Delaware Summer School on Inverse Scattering Fixed Frequency Inverse Source Problems

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1 Fourier Transform

Fourier Transform

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{ix\xi} f(x)dx$$

Properties

$$\begin{aligned}
(\widehat{(\frac{d}{dx})^n f}) &= (i\xi)^n \widehat{f} \\
\widehat{(ix)^n f} &= \left(\frac{d}{d\xi}\right)^n \widehat{f} \\
\widehat{f(x+M)} &= e^{iM\xi}\widehat{f}(\xi) \\
\widehat{e^{iMx}f(x)} &= \widehat{f}(\xi+M)
\end{aligned}$$

Schwartz Class

$$\mathcal{S} = \left\{ f \middle| \sup \left| x^m (\left(\frac{d}{dx}\right)^n f \right| < \infty \right\}$$

Inverse Fourier Transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}\widehat{f}(\xi)d\xi$$

Plancherel Equality

$$\int f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi$$

Theorem 1 (Paley-Wiener Theorem)**.** $f \in \mathcal{S}$

$$\text{supp } f \subset \mathbb{R}^+$$
$$\Longleftrightarrow$$
$$f \text{ extends to a bounded holomorphic function on } \mathbb{C}^+$$

Proof.

$$F(\zeta) = \frac{1}{2\pi}\int_0^\infty e^{it\zeta} f(t)dt$$

because $\Re(it\zeta) \leq 0$,

$$|F(\zeta)| = \frac{1}{2\pi}\int_0^\infty |f(t)|dt$$

The converse is harder.

If $F(\zeta)$ is holomorphic and bounded on $\mathbb{C}^+$, so is

$$F_\epsilon(\zeta) = \frac{F(\zeta)}{(1 - i\epsilon\zeta)^2}$$

and for $\zeta \in \mathbb{C}^+$,

$$|F_\epsilon(\zeta)| \leq \frac{1}{1 + |\epsilon\zeta|^2}$$

As $\epsilon \to 0$, $F_\epsilon(\zeta) \to F(\zeta)$, so $f_\epsilon(t) \to f(t)$, so if we prove that supp $f_\epsilon \subset \mathbb{R}^+$, then so is supp f.

$$f_\epsilon(t) = \int\limits_{\mathbb{R}} e^{-it\zeta} F_\epsilon(\zeta)d\zeta$$

For $t < 0$, and $\gamma(R)$ a semi-circular contour in the upper half plane,

$$
\begin{aligned}
\int_{\frown} e^{-it\zeta} F_\epsilon(\zeta) d\zeta &= 0 \\
\int_{-R}^{R} e^{-it\zeta} F_\epsilon(\zeta) d\zeta &= \int_{\frown} e^{-it\zeta} F_\epsilon(\zeta) d\zeta \\
&\leq \frac{\pi R}{1 + |\epsilon R|^2} \to 0 \text{ as } R \to \infty
\end{aligned}
$$

□

Corollary 2 (Paley-Wiener Theorem II)**.** $f \in \mathcal{S}$

$$
\text{supp } f \subset [-M, M]
$$
$$
\Longleftrightarrow
$$
$$
\widehat{f} \textit{ extends to a bounded holomorphic function on } \mathbb{C} \textit{ with}
$$
$$
|\widehat{f}(\zeta)| \leq K e^{M|\Im(\zeta)|}
$$

Proof. Apply the previous theorem to $f(x+M)$ and $f(x-M)$, recalling the formula for $\widehat{f(x+M)}$ □

2 The One Dimensional Wave equation

$$
U_{tt} - U_{xx} = -F(x,t) \quad [= -F(x)\delta(t)]
$$

Initial Conditions

$$
\begin{aligned}
U(x,0) &= 0 \\
U_t(x,0) &= 0
\end{aligned}
\qquad\qquad [U(x,t) \equiv 0 \quad \text{for } t < 0]
$$

Fourier Transform in time

$$\begin{aligned} u(x,k) &= \int_{-\infty}^{\infty} e^{ikt} U(x,t)dt \\ f(x,k) &= \int_{-\infty}^{\infty} e^{ikt} F(x,t)dt \end{aligned}$$

Helmholtz Equation

$$u_{xx} + k^2 u = f(x) \quad [= f(x,k)]$$

What about intial conditions ?

$$\begin{aligned} u(x,k) &= \int_{0}^{\infty} e^{ikt} U(x,t)dt \\ f(x,k) &= \int_{0}^{\infty} e^{ikt} F(x,t)dt \end{aligned}$$

Because $u(x,t) \equiv 0$ for $t < 0$, $u(x,k)$ extends to a bounded holomorphic function on $\mathbb{C}^+$.

Corollary 3 (Corollary of Paley-Wiener Theorem)**.** *There exists at most one causal solution to the Helmholtz equation.*

Proof.

$$v'' + k^2 v = 0$$
$$v = Ae^{ikx} + Be^{-ikx}$$

Exercise Show that there is no choice of A and B such that

$$v = Ae^{i\zeta x} + Be^{-i\zeta x}$$

remains bounded for all $x \in \mathbb{R}$ and all $\zeta \in \mathbb{C}^+$. □

Constructing the Causal Solution

$$u'' + k^2 u = f(x)$$

Fourier Transform in x,

$$\widehat{u}(\xi, k) = \int_{-\infty}^{\infty} e^{-ix\xi} u(x,k)dx$$

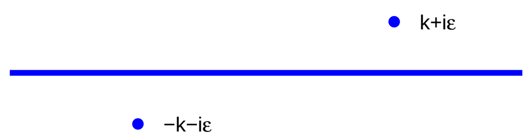
$$\begin{aligned}\left(-\xi^2+k^2\right)\widehat{u} &= \widehat{f}(\xi)\\ \widehat{u} &= \frac{\widehat{f}(\xi)}{k^2-\xi^2}\\ u(x,k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}\frac{\widehat{f}(\xi)}{k^2-\xi^2}d\xi\end{aligned}$$

What does $\int \frac{1}{k^2-\xi^2}$ mean? Because $\frac{1}{k^2-\xi^2}$ is not an integrable function, the formula for u could mean different things. We will use causality.

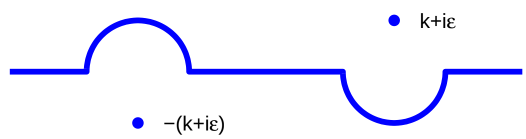
$$u(x,k+i\epsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}\frac{\widehat{f}(\xi)}{(k+i\epsilon)^2-\xi^2}d\xi$$

For $\epsilon > 0$, the denominator never vanishes, so we know exactly what this means.

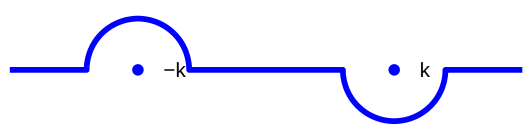
The integrand is an analytic function of ξ (for compactly supported $f(x)$), so we may view this as a contour integral.



We deform the contour of integration,

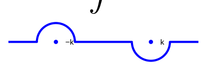


and then let $k + i\epsilon$ return to k.



Theorem 4. *The unique causal solutiion to the Helmholtz solution is given by:*

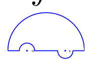
$$u(x,k) = \frac{1}{2\pi}\int e^{ix\xi}\frac{\widehat{f}(\xi)}{k^2-\xi^2}d\xi$$



Evaluate by residues for $x > \text{supp } f$

Suppose $x > \text{supp } f$ (this means $x > M = \sup_{x \in \text{supp } f} x$)

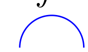
$$\int e^{ix\zeta} \frac{\widehat{f}(\zeta)}{k^2 - \zeta^2} d\zeta = 2\pi i \sum \text{residues}$$



Only one residue at $\zeta = k$.

$$= 2\pi i \frac{-e^{ixk} \widehat{f}(k)}{2k}$$

$$u = \frac{e^{ixk} \widehat{f}(k)}{2ik} + \int e^{ix\zeta} \frac{\widehat{f}(\zeta)}{k^2 - \zeta^2} d\zeta$$



But, for $\zeta \in \mathbb{C}^+$,

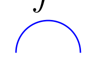
$$\begin{aligned} |e^{ix\zeta} \widehat{f}| &\leq K e^{-x\Im\zeta} e^{M\Im\zeta} \\ &= K e^{(M-x)\Im\zeta} < K \qquad \text{for } x > M \end{aligned}$$

and

$$\left| \frac{d\zeta}{k^2 - \zeta^2} \right| \leq \left| \frac{R d\theta}{R^2 - k^2} \right|$$

so

$$\int e^{ix\zeta} \frac{\widehat{f}(\zeta)}{k^2 - \zeta^2} d\zeta \xrightarrow{R \to \infty} 0$$



Theorem 5. *There exists a unique causal solution to*

$$u'' + k^2 u = f$$

For x outside the convex hull of the support of f,

$$u^+(x, k) = \frac{1}{2ik} e^{ik|x|} \widehat{f}(\pm k)$$

with the $\pm$ sign equal to minus if $x < \operatorname{supp} f$. u^+ satisfies the Sommerfeld Radiation Condition (SRC):

$$\operatorname{sgn}(x)\frac{d}{dx}u^+ = iku$$

The SRC solution is unique.

Proof. Check that

$$v = Ae^{ikx} + Be^{-ikx}$$

cannot satisfy the SRC. □

Fundamental Causal/SRC Solution

We could have written u as a convolution of f with a fundamental solution,

$$u^+(x,k) = \frac{1}{2ik}\int_{\mathbb{R}} e^{ik|x-y|} f(y)dy$$

and then noted that, for x outside the convex hull of the support of f,

$$e^{ik|x-y|} = e^{ik|x|}e^{\pm iky}$$

3 The Two Dimensional Helmholtz Equation

$$\left(\Delta + k^2\right)u = f$$

Fourier Transform

$$\widehat{u} = \int_{\mathbb{R}^2} e^{-i(x\xi+y\eta)}u(x,y)dxdy$$

$$\widehat{u} = \frac{\widehat{f}(\xi,\eta)}{\xi^2-(k^2-\eta^2)}$$

$\frac{1}{\xi^2-(k^2-\eta^2)}$ is not integrable. The denominator vanishes simply on the circle of radius k.

Causality

$$\widehat{u} = \lim_{\epsilon \downarrow 0} u(x, k + i\epsilon)$$

$$\widehat{u(x,k+i\epsilon)} = \frac{\widehat{f}(\xi, \eta)}{\xi^2 - ((k + i\epsilon)^2 - \eta^2)}$$

No zeros in denominator.
Define:

$$\begin{aligned} \omega(\zeta) &= \sqrt{\zeta^2 - \eta^2} \\ \omega_\epsilon &= \omega(k + i\epsilon) \\ \omega &= \omega(k) \end{aligned}$$

Properties of ω We make a precise defintion of ω, for each fixed real η.

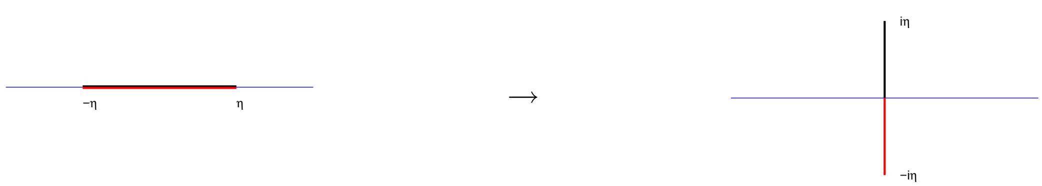
$$\omega(\zeta) \quad = \quad \sqrt{\zeta^2 - \eta^2}$$

For ζ large, we choose $\omega \sim \zeta$

$$= \quad \zeta \sqrt{1 - \left(\frac{\eta}{\zeta}\right)^2}$$

ω is holomorphic in $\mathbb{C} \setminus [-\eta, \eta]$; it has a branch cut along the real interval $[-\eta, \eta]$. As a function of ζ, ω maps each of the quarter planes to itself, In particular,

$$\Im\zeta \geq 0 \Rightarrow \Im\omega(\zeta) \geq 0$$

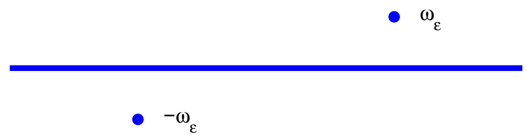


Exercise Use the maximum prinicple for harmonic functions to show that $\Im\omega(\zeta) \geq \Im\zeta$ for all $\zeta \in \mathbb{C}^+$.

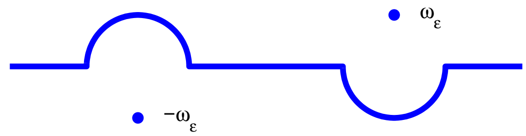
$$u_\epsilon = \int_{\mathbb{R}} e^{i\eta y} \left[\int_{\mathbb{R}} e^{i\xi x} \frac{\widehat{f}(\xi,\eta)}{\xi^2 - \omega_\epsilon^2} d\xi \right] d\eta$$

We treat the inner integral, as we did in the one dimensional case.

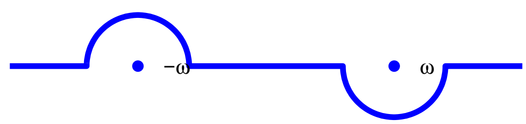
Recall that $\widehat{f}$ extends to be holomorphic in ξ (and in η) because f is compactly supported.



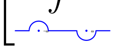
We deform the contour of integration,



and then let ϵ decrease to zero so that ω_ϵ returns to ω.



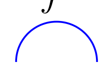
$$u(x,y) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} e^{i\eta y} \left[\int e^{i\xi x} \frac{\widehat{f}(\xi,\eta)}{\xi^2 - \omega^2} d\xi \right] d\eta$$



<u>**Evaluate by Residues**</u>

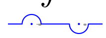
For $x > \text{supp}\, f$,

$$\int e^{i\xi x} \frac{\widehat{f}(\xi,\eta)}{\xi^2 - \omega^2} d\xi \underset{R\to\infty}{\longrightarrow} 0$$



so

$$\begin{aligned}\int e^{i\xi x}\frac{\widehat{f}(\xi,\eta)}{\xi^2-\omega^2}d\xi &= 2\pi i\,(\text{Residue at }\omega)\\ &= 2\pi i\frac{e^{i\omega x}\widehat{f}(\omega,\eta)}{2\omega}\end{aligned}$$



$$u=\frac{1}{2\pi}\int_{\mathbb{R}} e^{i(\eta y+\omega x)}\frac{\widehat{f}(\omega,\eta)}{2i\omega}d\eta$$

Integrable Singularity Note that $\frac{1}{\omega}=\frac{1}{\sqrt{k^2-\eta^2}}$ is singular at $\eta=\pm k$, but the singularity is integrable. This means that the integral above is unambiguously defined.

The Hankel Contour

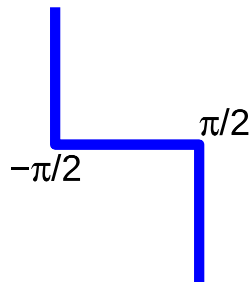
$$u=\frac{1}{2\pi}\int_{\mathbb{R}} e^{i(\eta y+\omega x)}\frac{\widehat{f}(\omega,\eta)}{2i\omega}d\eta$$

$$\eta^2+\omega^2=k^2$$

so a trigonometric substitution will remove the singularity.

$$\begin{aligned}\eta &= k\sin\theta\\ \omega &= k\cos\theta\\ d\theta &= \frac{d\eta}{\omega}\end{aligned}$$

Its obvious that this change of variables takes $\eta\in[-1,1]$ to $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$, but we need to make θ complex to cover the intervals $(-\infty,-1]$ and $[1,\infty)$.



$$\gamma(t) = \begin{cases} -\frac{\pi}{2} - i(t+\frac{\pi}{2}) & t < -\frac{\pi}{2} \\ t & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ \frac{\pi}{2} - i(t-\frac{\pi}{2}) & t > \frac{\pi}{2} \end{cases}$$

$$\sin\gamma(t) = \begin{cases} -\cosh(t+\frac{\pi}{2}) & t < -\frac{\pi}{2} \\ \sin t & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ \cosh(t-\frac{\pi}{2}) & t > \frac{\pi}{2} \end{cases} \qquad \cos\gamma(t) = \begin{cases} -i\sinh(t+\frac{\pi}{2}) & t < -\frac{\pi}{2} \\ \cos t & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ yi\sinh(t-\frac{\pi}{2}) & t > \frac{\pi}{2} \end{cases}$$

For $x > \text{supp } f$

$$u(x,y) = \frac{1}{2\pi i}\int_\gamma e^{ik(x\cos\theta + y\sin\theta)}\widehat{f}(k\cos\theta, k\sin\theta)d\theta$$

or, in polar coordinates

$$u(r,\phi) = \frac{1}{2\pi i}\int_\gamma e^{ikr\cos(\theta-\phi)}\widehat{f}(k,\theta)d\theta$$

This is only valid, and (the integral can only be shown to converge), for $r\cos\phi > \text{supp } f$. But we can shift the contour of integration:

$$\begin{aligned} u(r,\phi) &= \frac{1}{2\pi i}\int_{\gamma_\phi} e^{ikr\cos(\theta-\phi)}\widehat{f}(k,\theta)d\theta \\ &= \frac{1}{2\pi i}\int_{\gamma_0} e^{ikr\cos\theta}\widehat{f}(k,\theta+\phi)d\theta \end{aligned}$$

This integral converges and gives $u(r,\phi)$ for all points outside the convex hull of the support of f — although I haven't quite explained why this is so.

Method of Steepest Descent

We want to evaluate (approximate)

$$u(r,\phi) \;=\; \frac{1}{2\pi i}\int\limits_{\gamma_0} e^{kr\Phi(\theta)}\widehat{f}(k,\theta+\phi)d\theta$$

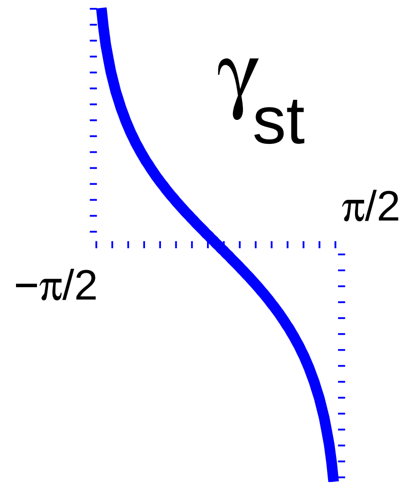
where $\Phi(\theta) = i\cos\theta$ as $r \longrightarrow \infty$ for fixed k.

1. Φ is imaginary so e^{Φ} oscillates between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$
2. Φ is real and negative on the vertical segments so e^{Φ} decays (rapidly)
3. We expect only the stationary point at $\theta = 0$ to contribute significantly for large r.

Steepest Descent Contour

1. passes through stationary point
2. imaginary Φ is constant so no oscillation in e^{Φ}. Nothing cancels
3. real Φ gets more negative as we move away from the stationary point, so only a small neighborhood of the stationary point will contribute significantly.

$$\begin{aligned}\theta &= \sigma + i\beta \\ \Phi(\theta) &= i\cos(\sigma+i\beta) = i\cos(\sigma)\cosh(\beta) + \sin(\sigma)\sinh(\beta)\end{aligned}$$



$$\begin{aligned}
\gamma_{st} &= \{\Im\Phi(\sigma + i\beta) = \Im\Phi(0) = 1\} \\
&= \{\cos\sigma \cosh\beta = 1\} \\
\cosh\beta &= \frac{1}{\cos\sigma} \\
e^{\beta} + e^{-\beta} &= \frac{2}{\cos\sigma} \\
0 &= (e^{\beta})^2 - \frac{2}{\cos\sigma}(e^{\beta}) + 1 \\
\beta &= \log\left(\frac{1 - \sin\sigma}{\cos\sigma}\right)
\end{aligned}$$

On γ_{st}, $\Phi = i + \sin\sigma \sinh\beta$

$$\begin{aligned}
u(r,\phi) &= \frac{e^{ikr}}{2\pi i} \int\limits_{\gamma_{st}} e^{kr \sin\sigma \sinh\beta} \widehat{f}(k, \sigma + i\beta + \phi), d(\sigma + i\beta) \\
&= \frac{e^{ikr}}{2\pi i} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-kr\frac{\sin^2\sigma}{\cos\sigma}} \widehat{f}(k, \sigma + i\beta + \phi)(1 - \frac{i}{\cos\sigma}) d\sigma
\end{aligned}$$

We could use these exact formulas to see the asymptotic behavior of u. Instead, we proceed in a way that is a little less explicit, but is a better template for applying the method to other problems.

We divide the contour integral into two parts; The δ in the calculation below will wind up being about $\frac{\pi}{4}$ and the corresponding ϵ about $\frac{1}{\sqrt{2}}$.

$$\left| \int\limits_{\gamma\cap\{|\theta|>\delta\}} e^{ikr\cos\theta}\widehat{f}(k,\theta+\phi)d\theta \right| \leq \int\limits_{\gamma\cap\{|\theta|>\delta\}} e^{-kr\sin\sigma\sinh\beta}|\widehat{f}(k,\theta+\phi)||d\theta|$$

$$\leq \int\limits_{\gamma\cap\{|\theta|>\delta\}} e^{-kr\sin\sigma\sinh\beta}e^{kR\sinh\beta}||f||_{L^1}|d\theta|$$

where R is the radius of the smallest ball containing the support of f

$$= \int\limits_{\gamma\cap\{|\theta|>\delta\}} e^{(-kr\sin\sigma+kR)\sinh\beta}|d\theta| \quad ||f||_{L^1}$$

On γ_{st}, $|\theta|>\delta$ implies that $|\sigma|>\epsilon$ and $|\beta|>\epsilon$, and $|d\theta|\leq d\beta$ (you can check this directly from the formula, or believe the picture, or think about what a steepest descent contour has to do) so that

$$\begin{aligned} &\leq \int_\epsilon^\infty e^{(-kr\sin\epsilon+kR)\sinh(\beta)}d\beta \\ &\leq \int_\epsilon^\infty e^{(-kr\sin\epsilon+kR)\sinh(\beta)}\frac{\cosh\beta}{\cosh\epsilon}d\beta \\ &= \frac{e^{(-kr\sin\epsilon+kR)\sinh(\epsilon)}}{(kr\sin\epsilon-kR)\cosh\epsilon} \end{aligned}$$

The conclusion is that this part of the integral is exponentially decaying in kr. The other part contains the behavior that dominates when kr is large.

$$\int\limits_{\gamma\cap\{|\theta|<\delta\}} e^{ikr\cos\theta}\widehat{f}(k,\theta+\phi)d\theta = e^{ikr}\int\limits_{\gamma\cap\{|\theta|<\delta\}} e^{2ikr\sin^2\left(\frac{\theta}{2}\right)}\widehat{f}(k,\theta+\phi)d\theta$$

Here we can make the substitution $w=\sin\frac{\theta}{2}$ to obtain

$$= e^{ikr}\int\limits_{\gamma\cap\{|\theta|<\delta\}} e^{2ikrw^2}\widehat{f}(k,2\arcsin(w)+\phi)\frac{2}{\sqrt{1-w^2}}dw$$

and use a convergent series expansion,

$$
\begin{aligned}
&= e^{ikr} \int\limits_{\gamma\cap\{|\theta|<\delta\}} e^{2ikrw^2} \left(1+\frac{1}{2}w^2+\dots\right)\left(\widehat{f}(k,\phi)+\widehat{f}'(k,\phi)w+\dots\right) dw \\
&= e^{ikr} \left(\widehat{f}(k,\phi) \int\limits_{\gamma\cap\{|\theta|<\delta\}} e^{2ikrw^2} dw + \widehat{f}'(k,\phi) \int\limits_{\gamma\cap\{|\theta|<\delta\}} e^{2ikrw^2} w dw + \dots \right)
\end{aligned}
$$

Finally set $v = \sqrt{kr}w$ to reveal the dependence on kr

$$
= e^{ikr} \left(\widehat{f}(k,\phi) \int\limits_{\gamma\cap\{|\theta|<\delta\}} e^{2iv^2} \frac{dv}{\sqrt{kr}} + \widehat{f}'(k,\phi) \int\limits_{\gamma\cap\{|\theta|<\delta\}} e^{2ikrw^2} \frac{vdv}{(kr)} + \dots \right)
$$

and remember that we are on the steepest descent contour[1] $v = \sqrt{i}\tau$

$$
\begin{aligned}
&= \frac{e^{ikr}\widehat{f}(k,\phi)\sqrt{i}}{\sqrt{kr}} \left(\int_{-kr \arcsin\delta}^{kr \arcsin\delta} e^{-2\tau^2} d\tau + \dots \right) \\
&= \frac{e^{ikr}\widehat{f}(k,\phi)\sqrt{i}}{\sqrt{kr}} \left(\int_{-\infty}^{\infty} e^{-2\tau^2} d\tau + \dots \right)
\end{aligned}
$$

[1]All of our changes of variables have been analytic in a ball about $\theta = 0$ and preserved the real axis. The steepest descent contour, originally defined to be the one on which the real part of an analytic function was constant and the imaginary part grew fastest, is still the one that keeps the real part of that analytic function, now described in terms of a different variable v^2, constant.

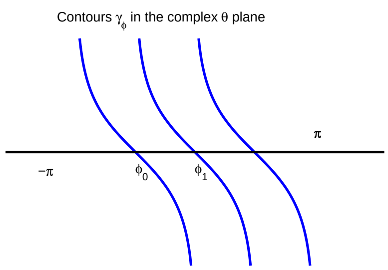
Summary

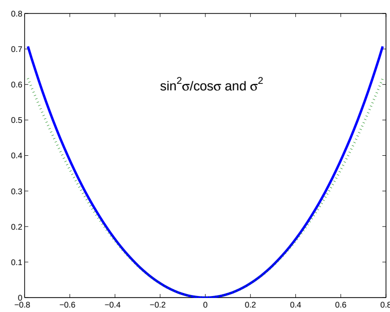
1. Define the Far Field α radiated by the source f:

$$\begin{aligned} \alpha(\theta) &:= \widehat{f}(k\cos\theta, k\sin\theta) \\ &= \widehat{f}(k,\theta) \\ &= \widehat{f}(k,\sigma+i\beta) \end{aligned}$$

2. Outside the convex hull of the support of f, the radiated field u is given by

$$u(r,\phi) \quad = \quad \langle\Gamma_\phi,\alpha\rangle \quad = \frac{1}{2\pi i}\int\limits_{\gamma_\phi} e^{ikr\cos(\theta-\phi)}\alpha(\theta)d\theta \quad \sim \frac{e^{ikr}e^{i\frac{\pi}{4}}}{\sqrt{2\pi kr}}\,\alpha(\phi)$$





3. α is entire and 2π periodic and

$$|\alpha(\sigma+i\beta)| \leq Ke^{s_{\mathrm{supp}\,f}(\sigma)\sinh(\beta)}$$

where $s_{\mathrm{supp}\,f}(\phi)$ is the support function of the set $\Omega = \mathrm{supp}\,f$.

The Support Function

Notation:

$$\Phi = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$$

D is a domain in $\mathbb{R}^2$.

$$
\begin{aligned}
s_D(\Phi) &:= \sup_{x \in D}(\Phi \cdot x) \\
\text{ch }(D) &= \left\{x \middle| x \cdot \Phi \leq s_D(\Phi)\right\}
\end{aligned}
$$

Fact:

$$
s_{\text{ch }(D)}(\Phi) = s_D(\Phi)
$$

<u>**Corollaries of item 3**</u>

1. <u>**Rellich's Lemma**</u> – $\alpha(\theta)$ vanishes (on any subset of $\mathbb{C}$ with a limit point) if and only if supp $u \subset$ ch supp f.
2. If $|\alpha(\phi + i\beta)|$ grows fast enough as β increases, i.e.

$$
|\alpha(\phi + i\beta)| \geq K e^{M \sinh(\beta)}
$$

supp f must include some points above the line $x \cdot \Phi = M$.

4 The Hankel function, Real analytcity and unique continuation

In the following section, we will make use of the fact, that a radiated waves with zero far field is identically zero off the support of its source. We have proven that this is true off the convex hull of the support of the source. The unique continuation property will allow us to extend this conclusion beyond the convex hull.

<u>**Unique Continuation Property**</u> A collection of functions has the unique continuation property if the following holds:

If u and all its derivatives vanish at a point, then u is identically zero.

Theorem 6. *Solutions to the free Helmholtz equation in a path-connected domain D have the UCP.*

Sketch of Proof.

Lemma 7. *Real analytic functions have the UCP.*

Lemma 8. *The Hankel function $H_0(|x|)$ is real analytic in $\mathbb{R}^2 \setminus 0$. Its Taylor series expansion at a point $p \in \mathbb{R}^2$ has a radius of convergence $r(p) = |p|$.*

Lemma 9. *If $v^0(x)$ satisfies*

$$\left(\Delta + k^2\right) v^0 = 0 \qquad \text{in } D$$

and

$$\int_{\partial D} |v^0(y)| ds_y < \infty \tag{4.1}$$

and

$$\int_{\partial D} |\frac{\partial v^0}{\partial \nu}(y)| ds_y < \infty \tag{4.2}$$

then

$$v^0(x) = \int_{\partial D} \left(H_0 \frac{\partial v^0}{\partial \nu} - v^0 \frac{\partial H_0}{\partial \nu} \right) ds \tag{4.3}$$

and v^0 is real analytic in D

□

proof that real analytic function have the UCP. .

Definition of Real Analytic A function is real analytic in a domain D if its Taylor series expansion has a nonzero radius of convergence at every point in D.
A sufficient condtion that a function is real analytic at p with radius of convergence $r(p)$[2] is that its partial derivatives satisfy:

$$|\partial x_1^n \partial x_2^m u(p)| \ \ \leq \ \ \frac{n!m!}{(r(p))^{n+m}} \tag{4.4}$$

[2] Actually, this condition guarantees convergence in the square $\{|x_1| < r\} \cap \{|x_2| < r\}$.

Lemma 10. *If If u is real analytic and $r(p)$ denotes the radius of convergence at a point p, then*

$$r(q) \geq r(p) - |p - q| \tag{4.5}$$

and $r(p)$ is a continuous function of p

Proof. To establish (4.5), we need to show the estimate (4.4) at the point q. We may assume $p = 0$. We will give the proof for a single variable,

$$\begin{aligned} u(q) &= \sum a_n q^n \\ \partial x^k u(q) &= \sum_{n=k}^{\infty} a_n \frac{n!}{(n-k)!} q^{n-k} \\ |\partial x^k u(q)| &\leq \sum_{n=k}^{\infty} (r(p))^n \frac{n!}{(n-k)!} |q|^{n-k} \end{aligned}$$

but we can recognize the series on the right as

$$\begin{aligned} &\leq \frac{\partial}{\partial q^k} \left(\frac{1}{r-q} \right) \\ &= \frac{k!}{(r-q)^k} \end{aligned}$$

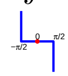
which is the estimate needed in (4.4), with $r - q$ in place of r.
Does the one-dimensional proof imply the two dimensional result by considering $u(q\cos\theta, q\sin\theta)$ as a function of the one-dimensional variable q, for every parameter θ?

Since p and q are interchangeable, continuity is an immediate consequence of (4.5). □

Suppose u is real analytic and vanishes to infinite order at p. Then u is identically zero inside the ball about p with radius $r(p)$. Let q be another point and connect p to q with a (compact) path of finite length.$r(p)$ has a minimum on the path. Cover the path with a finite number of balls of that minimum radius. Each centered at a point within the previous ball. Since u vanishes identically in the first ball, it vanishes to infinite order at the center of the second ball, etc. □

Proof that the Hankel function is real analytic.

$$H_0(kr) = \int e^{ikr\cos(\theta)} d\theta$$



As a function of r, $e^{ikr\cos(\theta)}$ is complex analytic in all of $\mathbb{C}$, so the integral over θ will also be analytic wherever it converges (the integral is a limit of Riemann sums, which we will show converge uniformly for r in compact subsets of $\mathbb{C} \setminus \{0\}$). The only place convergence is an issue is on the vertical segments. Along the segment $\gamma = -i\frac{\pi}{2} + t$ the integral is dominated by

$$\int_0^\infty e^{-re^{-\psi}} d\psi$$

which converges uniformly as long as $\Re r \geq \delta > 0$ is positive.The other vertical segment is estimated similarly.

Taylor series for complex analytic functions converge in the largest ball for which the function is analytic, so the series for $H_0(r - |p|)$ has radius of convergence $r(p) = |p|$.

Warning I have been a bit cavalier about equating the convergence of the Taylor series for $H_0(r - |p|)$ on the real axis with that for $H_0(|x - p|)$ in $\mathbb{R}^2$. I should really say something about composition or real analytic functions and square roots.

Modifying the contour from γ_0 to γ_ϕ, as we did on page ???, will give an analytic continuation of H_0 to any $r \in \mathbb{C} \setminus \{0\}$, but 0 is a branch point with a logarithmic singularity. We can see this by returning to

$$\begin{aligned}
\int_0^\infty e^{-re^{-\psi}} d\psi &= \int_0^\infty e^{-e^{\log r - \psi}} d\psi \\
&= \int_{-\log r}^\infty e^{-e^{-\tau}} d\tau \\
&= \int_{-\log r}^0 e^{-e^{-\tau}} d\tau + \int_0^\infty e^{-e^{-\tau}} d\tau
\end{aligned}$$

the second integral is a constant and the first is bounded (both above and below) by a multiple of $\log r$. ☐

Proof that v^0 is real analytic. .

The formula

$$v^0(x) = \int\limits_{\partial D} \left(H_0(|x-y|)\frac{\partial v^0}{\partial \nu} - v^0 \frac{\partial H_0(|x-y|)}{\partial \nu} \right) ds_y$$

shows $v^0(x)$ as a limit of Riemann sums of $H_0(|x-y_k|)$ and its derivative. Each of the functions $H_0(|x-y_k|)$ is real analytic at $x = p$ with radius of convergence $r_k = |p-y_k|$, All the y_k's are on the boundary of B, so all the individual Taylor series converge in any ball about $x = p$ which doesn't intersect the boundary. The finite sum of real analytic functions is real analytic, so each Riemann sum is real analytic in a ball of radius $d(p)$, where $d(p)$ is the distance from p to the boundary of D.
Passing to the limit requires a little care. One way to prove a function is real analytic is to estimate its derivatives. The derivatives of $H_0(|x-y|)$ satisfy

$$|\partial x_1^n \partial x_2^m H_0(|p-y|)| \leq \log|p-y| \frac{1}{|p-y|^{n+m}} n!m!$$

because of the fact that $H_0(r)$ is holomorphic (Cauchy integral formula) and the estimate $|H_0(r)| \leq \log(|r|)$. We can differentiate under the integral sign and check that

$$\begin{aligned}
|\partial x_1^n \partial x_2^m v^0(p)| &\leq \int\limits_{\partial B} \log|p-y| \left(\frac{1}{|p-y|^{n+m}} |\frac{\partial v^0}{\partial \nu}(y)| + \frac{1}{|p-y|^{n+m+1}} |v^0| \right) ds_y \\
&\leq \log(d(p)) \frac{1}{d(p)^{n+m+1}} n!m! \int\limits_{\partial D} \left(|v^0(y)| + |\frac{\partial v^0}{\partial \nu}(y)| \right) ds_y
\end{aligned}$$

where $d(p)$ is the distance to the boundary. This shows that as long as we know that the boundary values of v^0 are integrable, v^0 is analytic in the interior. ☐

5 Inverse Source Problem

Question: Deduce some useful information about f from its far field, $\alpha(\theta) = \widehat{f}(k\cos(\theta), k\sin(\theta))$.

5.1 Non-Radiating Sources, and Equivalent Sources

Reminder: All sources are compactly supported.

The radiated wave is the causal/outgoing solution to the Helmholtz equation outside the support of the source.

The far field is the asymptotic behavior of the radiated wave.

Non-radiating Source: A source is non-radiating if the following equivalent conditions are satisfied:

1. The far field vanishes.
2. The radiated wave vanishes.

Equivalent Sources: Two sources are equivalent if they radiate the same wave.

***D*-Free waves** satisfy the free Helmholtz equation in the domain D.

$$\left(\Delta + k^2\right) v^0 = 0$$

Theorem 11. *A source $f \in L^2(D)$ is non-radiating if and only if either of the conditions below hold*

1. $f = (\Delta + k^2)\,\Phi_{00}$
2. *f is orthogonal to all D- free waves*

where the notation Φ_{00} means that ($\Phi \in H_0^2(D)$) Φ and its first order and second order derivatives are square integrable and Φ vanishes outside D.

Proof. If f is non-radiating, then the radiated wave is the Φ_{00}. If 1 holds then Φ_{00} satisfies the (SRC), so is the unique outgoing solution to the Helmholtz equation, so $f$is non-radiating.

$$
\begin{aligned}
\int_D v^0 f &= \int_D v^0 \left(\Delta + k^2\right) u^+ \\
&= \int_D \left(\Delta + k^2\right) v^0 u^+ + \int_{\partial D} v^0 \frac{\partial u^+}{\partial \nu} - u^+ \frac{\partial v^0}{\partial \nu} \\
&= 0 + \int_{\partial D} v^0 \frac{\partial u^+}{\partial \nu} - u^+ \frac{\partial v^0}{\partial \nu}
\end{aligned}
$$

If f is non-radiating, then u^+ and its first derivatives vanish on ∂D, so 2 holds. If 2 holds, then choose $v^0 = e^{ik\Theta \cdot x}$ to see that

$$
\begin{aligned}
0 &= \int e^{ik\Theta \cdot x} f(x) dx \\
&= \widehat{f}(k\Theta)
\end{aligned}
$$

so the far field vanishes and f is non-radiating. □

Theorem 12. *Every source supported in D has an equivalent D-free source (supported in D).*

Proof. The fourth order PDE,

$$
\left(\Delta + k^2\right)^2 \Phi_{00} = \left(\Delta + k^2\right) f \tag{5.1}
$$

has a unique solution (this requires a proof I haven't given – the unbounded operator is self adjoint, so has compact resolvent; if zero were an eigenvalue, this would violate uniqueness for the Cauchy problem for $(\Delta + k^2)$).
Because $(\Delta + k^2)\,\Phi_{00}$ is non-radiating, $f - (\Delta + k^2)\,\Phi_{00}$ radiates the same far field as $f$and is $D - free$ because of (5.1). □

Theorem 13. *Fix any open neighborhood of ∂D and call it $N_\epsilon(\partial D)$. Every source supported in D has an equivalent source supported in $N_\epsilon(\partial D)$.*

Proof. Let u^+ be the radiated wave and let ϕ_ϵ be a smooth function which is identically equal to one in the unbounded component of $\mathbb{R}^2 \setminus (N_\epsilon(\partial D)$ and identically equal to zero in all bounded components of $\mathbb{R}^2 \setminus (N_\epsilon(\partial D)$. It is easy to check that $f = (\Delta + k^2)\,(\phi u^+)$ is supported in $N_\epsilon(\partial D)$ and that ϕu^+, which has the same far field as u^+, is the unique outgoing solution. □

5.2 The Support of a Far Field

We want to say something about the support of a source that radiates a given far field. The theorems above show that:

1. If f radiates α and $\operatorname{supp} f \subset D$, then there exists another source (e.g. the unique D-free source) whose support is all of D. Hence we can always make the support bigger without changing the far field.

2. If $\operatorname{supp} f$ is properly contained in the interior of D, so that $\operatorname{supp} f \cap N_\epsilon(\partial D) = \emptyset$, then we can find another source with support that is disjoint from the support of f. In fact, we can find a combination of single and double layer potentials supported exactly on (∂D).

Carrier A set D carries a far field α if, for every $\epsilon > 0$, there exists a source f, supported in $N_\epsilon(D)$, that readiates α.

Minimal Carrier D is a minimal carrier for α if no proper subset of D carries α.

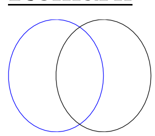
Theorem 14. *If α has a compact carrier, then α has a unique minimal conxex carrier.*

Minimal *Convex* Carrier D is a minimal *convex* carrier for α if no proper *convex* subset of D carries α.

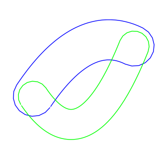
The convex support of a Far Field is the unique minimal convex carrier.

Lemma 15. *If Ω_1 and Ω_2 carry α, and $\mathbb{R}^n \setminus (\Omega_1 \cup \Omega_2)$ is connected, then $\Omega_1 \cap \Omega_2$ carries α.*

Remark – $\mathbb{R}^n \setminus (\Omega_1 \cup \Omega_2)$ is connected means:

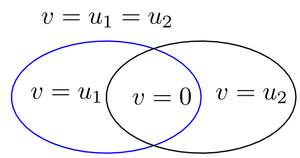


and not



Proof.

$$\begin{aligned}(\Delta + k^2)u_1 &= F_1 \\ (\Delta + k^2)u_2 &= F_2\end{aligned}$$



$$v = \begin{cases} \phi u_1, & x \in \mathbb{R}^n \backslash \Omega_1 \\ \phi u_2, & x \in \mathbb{R}^n \backslash \Omega_2 \\ 0, & x \in \Omega_1 \cap \Omega_2 \end{cases}$$

$$F_3 = (\Delta + k^2)v$$

v is a well-defined function as long as $u_1 = u_2$ on $\mathbb{R}^n \setminus (\Omega_1 \bigcup \Omega_2)$, which follows from Rellich's lemma and unique continuation. □

Proof. **of theorem 14**
The unique minimal convex set must be the intersection of all convex sets that carry α.

$$\text{csupp } \alpha := \bigcap_{D \in \mathcal{D}} D$$

We must show that csupp α carries α. We first will show that, for any $\epsilon > 0$, there is a finite subcollection such that

$$N_\epsilon(\text{csupp } \alpha) \supset \bigcap_{n=1}^{N} D_n \tag{5.2}$$

Once we have established (5.2), then we can use induction and the intersection lemma to complete the proof. Convexity is used in two places:

1. The intersection of convex sets is convex.

2. The complement of the union of two convex sets is connected. *The complement of the union of three convex sets need not be connected. A triangle is the union of three line segments*

We will deduce (5.2) from the fact that every open cover of a compact set has a finite subcover.

Since α has at least one compact carrier, say B, csupp $\alpha \subset B$

$$\text{csupp }\alpha = \bigcap_{D\in\mathcal{D}} D\cap B$$

Taking complements

$$\begin{aligned} B\setminus \text{csupp }\alpha &= \bigcup_{D\in\mathcal{D}} B\setminus(D\cap B) \\ B\setminus N_\epsilon(\text{csupp }\alpha) &\subset \bigcup_{D\in\mathcal{D}} B\setminus(D\cap B) \end{aligned}$$

Now $B\setminus N_\epsilon(\text{csupp }\alpha)$ is compact and $B\setminus(D\cap B)$ are open (as subsets of B), so a finite subcollection covers, i.e.

$$B\setminus N_\epsilon(\text{csupp }\alpha) \subset \bigcup_{n=1}^{N} B\setminus(D_n\cap B)$$

or, taking complements

$$N_\epsilon(\text{csupp }\alpha) \supset \bigcap_{n=1}^{N}(D_n\cap B)$$

□

Unions of Well Separated Convex Sets - A set is UWSCS if

$$B=\bigcup B_k$$
$$\text{Each } B_k \text{ is convex}$$
$$\text{diam}(B_k) < \operatorname*{dist}_{j\neq k}(B_k, B_j)$$

Theorem 16. *If α has a compact carrier, then α has a unique minimal UWSCS carrier.*

5.3 Thin Sources — Single and Double layers

Suppose that D is a smooth domain, and we have two smooth functions, one defined inside D and the other defined outside D.

$$v = \begin{cases} v_1 & \text{for } x \in D \\ v_2 & \text{for } x \in \mathbb{R}^2 \setminus D \end{cases}$$

We want to define $(\Delta + k^2)\, v$. If v^ϵ are smooth functions (across ∂D) that approximate v, and Φ is an arbitrary smooth compactly supported function, then

$$\int_{\mathbb{R}^2} \Phi \left(\Delta + k^2\right) v^\epsilon \;\; = \;\; \int_{\mathbb{R}^2} v^\epsilon \left(\Delta + k^2\right) \Phi$$

but the limit on the right hand side always exists, so we may define:

$$\langle \left(\Delta + k^2\right) v, \Phi \rangle \;\; := \;\; \int_{\mathbb{R}^2} v \left(\Delta + k^2\right) \Phi$$

We have defined $(\Delta + k^2)\, v$ by saying what is does to a smooth function Φ. This makes sense for any integrable function v, but for a v which is smooth inside and outside D, we can compute a more explicit formula.

$$\begin{aligned}
\langle \left(\Delta + k^2\right) v, \Phi \rangle \;\; &= \;\; \int_{\mathbb{R}^2} v \left(\Delta + k^2\right) \Phi \\
&= \;\; \int_{D} v \left(\Delta + k^2\right) \Phi + \int_{\mathbb{R}^2 \setminus D} v \left(\Delta + k^2\right) \Phi \\
&= \;\; \int_{D} \Phi \left(\Delta + k^2\right) v + \int_{\partial D} \Phi \frac{\partial v}{\partial \nu} - v \frac{\partial \Phi}{\partial \nu} \\
&\quad\;\; + \int_{\mathbb{R}^2 \setminus D} \Phi \left(\Delta + k^2\right) v - \int_{\partial D} \Phi \frac{\partial v}{\partial \nu} - v \frac{\partial \Phi}{\partial \nu} \\
&= \;\; \int_{D} \Phi \left(\Delta + k^2\right) v_1 + \int_{\mathbb{R}^2 \setminus D} \Phi \left(\Delta + k^2\right) v_2 + \int_{\partial D} \Phi [\frac{\partial v}{\partial \nu}] - [v] \frac{\partial \Phi}{\partial \nu}
\end{aligned}$$

where $[v] = v_1 - v_2$ means the jump in v across ∂D.

If we choose v_1 and v_2 to be solutions to the free Helmholtz equation in D and $\mathbb{R}^2 \setminus D$, respectively, then

$$\langle \left(\Delta + k^2\right) v, \Phi \rangle = \int\limits_{\partial D} \Phi [\frac{\partial v}{\partial \nu}] - [v] \frac{\partial \Phi}{\partial \nu}$$

which we write as

$$\left(\Delta + k^2\right) v = [\frac{\partial v}{\partial \nu}] \delta_{\partial D} + [v] \delta'_{\partial D}$$

For a smooth curve γ, we define the dirac delta on γ by

$$\begin{aligned} \langle \delta_\gamma, \Phi \rangle &= \int\limits_\gamma \Phi d\gamma = \int_0^L \Phi(\gamma(s)) ds \\ \langle \delta'_\gamma, \Phi \rangle &= \int\limits_\gamma \frac{\partial \Phi}{\partial \nu} d\gamma = \int_0^L \frac{\partial \Phi}{\partial \nu}(\gamma(s)) ds \end{aligned}$$

where s is the arclength parameter.

A thin source is a distribution that is supported on a set of measure zero (a single or a double layer on a curve) such that the solution to the Helmholtz equation is a (locally L^1) function.

Example of a thin source

$$f = a\delta_\gamma + b\delta'_\gamma$$

where a and b are functions defined on γ.

Theorem 17. *A thin source ω is non-radiating if and only if*

1. supp ω *is the boundary of a bounded open set D*

2. ω is the Cauchy data of a D-free wave restricted to ∂D, i.e.

$$\omega = \frac{\partial v^0}{\partial \nu}\delta_{\partial D} + v^0 \delta'_{\partial d}$$

where

$$\left(\Delta + k^2\right) v^0 = 0 \quad \text{inside } D$$

Proof in the case that $\operatorname{supp}\omega$ *is a smooth curve.* The if direction is easy, if we start with a D and a v^0, then define

$$u = \begin{cases} v^0 & x \in D \\ 0 & x \in \mathbb{R}^2 \setminus D \end{cases}$$

then the jump formula tells us that $\left(\Delta + k^2\right) u = \omega$ and because $u = 0$ outside D, it satisfies the (SRC).

For the other direction, observe that $\mathbb{R}^2 \setminus \operatorname{supp}\omega$ has one unbounded and (possibly) one or more bounded components. If ω is non- radiating, then the solution $u$to

$$\left(\Delta + k^2\right) u = \omega$$

is identically zero on the unbounded component, and satisfies $\left(\Delta + k^2\right) u = 0$ in the open bounded components D.

If D has a smooth boundary, then

$$\left(\Delta + k^2\right) u = \omega = [\frac{\partial u}{\partial \nu}]\delta_{\partial D} + [u]\delta'_{\partial D}$$

but u is zero outside D, so the jump across D is just the boundary values of u from the inside.

If $\mathbb{R}^2 \setminus \operatorname{supp}\omega$ contains a curve which is not part of the boundary of a bounded component, then $u \equiv 0$ on both sides of that curve $\operatorname{supp}\omega$, so the jumps are zero, hence $\omega \equiv 0$.

The same proof holds without smoothness assumptions, but we need to use the language of distributions to replace the jump formulas.

□

Arcs — A set γ of measure zero is an arc if $\mathbb{R}^2 \setminus \gamma$ is connected.
Arcs are the opposite of boundaries.

Theorem 18. *Arcs are minimal carriers, i.e. if ω radiates α and $\gamma = \operatorname{supp} \omega$ is is an arc, then no source supported on a subset of γ can radiate α.*

Proof. If u is any solution to

$$\left(\Delta + k^2\right) u = \omega$$

then, the jump formula tells us that

$$\omega = [\frac{\partial u}{\partial \nu}]\delta_\gamma + [u]\delta'_\gamma$$

Let ω_2 and be another source supported in γ that radiates α, and u_2 the radiated wave.

$$\left(\Delta + k^2\right) u_2 = \omega_2$$

Then, the jump formula tells us that

$$\omega_2 = [\frac{\partial u_2}{\partial \nu}]\delta_\gamma + [u_2]\delta'_\gamma$$

but Rellich's lemma and unique continuation tell us that $u = u_2$ on $\mathbb{R}^2 \setminus \gamma$, so $\omega_2 = \omega$, and therefore $\operatorname{supp} \omega_2 = \operatorname{supp} \omega = \gamma$. □

Examples of Equivalent Minimal Sources.

1. The Cauchy data of a free solution, restricted to a boundary ∂D is a non-radiating source ω.

2. Cut the boundary into 2 halves; $\partial D = \gamma_1 \cup \gamma_2$.

3. ω_1 and $-\omega_2$ are equivalent sources.

4. γ_1 and γ_2 are minimal carriers.

5. Ergo, there cannot exist a unique minimal carrier.

Example 1 Let $v^0 = e^{ik\Theta\cdot x}$ or $v^0 = J_0(kx)$,then

$$\omega = \frac{\partial v^0}{\partial \nu}\delta_\gamma + v^0\delta'_\gamma$$

is nonradiating, so

$$\omega_1 = \frac{\partial v^0}{\partial \nu}\delta_{\gamma_1} + v^0\delta'_{\gamma_1}$$

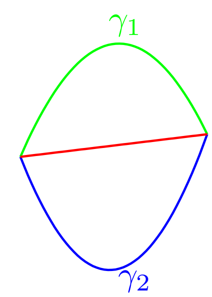
and

$$-\omega_2 = -\frac{\partial v^0}{\partial \nu}\delta_{\gamma_2} - v^0\delta'_{\gamma_2}$$

are equivalent minimal sources. Moreover,

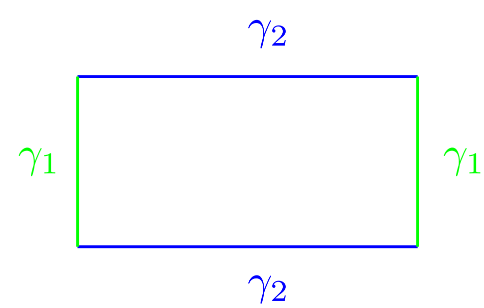
$$\omega_3 = \frac{\partial v^0}{\partial \nu}\delta_{\gamma_3} + v^0\delta'_{\gamma_3}$$

is also equivalent,and γ_3 is the unique minimal convex carrier.



Example 2

We define ω_1 and $-\omega_2$ just as in the previous example. Both sources are equivalent and both γ_1 and γ_2 are minimal carriers that are unions of convex sets. Only γ_1, however, is well-separated.



6 Computing the Support of a Far Field

The Restricted Restricted Fourier Transform and its Singular Value Decomposition

$$\begin{array}{rcl} L^2(B_R(0)) & \xrightarrow{\mathcal{F}_R} & L^2(S^1) \\ f(r,\phi) & \xmapsto{\mathcal{F}_R} & \widehat{f}(k,\theta) \end{array}$$

Theorem 19.

$$\mathcal{F}_R \;=\; \sum \left[\frac{2\pi}{k^2}\sigma_n(kR)\right]\left[\frac{e^{in\theta}}{\sqrt{2\pi}}\right]\otimes\left[\frac{k^2 e^{in\phi}J_n(kr)}{\sqrt{2\pi}\sigma_n(kR)}\right]$$

So that the singular values of $\mathcal{F}_R$ *are* $\frac{2\pi}{k^2}\sigma_n(kR)$ *with*

$$\sigma_n^2(s) = \int_0^s |J_n(r)|^2 r dr$$

Proof.

$$\begin{aligned}\mathcal{F}_R f &= \int_0^R \int_0^{2\pi} e^{ikr\cos(\theta-\phi)} f(r,\phi) r dr d\phi \\ \mathcal{F}_R^* \alpha &= \int_0^{2\pi} e^{-ikr\cos(\theta-\phi)}\alpha(\theta) d\theta\end{aligned}$$

We apply $\mathcal{F}_R^*$ to $e^{in\theta}$

$$\begin{aligned}\mathcal{F}_R^* e^{in\theta} &= \int_0^{2\pi} e^{-ikr\cos(\theta-\phi)} e^{in\theta} d\theta \\ &= \int_0^{2\pi} e^{-ikr\cos(\psi)} e^{in\psi} d\psi e^{in\phi} \\ &= (-i)^n J_n(kr) e^{in\phi}\end{aligned}$$

We conclude that

$$\mathcal{F}_R \;=\; \sum e^{in\theta}\otimes J_n(kr)e^{in\phi}$$

and normalize the right and left eigenfunctions

$$= \;\sum 2\pi\beta_n \frac{e^{in\theta}}{\sqrt{2\pi}} \otimes \frac{e^{in\phi}J_n(kr)}{\sqrt{2\pi}\beta_n}$$

where

$$\begin{aligned}
\beta_n^2 &= \int_0^R |J_n(kr)|^2 r dr \\
&= \frac{1}{k^2}\int_0^R |J_n(kr)|^2 kr d(kr) \\
&= \frac{1}{k^2}\int_0^{kR} |J_n(s)|^2 s ds \\
&= \frac{1}{k^2}\sigma_n(kR)
\end{aligned}$$

□

Corollary 20. *$\alpha(\theta)$ is the restricted Fourier transform of a function $f \in L^2(B_R(0))$, if and only if*

$$\alpha(\theta) = \sum \alpha_n e^{in\theta}$$

with

$$\sum |\tfrac{\alpha_n}{\sigma_n(kR)}|^2 < \infty$$

and

$$\sum |\tfrac{\alpha_n^p}{\sigma_n(kR)}|^2 = ||f||_{L^2(B_R(0))}$$

Corollary 21. *$\alpha(\theta)$ is the restricted Fourier transform of a function $f \in L^2(B_R(p))$, if and only if*

$$e^{i|p|\cos(\theta-\phi_p)}\alpha(\theta) = \sum \alpha_n^p e^{in\theta} \tag{6.1}$$

with

$$\sum |\tfrac{\alpha_n^p}{\sigma_n(kR)}|^2 < \infty \tag{6.2}$$

Proof. The Fourier Transform of a translated function is the Fourier transform of the function multiplied by a phase factor, i.e.

$$\widehat{f(x-p)} = \widehat{f} e^{ip\cdot\xi}$$

or, in polar coordinates

$$= \widehat{f} \times e^{i\rho|p|\cos(\theta-\phi_p)}$$

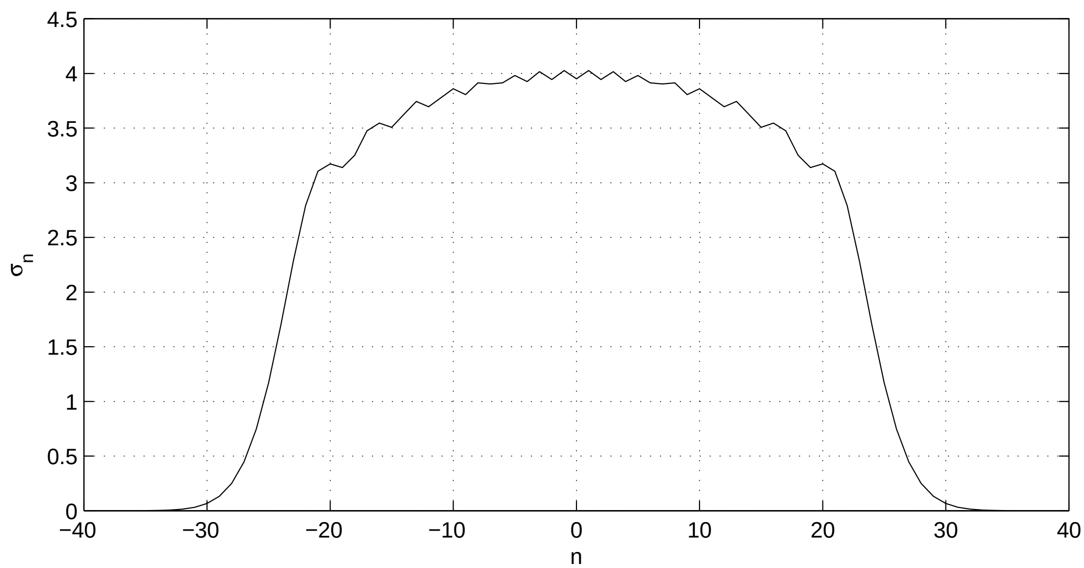
Setting $\rho = k$ gives formula for the restricted Fourier transform. □

Corollary 22. *$\alpha(\theta)$ is the restricted Fourier transform of a function $f \in \bigcap_{j=1}^{N} L^2(B_{R_j}(p_j))$, if and only if (6.1) and (6.2) hold for every j.*

Proof. We just combine the intersection lemma 15 with the previous corollary 21. ☐

Rapid Transition to Evanescence

Below is a plot of $\sigma_n(25)$ as a function of n.

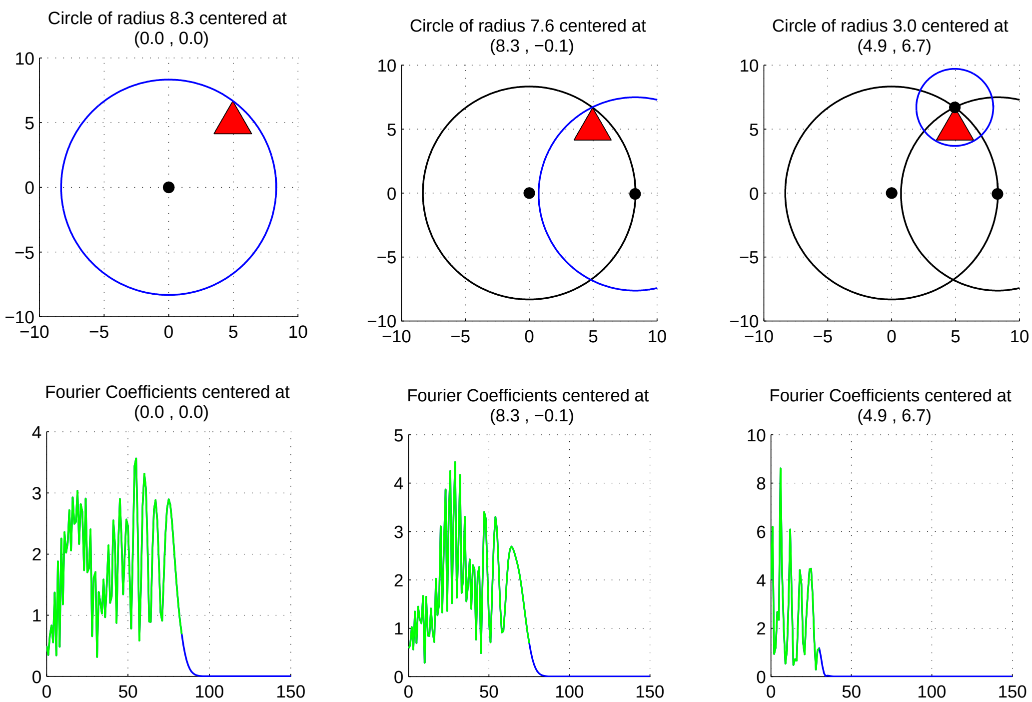


- $\sigma_n(R)$ uniformly big if $n < R$
- $\sigma_n(R)$ uniformly small if $n > R$
- Uniform contrast between big and small

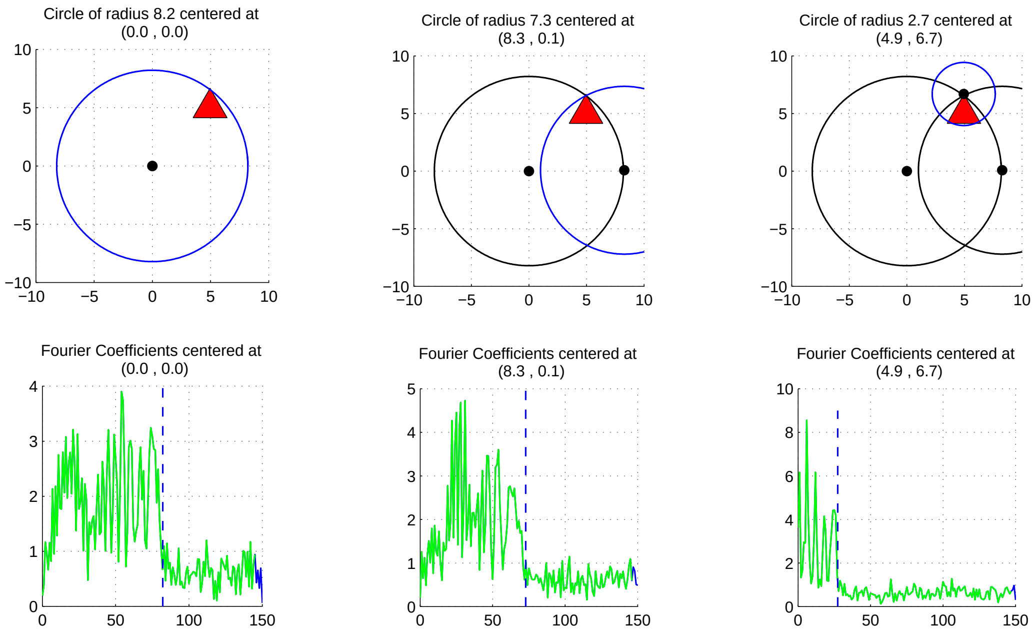
An Algorithm based on the Rapid Transition

- Expand the far field $\alpha(\theta)$ in a Fourier series
- Plot the modulus of the Fourier coefficients.
- Find the value of n where they become effectively zero.
- The convex support of the far field is contained in the ball of radius $R = \frac{n}{k}$ centered at zero.
- Replace $\alpha(\theta)$ by $e^{ik\Theta \cdot c}\alpha = e^{ik|c|\cos(\theta - \phi_c)}\alpha(\theta)$ and repeat.

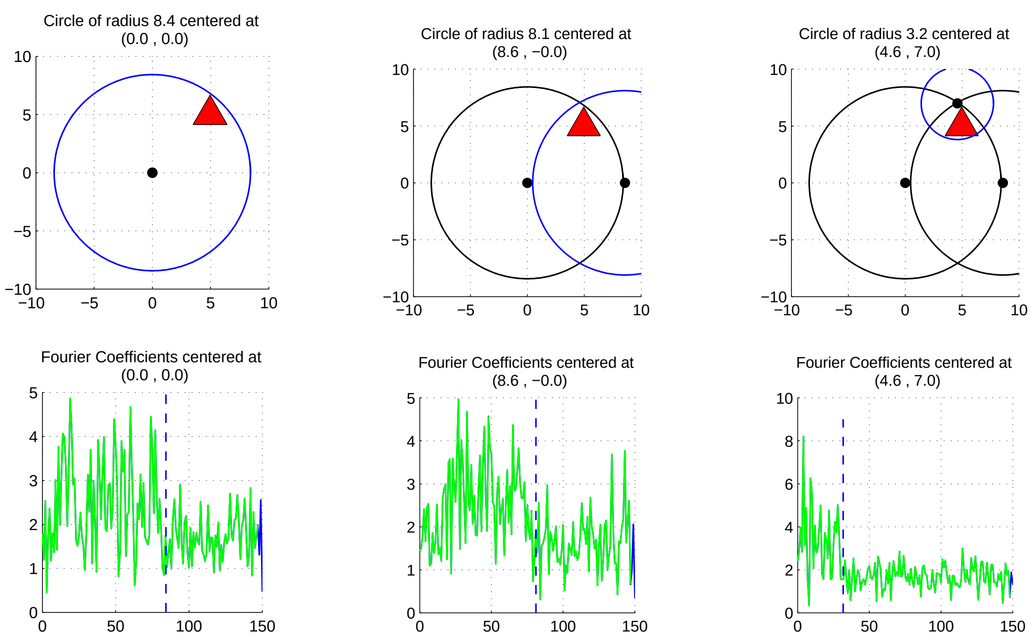
Locating the Scattering Support (k = 10)



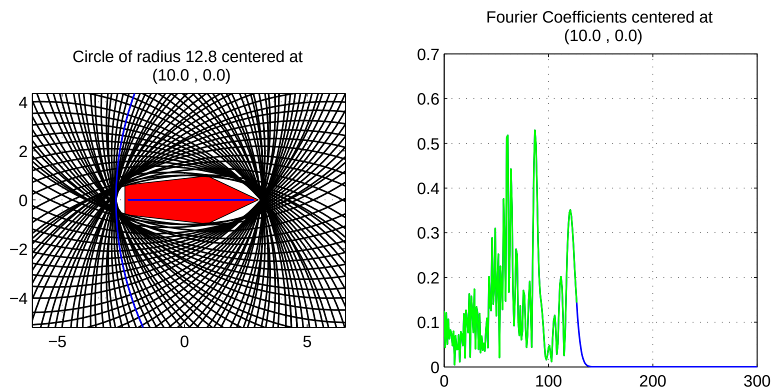
20% additive gaussian noise (k = 10)



50% additive gaussian noise (k = 10)



A Pentagon and 100 circles



Some Analytical Justification

Lemma 23. *For fixed s and $n \geq 0$, $\sigma_{2n+1}(s)$ and $\sigma_{2n}(s)$ are monotone decreasing functions of n.*

Proof.

$$\begin{aligned}
J_n' &= \frac{J_{n+1} - J_{n-1}}{2} \\
\frac{n}{r} J_n &= \frac{J_{n+1} + J_{n-1}}{2} \\
\frac{n}{2r}(J_n^2)' &= \frac{J_{n+1}^2 - J_{n-1}^2}{4} \\
\int_0^R \frac{n}{2r}(J_n^2)' r dr &= \frac{\sigma_{n+1}^2 - \sigma_{n-1}^2}{4} \\
2nJ_n^2(R) &= \sigma_{n+1}^2 - \sigma_{n-1}^2 \\
\sigma_{n-1}^2(R) &= 2nJ_n^2(R) + \sigma_{n+1}^2 \\
&= 2nJ_n^2(R) + (2n+2)J_{2n+2}^2(R) + \ldots
\end{aligned}$$

□

Lemma 24.

$$\sigma_n^2(R) = (RJ_n'(R))^2 + (R^2 - n^2)J_n^2(R)$$

Proof.

$$\left(r\frac{d}{dr}\right)^2 J_n + (k^2r^2 - n^2)J_n \;=\; 0$$

multiply by $2\left(r\frac{d}{dr}\right)J_n$

$$\left(r\frac{d}{dr}\right)\left(r\frac{d}{dr}J_n\right)^2 + \left(r\frac{d}{dr}\right)\left((k^2r^2 - n^2)J_n^2\right) \;=\; 2k^2r^2J_n^2$$

multiply by $\frac{1}{r}$ and integrate from 0 to R.

$$\left(r\frac{d}{dr}J_n\right)^2\Big|_{r=R} + (k^2R^2 - n^2)J_n^2(R) \;=\; 2k^2\sigma_n^2$$

□

Lemma 25. *For $n < R$, with $\cos\alpha = \frac{n}{R}$, and consequently $n\tan\alpha = R\sin\alpha = \sqrt{R^2 - n^2}$,*

$$H_n(R) \;=\; e^{i(n(\tan\alpha-\alpha)+\frac{\pi}{4})}\sqrt{\frac{1}{2\pi n\tan\alpha}}\left(1 + O\left(\frac{1}{\sqrt{n\tan^3\alpha}}\right)\right)$$

$$J_n(R) \;=\; \cos(n(\tan\alpha - \alpha) + \frac{\pi}{4})\sqrt{\frac{1}{2\pi n\tan\alpha}}\left(1 + O\left(\frac{1}{\sqrt{n\tan^3\alpha}}\right)\right)$$

$$RH_n'(R) \;=\; e^{i(n(\tan\alpha-\alpha)+\frac{\pi}{4})}\; iR\sin\alpha\sqrt{\frac{1}{2\pi n\tan\alpha}}\left(1 + O\left(\frac{1}{\sqrt{n\tan^3\alpha}}\right)\right)$$

$$RJ_n'(R) \;=\; R\sin\alpha\sin(n(\tan\alpha - \alpha) + \frac{\pi}{4})\sqrt{\frac{1}{2\pi n\tan\alpha}}\left(1 + O\left(\frac{1}{\sqrt{n\tan^3\alpha}}\right)\right)$$

$$\sigma_n^2(R) \;=\; \frac{1}{\pi}n\tan\alpha\left(1 + O\left(\frac{1}{\sqrt{n\tan^3\alpha}}\right)\right)$$

(very sketchy) Sketch of Proof. Set $R = \frac{n}{\cos\alpha}$ and apply the method of steepest descent to the integral which defines the Hankel function

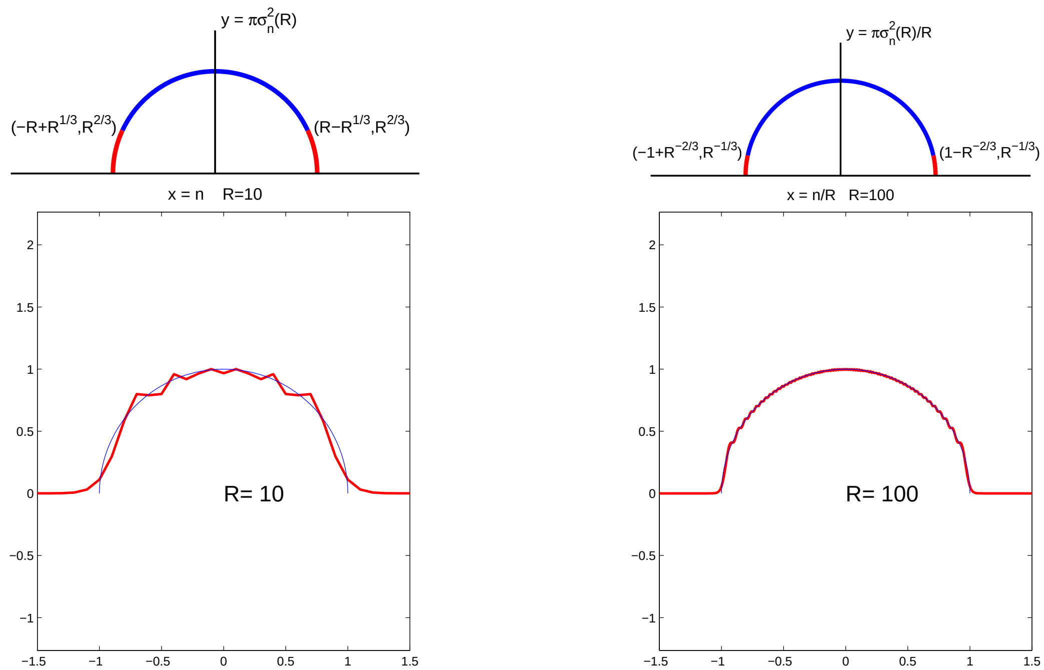
$$H_n = \int_{\gamma_1} e^{i(R\sin(\theta)-n\theta)}d\theta$$

where γ_1 is the contour γ_{st} we used before shifted by $\frac{\pi}{2}$. □

Theorem 26. *There is a constant K, such that, for any constant M, and $n < R - MR^{\frac{1}{3}}$*

$$\left(1 - \frac{K}{M^2}\right)\sqrt{R^2 - n^2} \leq \pi\sigma_n^2(R) \leq \left(1 + \frac{K}{M^2}\right)\sqrt{R^2 - n^2}$$

The theorem says: If we plot $\sigma_n^2(R)$ for fixed R as a function of n, it really looks like a semicircle of radius R on the blue part.



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