

and has asymptotics of the form

$$u \sim \frac{e^{ikr}}{r^{\frac{n-1}{2}}}\alpha(\Theta) + \frac{e^{-ikr}}{r^{\frac{n-1}{2}}}i\alpha(-\Theta).$$

Therefore, if two fields have the same far field, they have the same incident far field and the same outgoing far field. Thus, for the Helmholtz equation, two sources radiate the same outgoing far field if and only if every field radiated by one source is also radiated by the other.

Finally, we generalize our definition of unbounded and bounded open subsets of a domain D, just as we did far fields.

Definition 25. *An open set in D is unbounded if it intersects the complement of every compact subset of D.*

Definition 26. *Let $O \subset D$ be an open set and O^∞ denote the unbounded connected components of O. We say that O has no bounded connected component if $O = O^\infty$.*

If D is connected, then the complement of a compact set has only one unbounded component. If D is $\mathbb{R}^n$, then O^∞ is exactly what it was before. If D is a bounded domain, O^∞ is the union of open components which intersect every neighbourhood of ∂D.

5. Conclusions

We have defined the UWSC-support of a far field. It carries a far field and must be a subset of the UWSC-hull of the support of any source that radiates that field. This is a very general notion. In particular it is not wavelength dependent. We have not proposed any algorithm to compute the UWSC-support. The articles [13, 14] have illustrated that it is possible to compute the convex scattering support. Computations in [2, 3] show that very small and very well separated sources can be effectively computed from far field data. We consider the task of understanding the relationship between wavelength, size and separation that is necessary to produce a stable algorithm an important task, but one which we have not undertaken here.

We have discussed only the inverse source problem, not the inverse scattering problem, which is the one considered in [2, 3, 5, 11]. The inverse source problem is linear, which makes our analysis simpler and more complete. The Born, or linear, approximation to the inverse scattering problem is an inverse source problem, so our results apply directly to that. In [6, 12, 13], we applied the notion of convex scattering support to inverse scattering problems. We expect similar applications of the UWSC-support, and we intend to address this in future work. We have not touched on the issue of how to incorporate *a priori* information about the source or scatterer.

In $\mathbb{R}^2$, we have argued convincingly that we need both convexity and the well-separated property in order to unambiguously define the support of a far field. In three (or more) dimensions, we have made an equally strong case for convexity, but the necessity of the well-separated condition is not apparent.

Acknowledgments

The research was supported by NSF grant DMS-0355455 and ONR grant N00014-05-1-0716.

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