Discreteness of Transmission Eigenvalues via Upper Triangular Compact Operators*

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June 3, 2011

*This research was supported in part by NSF grant DMS-1007447

Abstract

Transmission eigenvalues are points in the spectrum of the interior transmission operator, a coupled 2x2 system of elliptic partial differential equations, where one unknown function must satisfy two boundary conditions and the other must satisfy none.

We show that the interior transmission eigenvalues are discrete and depend continuously on the contrast by proving that the interior transmission operator has *upper triangular compact resolvent*, and that the spectrum of these operators share many of the properties of operators with compact resolvent. In particular, the spectrum is discrete and the generalized eigenspaces are finite dimensional.

Our main hypothesis is a coercivity condition on the *contrast* that must hold only in a neighborhood of the boundary.

1 Introduction

The time-harmonic scattering of waves by a penetrable scatterer in a vacuum can be modeled with the Helmholtz equation. The *total wave* u satisfies the perturbed Helmholtz equation

$$\left(\Delta + k^2(1+m)\right) u = 0 \quad \text{in } \mathbb{R}^n \tag{1.1}$$

where the contrast, $m(x)$, denotes the deviation of the square of the index of refraction from the constant background; i.e. $n^2(x) = 1+m(x)$. The relative (far field) scattering operator, s^+, records the correspondence between the asymptotics of solutions to of the free Helmholtz equation to those of (1.1). If the operator s^+ has a nontrivial kernel (null space) at wavenumber k, then we say that k is a transmission eigenvalue [4] [6]. Certain inverse scattering methods are known to succeed only at wavenumbers that are not transmission eigenvalues [4]. If a scatterer $m(x)$ is supported in a bounded domain D, then, if k^2 is a transmission eigenvalue, k^2 must be an *interior transmission eigenvalue* as defined below. It is possible to make an extended definition of the scattering operator s^+ such that the two are equivalent [6]. A precise relationship between interior transmission eigenvalues and the scattering operator has yet to be discovered.

Definition 1. *A wavenumber k^2 is called an interior transmission eigenvalue of m in the domain D if there exists a non-trivial pair (V, W) solving*

$$\begin{aligned} \Delta W(x) + k^2(1+m)W = 0 \qquad & in \qquad D \\ \Delta V + k^2 V = 0 \qquad & in \qquad D \\ W = V,\ \frac{\partial W}{\partial \nu} = \frac{\partial V}{\partial \nu} \qquad & on \qquad \partial D \end{aligned}$$

If we set $u = W - V$, $v = k^2 V$, and $\lambda = -k^2$, then the interior transmission eigenvalue problem can be rewritten as the coupled 2x2 system of elliptic partial differential equations below. Its distinguishing feature is that u must satisfy two boundary conditions and v satisfies none.

$$(\Delta - \lambda(1+m))\, u + mv = 0$$

$$(\Delta - \lambda)\, v = 0 \tag{1.2}$$

$$u = 0\ ; \quad \tfrac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial D \tag{1.3}$$

Thus the interior transmission eigenvalues are the spectrum of the generalized eigenvalue problem:

$$B - \lambda I_m := \begin{pmatrix} \Delta_{00} & m \\ 0 & \Delta_{--} \end{pmatrix} - \lambda \begin{pmatrix} (1+m) & 0 \\ 0 & 1 \end{pmatrix} \tag{1.4}$$

Where we use the notation Δ_{00} and Δ_{--} to denote the fact that u must satisfy two boundary conditions and v needn't satisfy any. In other words, we treat B as an unbounded operator on $L^2(D) \oplus L^2(D)$ with domain:

$$B : H_0^2(D) \oplus \left\{ v \in L^2(D) : \Delta v \in L^2(D) \right\} \longrightarrow L^2(D) \oplus L^2(D)$$

The Hilbert space H_0^2 is the completion of C_0^∞ in the norm

$$||u||_2^2 = ||u||^2 + ||\Delta u||^2$$

and $||u||$means the $L^2(D)$ norm. We will always assume that $\Re(1+m) > \delta > 0$, so we may also describe the spectrum of the generalized eigenvalue problem as the spectrum of $I_m^{-1}B$. Neither B nor $I_m^{-1}B$ is self-adjoint; neither has a compact resolvent, but we will show that, as long as D is bounded, both resolvents are *upper triangular compact*, and that this is enough to reach most of the conclusions we know for operators with compact resolvents, including the facts that the nonzero spectrum is discrete and consists of eigenvalues of finite multiplicity. Our main theorem is:

Theorem 2. *Suppose that there are real numbers $m^* \geq m_* > 0$ and a unit complex number $e^{i\theta}$ in the open right half plane (i.e $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$) such that*

1. *$\Re(e^{i\theta}m(x)) > m_*$ in some neighborhood N of ∂D, or that $m(x)$ is real in all of D, and satisfies $m(x) \leq -m_*$ in some neighborhood N of ∂D.*

2. *$|m(x)| < m^*$ in all of D.*

3. *$\Re(1 + m(x)) \geq \delta > 0$ in all of D.*

Then the spectrum of (1.4) *consists of a (possibly empty) discrete set of eigenvalues with finite dimensional generalized eigenspaces. Eigenspaces corresponding to different eigenvalues are linearly independent. The eigenvalues and the generalized eigenspaces depend continuously on m in the $L^\infty(D)$ topology.*

We will obtain this as a corollary of

Theorem 3. *Suppose that there are real numbers $m^* \geq m_* > 0$ and a unit complex number $e^{i\theta}$ such that*

1. *$\Re(e^{i\theta}m(x)) > m_*$ in some neighborhood of ∂D.*

2. *$|m(x)| < m^*$ in all of D.*

Then the spectrum of B consists of a (possibly empty) discrete set of eigenvalues with finite dimensional generalized eigenspaces. Eigenspaces corresponding to different eigenvalues are linearly independent.The eigenvalues and the generalized eigenspaces depend continuously on m in the $L^\infty(D)$ topology.

Remark 4. *If we assume that m is continuous in the closure of D, then we need only state the conditions in item 1 of each theorem on ∂D, and they will necessary hold in some neighborhood.*

Our statement about the independence of eigenfunctions corresponding to different eigenvalues is, of course, trivially true for any linear operator. We state it explicitly because it is not obvious in other formulations of the interior transmission problem.The continuity of the eigenvalues and eigenfunctions will follow from the continuity of the spectral projections in the proposition below. We use the subscript in B_m to denote the dependence on the contrast $m(x)$.

Proposition 5. *Let A represent the operator B_m, or $I_m^{-1} B_m$ and let γ be a bounded rectifiable curve in the complex plane that avoids the spectrum of A. Then the spectral projection*

$$P_\gamma(m) := \tfrac{1}{2\pi i} \int_\gamma (A - \lambda I)^{-1} \, d\lambda \tag{1.5}$$

viewed as an operator valued function of $m \in L^\infty(D)$ with values in the space of operators mapping:

$$P_\gamma : L^2(D) \oplus L^2(D) \longrightarrow H_0^2(D) \oplus L^2(D)$$

is continuous at m.

There are many different ways to state the continuity of eigenvalues and eigenspaces, but all can be inferred from proposition 5 by choosing the curve γ appropriately. Typically, the most useful choice is a small circle surrounding an isolated eigenvalue. The continuity of P_γ implies, for example, that the dimension of its range is constant for small perturbations, so that a simple eigenvalue must remain simple, and the total algebraic multiplicity of the eigenvalues inside γ cannot change.

Remark 6. *If we modify the background of our interior transmission eigenvalue to be a variable real valued index of refraction, $n(x)$, that is bounded away from zero, and even replace the Laplacian with a real divergence form operator, i.e.*

Definition 7. *A wavenumber k^2 is called an interior transmission eigenvalue of m in the domain D with real background (n, γ) if there exists a non-trivial pair (V, W) solving*

$$\begin{aligned} \nabla \cdot \gamma \nabla W(x) + k^2 n^2(x)(1 + m(x))W = 0 \qquad & in \qquad D \\ \nabla \cdot \gamma \nabla V + k^2 n^2(x) V = 0 \qquad & in \qquad D \\ W = V, \ \frac{\partial W}{\partial \nu} = \frac{\partial V}{\partial \nu} \qquad & on \qquad \partial D \end{aligned}$$

As long as $n(x) \geq \delta > 0$ and γ is real, positive definite, and satisfies $\gamma \geq \delta I$ both theorem 2 and theorem 3 continue to hold.The proofs of all the estimates are correct line by line with just the obvious substitutions. The resolvent in (1.4) becomes

$$B - \lambda I_m := \frac{1}{n^2(x)} \begin{pmatrix} \nabla \gamma \nabla_{00} & n^2 m \\ 0 & \nabla \gamma \nabla_{--} \end{pmatrix} - \lambda \begin{pmatrix} (1+m) & 0 \\ 0 & 1 \end{pmatrix}$$

which enjoys the same duality (e.g. (2.21)). As the compactness properties follow directly from the estimates, they are also the same.

The transmission eigenvalue was introduced by Colton and Monk in 1988 [5]. In 1989, Colton, Kirsch, and Päivärinta [3] proved that the set of real transmission eigenvalues was discrete. That this set was non-empty was first proved in 2008 by Päivärinta and the author [8], under the hypothesis that m was large enough. In 2010, Cakoni, Gintides, and Haddar removed that restriction and showed that the set of real transmission eigenvalues was infinite [2].

Our results differ from previous work because we assume relatively little about the contrast m in the interior of D. Most of the previous work that we are aware of requires that $m(x) > m_* > 0$ or $m(x) < -m_* < 0$ in the whole domain. An exception is [1], which only assumes $m(x) \geq 0$ and allows cavities (i.e. $m(x) = 0$ on an open subset of D).Some of these results on real transmission eigenvalues have been extended to general elliptic operators in [7].

2 A Priori Estimates

Because v in (1.2) satisfies no boundary conditions, we don't have a direct estimate in terms of $||g||$. However, we still have the local elliptic estimates. We prove a simple version of these estimates below to show that, for large positive λ, Most of $||v||$ is concentrated near the boundary. The function ρ in the proposition will be either 1 or $(1+m)$ in our applications.

Proposition 8. *Suppose that $\Re(\rho) > \delta > 0$ and $\phi(x) \in C_0^\infty(D)$ be real valued, with $0 \leq \phi \leq 1$ (in our applications, we will take $\phi(x) = 1$ outside a neighborhood N of ∂D). If*

$$(\Delta - \lambda\rho)\, v = g \quad \text{in } D$$

then, there is a constant $K(\phi, \delta)$ such that, for sufficiently large positive λ,

$$
\begin{aligned}
||\phi v||^2 &\leq \frac{K}{\lambda - K}\left(||(1-\phi)v||^2 + ||\phi g||^2\right) && (2.1)\\
||v||^2 &\leq K\left(||(1-\phi)v||^2 + \frac{||\phi g||^2}{\lambda - K}\right) && (2.2)\\
||\nabla(\phi v)||^2 &\leq K\left(||v||^2 + +||\phi g||^2\right) && (2.3)
\end{aligned}
$$

Proof.

$$
\begin{aligned}
\int_D \overline{v}\phi^2\left(\Delta - \lambda\rho\right)v &= \int_D \phi^2\overline{v}g\\
-\int \nabla(\phi^2\overline{v})\cdot\nabla v - \lambda\int\rho|\phi v|^2 &= \int_D \phi\overline{v}\phi g\\
-\int \nabla(\phi\overline{v})\cdot\phi\nabla v - \phi\overline{v}\nabla\phi\cdot\nabla v - \lambda\int\rho|\phi v|^2 &=\\
-\int \nabla(\phi\overline{v})\cdot\nabla(\phi v) + \int\nabla(\phi\overline{v})\cdot v\nabla\phi - \phi\overline{v}\nabla\phi\cdot\nabla v - \lambda\int\rho|\phi v|^2 &=\\
-\int |\nabla(\phi v)|^2 + \int|\nabla\phi|^2\,|v|^2 + \frac{1}{2}\int\nabla\phi^2\cdot(v\nabla\overline{v}\cdot - \overline{v}\nabla v) - \lambda\int\rho|\phi v|^2 &=
\end{aligned}
$$

Taking real parts removes the third term on the left hand side

$$
-\int|\nabla(\phi v)|^2 + \int|\nabla\phi|^2\,|v|^2 - \lambda\int\rho|\phi v|^2 = \Re\left(\int_D \phi\overline{v}\phi g\right)
$$

Rearranging yields

$$
\begin{aligned}
||\nabla(\phi v)||^2 + \lambda\int\rho|\phi v|^2 &= \int|v|^2|\nabla\phi|^2 - \Re\left(\int_D\phi\overline{v}\phi g\right)\\
||\nabla(\phi v)||^2 + \lambda\delta||\phi v||^2 &\leq K(\phi)\left(||v||^2 + ||\phi g||^2\right)
\end{aligned}
$$

which immediately yields (2.3) and also

$$
\begin{aligned}
||\phi v||^2 &\leq \frac{K}{\lambda\delta}\left(||v||^2 + ||\phi g||^2\right)\\
&\leq \frac{2K}{\lambda\delta}\left(||\phi v||^2 + ||(1-\phi)v||^2 + ||\phi g||^2\right)
\end{aligned}
$$

which becomes (2.1) after subtracting the term $||\phi v||^2$ from both sides. Adding $||(1-\phi)v||^2$ to both sides yields (2.2). $\square$

Corollary 9. *Suppose that ϕ, v, and ρ satisfy the hypothesis of proposition 8, that, for some unit complex number $e^{i\theta}$ and some neighborhood N of ∂D, $\Re(e^{i\theta}m(x)) > m_*$, and that $\phi(x) = 1$ in $D \setminus N$. Then, for sufficiently large positive λ,*

$$||v||^2 \leq K\left(\left|\int m|v|^2\right| + \frac{||\phi g||^2}{\lambda - K}\right) \tag{2.4}$$

and

$$\left|\int m\phi^2|v|^2\right| \leq \frac{K}{\lambda - K}\left(\left|\int (1-\phi^2)m|v|^2\right| + ||\phi g||^2\right) \tag{2.5}$$

Proof.

$$\begin{aligned}
\int |v|^2 &\leq \left|\int (1-\phi^2)|v|^2\right| + \left|\int \phi^2|v|^2\right| \\
&\leq \left|\int \Re\left(\frac{e^{i\theta}m}{m_*}\right)(1-\phi^2)|v|^2\right| + \left|\int \phi^2|v|^2\right| \\
&\leq \left|\Re\left(\frac{e^{i\theta}}{m_*}\int m(1-\phi^2)|v|^2\right)\right| + \left|\int \phi^2|v|^2\right| \\
&\leq \left|\frac{1}{m_*}\int m(1-\phi^2)|v|^2\right| + \left|\int \phi^2|v|^2\right| \\
&\leq \left|\frac{1}{m_*}\int m|v|^2\right| + \left|\frac{1}{m_*}\int \phi^2 m|v|^2\right| + \left|\int \phi^2|v|^2\right| \\
&\leq \left|\frac{1}{m_*}\int m|v|^2\right| + \frac{m_* + m^*}{m_*}||\phi v||^2
\end{aligned}$$

Applying (2.1),

$$\begin{aligned}
&\leq \left|\frac{1}{m_*}\int m|v|^2\right| + \left(\frac{m_* + m^*}{m_*}\right)\frac{K}{\lambda - K}\left(||(1-\phi)v||^2 + ||\phi g||^2\right) \\
&\leq \left|\frac{1}{m_*}\int m|v|^2\right| + \left(\frac{m_* + m^*}{m_*}\right)\frac{K}{\lambda - K}\left(||v||^2 + ||\phi g||^2\right)
\end{aligned}$$

so, with a different constant and λ large enough, we obtain (2.4).

$$||v||^2 \leq \widetilde{K}\left(\left|\int m|v|^2\right| + \frac{||\phi g||^2}{\lambda - K}\right)$$

We make a similar calculation to establish (2.5)

$$\left|\int \phi^2 m|v|^2\right| \;\leq\; m^* \int \phi^2|v|^2$$

Applying (2.1) gives

$$\leq \frac{K}{\lambda - K}\left(\int (1-\phi)^2|v|^2 + ||\phi g||^2\right)$$

Because $0 \leq \phi \leq 1$

$$\begin{aligned}
&\leq \frac{K}{\lambda - K}\left(\int (1-\phi^2)|v|^2 + ||\phi g||^2\right) \\
&\leq \frac{K}{\lambda - K}\left(\left|\int \Re\left(\frac{e^{i\theta} m}{m_*}\right)(1-\phi^2)|v|^2\right| + ||\phi g||^2\right) \\
&\leq \frac{K}{\lambda - K}\left(\frac{1}{m_*}\left|\int m(1-\phi^2)|v|^2\right| + ||\phi g||^2\right)
\end{aligned}$$

□

Next, we derive some a priori estimates for the resolvent of B. Suppose that λ is large enough, that

$$(\Delta - \lambda)u + mv \;=\; f \tag{2.6}$$

$$(\Delta - \lambda)v \;=\; g \tag{2.7}$$

and that u and $\frac{\partial u}{\partial \nu}$ vanish on ∂D. Multiplying the complex conjugate of (2.7) by u yields

$$\int u(\Delta - \lambda)\overline{v} \;=\; \int \overline{g}u$$

and integrating by parts

$$\int \overline{v}(\Delta - \lambda) = \;=\; \int \overline{g}u \tag{2.8}$$

Multiplying (2.6) by $\overline{v}$ and inserting (2.8) yields

$$\int m|v|^2 \;=\; \int f\overline{v} - \int \overline{g}u \tag{2.9}$$

from which we conclude (using (2.4)) that

$$||v||^2 \;\leq\; K\left(||f||^2 + \frac{||g||^2}{\lambda - K} + ||g||\,||u||\right) \tag{2.10}$$

Next we multiply (2.6) by $\overline{u}$

$$\int \overline{u}(\Delta - \lambda)u + \int mv\overline{u} \;=\; \int f\overline{u}$$

and integrate by parts

$$-\int |\nabla u|^2 - \lambda \int |u|^2 \;=\; \int f\overline{u} - \int mv\overline{u}$$

$$\int |\nabla u|^2 + (\lambda - K)\int |u|^2 \;\leq\; K\left(||f||^2 + ||v||^2\right) \tag{2.11}$$

so that

$$||u||^2 \;\leq\; \frac{K}{\lambda - K}\left(||f||^2 + ||v||^2\right) \tag{2.12}$$

and

$$||\nabla u||^2 \;\leq\; K\left(||f||^2 + ||v||^2\right) \tag{2.13}$$

Using (2.12) in (2.10), gives

$$||v||^2 \;\leq\; K\left(||f||^2 + \frac{||g||^2}{\lambda}\right) \tag{2.14}$$

so that (2.12) and (2.13) become

$$||u||^2 \;\leq\; \frac{K}{\lambda - K}\left(||f||^2 + \frac{||g||^2}{\lambda}\right) \tag{2.15}$$

and

$$||\nabla u||^2 \;\leq\; K\left(||f||^2 + \frac{||g||^2}{\lambda}\right) \tag{2.16}$$

It now follows from (2.6) that

$$||\Delta u|| \;\leq\; K\left(||f|| + \frac{||g||}{\lambda}\right) \tag{2.17}$$

and from (2.7) that

$$||\Delta v|| \;\leq\; K\left(\lambda||f|| + ||g||\right) \tag{2.18}$$

It also folows from (2.4) that

$$\int |\nabla(\phi v)|^2 \;\leq\; K\left(||f||^2 + ||g||^2\right) \tag{2.19}$$

The proposition below is a direct consequence of this list of inequalities.

Proposition 10. *For λ real, positive, and large enough,*

1. $(B - \lambda I) : H_0^2(D) \oplus \left\{v \in L^2(D) : \Delta v \in L^2(D)\right\} \longrightarrow L^2(D) \oplus L^2(D)$ *is invertible.*

2. If we write the the resolvent in block diagonal form,

$$(B - \lambda I)^{-1} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \tag{2.20}$$

then R_{11}, R_{12}, and R_{22} are compact. If $\phi(x)$ is a smooth function vanishing in a neighborhood of ∂D and equal to 1 on $D \setminus N$, them ϕR_{21} is compact.

3. $||R_{11}|| + ||R_{12}|| + ||R_{22}|| + ||\phi R_{21}|| \leq \dfrac{K}{\lambda}$

Proof. The combination of (2.14) through (2.18) show that $B - \lambda I$ is one to one and has closed range. If we note that the adjoint of B is

$$(B - \lambda I)^* = \begin{pmatrix} \Delta_{--} - \lambda I & \overline{m} \\ 0 & \Delta_{00} - \lambda I \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{(B - \lambda I)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.21}$$

we see that B^* is just B with m replaced by $\overline{m}$ and u and v interchanged. Therefore (2.14) and (2.15) ensure uniqueness for B^* and consequently existence for B. Now $R_{11}f$ in (2.20) denotes the function u that satisfies (2.6) and (2.7) with $g = 0$, and $R_{12}g$ denotes the function u that satisfies (2.6) and (2.7) with $f = 0$. The estimates (2.15) and (2.16) imply that both are compact.

Similarly, $R_{22}g$ denotes the function v that satisfies (2.6) and (2.7) with $f = 0$ and $R_{21}f$ denotes the function v that satisfies (2.6) and (2.7) with $g = 0$ The combination of (2.14) and (2.19) imply the compactness of ϕR_{21} and ϕR_{22}. To see the compactness of R_{22}, suppose that we have a sequence $\{g_n\}$ converging weakly to zero. Now $u_n = R_{12}g_n$ and $\phi v_n = \phi R_{22}g_n$ converge strongly to zero. According to (2.9),

$$\int m|v_n|^2 = \int \overline{g_n} u_n \tag{2.22}$$

and the right side converges to zero. Hence

$$\int m(1 - \phi^2)|v_n|^2 = \int \overline{g_n} u_n - \int \phi^2 m|v_n|^2 \longrightarrow 0$$

and therefore the coercivity condition on m implies that $||(1-\phi)v_n||$ converge to zero so that R_{22} is compact.

The estimates in the last item follow from (2.15),(2.14), and the combination of (2.14) and (2.1). $\square$

We have constructed the resolvent $(B - \lambda I)^{-1}$ for large positive λ and have shown that it is *upper triangular compact*.

3 Upper Triangular Compact Operators

For an unbounded operator with a compact resolvent, discreteness of spectra follows from the resolvent identity

$$R(\mu) - R(\lambda) = (\mu - \lambda)R(\mu)R(\lambda) \tag{3.1}$$

rewritten as

$$R(\mu) = R(\lambda)\,(I - (\mu - \lambda)R(\lambda))^{-1} \tag{3.2}$$

and the analytic Fredholm theorem [4].

Proposition 11 (The Analytic Fredholm Theorem)**.** *Suppose that $R(\lambda)$ is an analytic compact operator valued function of λ for λ in some open connected set L. Then if $I - R(\lambda_0)$ is invertible for one $\lambda_0 \in L$, it is invertible for all but a discrete set of $\lambda \in L$.*

Remark 12. *Because $I - R(\lambda)$ must have index 0, it is enough to check that* $\ker(I - R(\lambda_0))$ *or* $\mathrm{coker}\,(I - R(\lambda_0))$ *is empty. We don't need to check both.*

Definition 13. *Suppose that R is a bounded operator mapping a Hilbert space H to itself. If the Hilbert space has a decomposition into a direct sum $H = \bigoplus_{j=1}^{n} H_j$, we say that R is* ***upper triangular compact*** *(UTC) (with respect to this decomposition) if the upper triangular blocks (including the diagonal) in the corresponding decomposition of $R = \sum_{jk=1}^{n} R_{jk}$ are compact.*

The proposition below asserts that the conclusions of the analytic Fredholm theorem continue to hold for operators that are upper triangular Fredholm.

Proposition 14 (The Upper Triangular Analytic Fredholm Theorem)**.** *Suppose that $R(\lambda)$ is an analytic UTC operator valued function of λ for λ in an open connected set L. Then if $I - R(\lambda_0)$ has empty kernel or cokernel for one λ_0, it is invertible for that λ_0 and for all but a discrete set of $\lambda \in L$.*

Proof. We perform block Gaussian elimination modulo compact operators. We subtract a multiple of the first row from each subsequent row, so that the resulting operators in the first column are compact. Specifically, we set

$$\begin{aligned} \widetilde{R_{n1}} &= R_{n1} - R_{n1}(I - R_{11}) \\ &= R_{n1}R_{11} \end{aligned}$$

and, for $m > 1$,

$$\widetilde{R_{nm}} = R_{nm} - R_{n1}R_{1m}$$

The new first column is compact below the diagonal, and the entries in the other columns have only been changed by the addition of a compact operator (R_{1m} for $m > 1$ are compact), so the new operator still has the form, identity plus UTC. After repeating for each subsequent column, we reach a point where all off diagonal blocks are compact. We have produced a factorization

$$I - R(z) = (I - L(z))(I - C(z))$$

with $L(z)$ strictly lower triangular, and $C(z)$ compact. The first factor is always invertible and we may invoke the analytic Fredholm theorem 11, and the remark which follows it, on the second factor. □

Theorem 15 (UTC Resolvent Theorem)**.** *Let B be a closed densely defined operator on an infinite dimensional Hilbert space and suppose that for one complex number λ_0, $(B - \lambda_0 I)$ is invertible and $(B - \lambda_0 I)^{-1}$ is UTC. Then the spectrum of B consists of a (possibly empty) discrete set of eigenvalues. with finite dimensional generalized eigenspaces.*

Proof. According to the resolvent identity (3.2), for any complex number τ, we may write

$$(B - \tau I) = (B - \lambda_0 I)\left(I - (B - \lambda_0 I)^{-1}(\tau - \lambda_0)\right)$$

Because $(B - \lambda_0 I)^{-1}$ is UTC, the UT analytic Fredholm theorem implies that factor on the right is invertible at all but a discrete set of points τ_n, and the dimension of the kernel is finite at all such points. □

4 Application of the UT Analytic Fredholm Theorem

Proof of Theorem 3. Let B be the operator defined in (1.4). Proposition 10 tells us that, if we choose λ_0 real, positive, and large enough, then $(B - \lambda_0 I)$ is invertible and $(B - \lambda_0 I)^{-1}$ is UTC, so theorem 15 guarantees that the spectrum of B is discrete and of finite multiplicity. □

Proof of Theorem 2.

$$(B - \lambda_0 I_m) \;=\; (B - \lambda_0 I)\left(I - \lambda_0 (B - \lambda_0 I)^{-1}\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}\right) \tag{4.1}$$

The factor on the right is of the form identity plus UTC, and therefore UT Fredholm of index zero. We will show that the kernel or cokernel of that factor is empty for a large positive λ_0. It will then follow that

$$(B - \lambda_0 I_m)^{-1} = \left(I - \lambda_0 (B - \lambda_0 I)^{-1}\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}\right)^{-1}(B - \lambda_0 I)^{-1}$$

is UTC – because the product of UTC and identity minus UTC is UTC. Multiplication by I_m also preserves UTC, so $(I_m^{-1}B - \lambda_0 I)^{-1}$ is the upper triangular compact resolvent of $I_m^{-1}B$ at λ_0. Theorem 15 now implies theorem 2. Therefore we may finish the proof of theorem 2 with:

Proposition 16. *Suppose that* $\Re(1 + m(x)) \geq \delta > 0$ *in* D*.*

1. *If* m *is real in* D *and* $m < -m_* < 0$ *in some neighborhood* N *of* ∂D*, then, for* λ *real and sufficiently large,* $\ker(B - \lambda I_m)$ *is empty.*

2. *If* $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$ *and* $\Re(e^{i\theta}m(x)) > m_* > 0$ *in some neighborhood* N *of* ∂D*, then, for* λ *real and sufficiently large,* coker $(B - \lambda I_m)$ *is empty.*

Proof. The kernel of $(B - \lambda I_m)$ consists of functions satisfying

$$(\Delta - \lambda)u + mv \;=\; \lambda m u \tag{4.2}$$

$$(\Delta - \lambda)v \;=\; 0 \tag{4.3}$$

with u and $\frac{\partial u}{\partial \nu}$ vanishing on ∂D. We multiply the conjugate of (4.3) by u and integrate by parts to obtain

$$\int \overline{v}(\Delta - \lambda)u \;=\; 0$$

Subtracting this from the integral of $\overline{v}$ times the (4.2) yields

$$\int m|v|^2 \;=\; \lambda \int m u \overline{v}$$

Because the left hand side is real

$$\int m|v|^2 \;=\; \lambda \int m \overline{u} v \tag{4.4}$$

We next multiply (4.2) by $\overline{u}$ and integrate by parts to find that

$$-\int |\nabla u|^2 - \lambda \int (1+m)|u|^2 \;=\; -\int m \overline{u} v$$

which becomes, after inserting (4.4)

$$\begin{aligned} -\int |\nabla u|^2 - \lambda \int (1+m)|u|^2 &= \frac{-1}{\lambda} \int m|v|^2 \\ &= \frac{-1}{\lambda} \left(\int (1-\phi^2) m|v|^2 + \int \phi^2 m|v|^2 \right) \end{aligned} \tag{4.5}$$

The estimate (2.5) with $g = 0$ and $\rho = 1$, implies that the real number

$$z \;=\; \frac{\int \phi^2 m|v|^2}{\int (1-\phi^2) m|v|^2} \tag{4.6}$$

satisfies

$$|z| \;\leq\; \frac{K}{\lambda} \tag{4.7}$$

so that for λ large enough

$$1 + z \;>\; 0$$

If we rewrite (4.5) as

$$-\int |\nabla u|^2 - \lambda \int (1+m)|u|^2 \;=\; \frac{-1}{\lambda} \left(\int (1-\phi^2) m|v|^2 \right) (1+z)$$

we see that, for λ large enough, the right hand side is positive, while the left hand side is negative, unless both u and v are identically zero.

Finally, the kernel of $(B - \lambda I_m)^*$ consists of functions satisfying

$$(\Delta - \lambda)v + \overline{m}u = 0 \tag{4.8}$$
$$(\Delta - \lambda)u = \lambda \overline{m}u \tag{4.9}$$

with v and $\frac{\partial v}{\partial \nu}$ vanishing on ∂D. We multiply (4.8) first by $\overline{v}$ and integrate

$$-\int |\nabla v|^2 - \lambda \int |v|^2 = -\int \overline{m}u\overline{v} \tag{4.10}$$

and then multiply the conjugate of (4.8) by u to obtain

$$\int u(\Delta - \lambda)\overline{v} = -\int m|u|^2$$

Multiplying (4.9) by $\overline{v}$ and integrating by parts gives

$$\int u(\Delta - \lambda)\overline{v} = \lambda \int \overline{m}u\overline{v}$$

Combining gives

$$-\int |\nabla v|^2 - \lambda \int |v|^2 = \frac{1}{\lambda}\int m|u|^2$$

we split the integral on the right into two parts

$$= \frac{1}{\lambda}\left(\int (1-\phi^2)m|u|^2 + \int \phi^2 m|u|^2\right)$$

We again employ (2.5) with $g = 0$ and $\rho = 1 + m$, to see that

$$-\int |\nabla v|^2 - \lambda \int |v|^2 = \frac{1}{\lambda}\left(\int (1-\phi^2)m|u|^2\right)(1+z) \tag{4.11}$$

where z defined as in (4.6), is complex, but still satisfies (4.7). This time, the left hand side is a negative real number, and the hypothesis guarantees that, for every x, $m(x)$ sits in an open half plane (a cone) that does not contain the negative real semi-axis. This means that the argument of the integral on the right hand side of (4.11) is bounded away from π. The argument of $1+z$ approaches zero as λ increases, so for large enough λ, the left hand side belongs to the negative real axis, while the right hand side cannot, unless u, and therefore also v, is identically zero. □

This finishes the proof of theorem 2. □

Proof of Proposition 5. Those familiar with spectral theory will recognize that the main step in the proof we give below verifies that, if p and m are two different contrasts, then B_p (resp. $I_p^{-1}B_p$) is a relatively bounded perturbation of B_m (resp. $I_m^{-1}B_m$).

Because the curve γ avoids the spectrum of B_m, the resolvent $(B_m - \lambda I)^{-1}$is a holomorphic, hence continuous, function on the compact set γ. Therefore

$$\Gamma(m) := \sup_{\lambda\in\gamma} |||\,(B_m - \lambda I)^{-1}\,||| < \infty$$

where we have used $|||A|||$ denote the norm of the operator as a mapping from $L^2(D) \oplus L^2(D)$ into $H_0^2(D) \oplus L^2(D)$ and will use $|A|$ to denote the norm of a mapping from $H_0^2(D) \oplus L^2(D)$ to $L^2(D) \oplus L^2(D)$. Now,

$$|B_p - B_m| \;=\; \left|\begin{pmatrix} 0 & (p-m) \\ 0 & 0 \end{pmatrix}\right| \le ||p-m||_\infty$$

so that

$$\begin{aligned} |||\,(B_p - \lambda I)^{-1}\,||| &= |||\,(B_m - \lambda I)^{-1}\left(I - (B_p - B_m)\,(B_m - \lambda I)^{-1}\right)^{-1}\,||| \\ &\le \Gamma(m)\,(1 - ||p-m||_\infty \Gamma(m))^{-1} \end{aligned}$$

so we may conclude the existence of $(B_p - \lambda I)^{-1}$for $||p-m||_\infty < 1/\Gamma(m)$ and further that

$$\begin{aligned} |||\,(B_p - \lambda I)^{-1} - (B_m - \lambda I)^{-1}\,||| &= |||\,(B_p - \lambda I)^{-1}\,(B_p - B_m)\,(B_m - \lambda I)^{-1}\,||| \\ &\le |||\,(B_p - \lambda I)^{-1}\,|||\;|B_p - B_m|\;|||\,(B_m - \lambda I)^{-1}\,||| \\ &\le \frac{\Gamma(m)}{(1 - ||p-m||_\infty \Gamma(m))}\,||p-m||_\infty\,\Gamma(m) \end{aligned}$$

and hence, letting $|\gamma|$ denote the length of γ and recalling the definition of the spectral projection from (1.5)

$$|||P_\gamma(p) - P_\gamma(m)||| \;\le\; \frac{|\gamma|\;||p-m||_\infty\;\Gamma^2(m)}{(1 - ||p-m||_\infty \Gamma(m))}$$

which establishes the continuity of P_γ in the Born approximation.

The proof for $I_m^{-1}B_m$ is analogous. We redefine

$$\Gamma(m) := \sup_{\lambda\in\gamma} |||\,\left(I_m^{-1}B_m - \lambda I\right)^{-1}\,||| < \infty$$

and compute the norm of the perturbation

$$
|I_p^{-1}B_p - I_m^{-1}B_m| \;=\; \left| \begin{pmatrix} \frac{m-p}{(1+m)(1+p)}\Delta_{00} & \frac{p-m}{(1+m)(1+p)} \\ 0 & 0 \end{pmatrix} \right| \leq \frac{||p-m||_\infty}{\delta^2}
$$

so that

$$
\begin{aligned}
||| \left(I_p^{-1}B_p - \lambda I\right)^{-1} ||| &= ||| \left(I_m^{-1}B_m - \lambda I\right)^{-1} \left(I - (I_p^{-1}B_p - I_m^{-1}B_m)\left(I_m^{-1}B_m - \lambda I\right)^{-1}\right)^{-1} ||| \\
&\leq \Gamma(m)\left(1 - \frac{||p-m||_\infty}{\delta^2}\Gamma(m)\right)^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
||| \left(I_p^{-1}B_p - \lambda I\right)^{-1} &- \left(I_m^{-1}B_m - \lambda I\right)^{-1} ||| \\
&= ||| \left(I_p^{-1}B_p - \lambda I\right)^{-1}\left(I_p^{-1}B_p - I_m^{-1}B_m\right)\left(I_m^{-1}B_m - \lambda I\right)^{-1} ||| \\
&\leq ||| \left(I_p^{-1}B_p - \lambda I\right)^{-1} ||| \, |I_p^{-1}B_p - I_m^{-1}B_m| \, ||| \left(I_m^{-1}B_m - \lambda I\right)^{-1} ||| \\
&\leq \frac{\Gamma(m)}{1 - \frac{||p-m||_\infty}{\delta^2}\Gamma(m)} \, \frac{||p-m||_\infty}{\delta^2} \, \Gamma(m) \\
&= \frac{\Gamma^2(m)||p-m||_\infty}{\delta^2 - ||p-m||_\infty}
\end{aligned}
$$

which establishes the continuity of the resolvent and hence the spectral projection.

□

5 Discussion

We have shown that the interior transmission eigenvalue problem, with some coercivity conditions on the contrast m, naturally leads to a simple class of closed operators with upper triangular compact resolvents, which share the properties of operators with compact resolvent.

For the Born approximation to the interior transmission eigenvalue problem (theorem 3), the coercivity condition on the values of m near the boundary seems pretty natural. We expect that this cannot be weakened too much. The conditions we require for theorem 2 are more ad hoc. They are required for our proof, but we see no strong reason to believe they are necessary.

If we could prove that these resolvents were not quasi-nilpotent, existence of transmission eigenvalues, and very likely completeness of generalized eigenspaces, would follow.

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