Transmission eigenvalues for degenerate and singular cases

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Abstract

We prove the existence of infinitely many real interior transmission eigenvalues for positive contrasts that are bounded above and below by a powers of the distance to the boundary, and give a lower bound for the counting function.

1 Introduction

In this paper we will study the real interior transmission eigenvalues. We will prove that positive values of the square of the wavenumber k^2 for which there is a non-trivial pair (U, V) solving

$$\Delta U(x) + k^2(1 + m(x))U(x) = 0, \quad x \in D, \tag{1.1}$$

$$\Delta V(x) + k^2 V(x) = 0, \quad x \in D, \tag{1.2}$$

$$U(x) = V(x), \quad \frac{\partial U}{\partial \nu}(x) = \frac{\partial V}{\partial \nu}(x), \quad x \in \partial D. \tag{1.3}$$

are discrete and infinite for a class of positive measurable contrasts m which vanish, or blow up, at the boundary of a bounded domain $D \subset \mathbb{R}^d$ like a power of the distance to the boundary. We will also give a lower bound on

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the counting function for continuous contrasts.

The interior transmission eigenvalue problem arises naturally in inverse scattering theory. If k^2 is not a transmission eigenvalue then the relative scattering, or far field, operator is injective with dense range. This condition guarantees the success of certain inverse scattering algorithms ([6], [8]).

The study of the interior transmission problem and transmission eigenvalues has a long history. The problem was first introduced in 1988 by Colton and Monk [7] in connection with an inverse scattering problem for the reduced wave equation. The discreteness of the set of transmission eigenvalues was established by Colton, Kirsch and Päivärinta [5]. Päivärinta and Sylvester [18] proved the first existence result in 2008. The existence of an infinite set of transmission eigenvalues was established by Cakoni, Gintides and Haddar [2]. We also mention some results on transmission eigenvalues for Maxwell's equations and for the Helmholtz equation in presence of cavities [3], [13], [4], as well as very recent results on transmission eigenvalues for elliptic operators of arbitrary order with constant coefficients of Hitrik, Krupchyk, Ola and Päivärinta [10], [11], [12].

The knowledge of the transmission eigenvalues uniquely determines a radial scatterer [14], [15], [18]. For non-radial scatterers, transmission eigenvalues have also been used to infer simple properties of the scatterer [1].

2 Existence of Transmission Eigenvalues

We will follow the approach introduced in [20] and followed in [18], making the substitution $u = U - V$, and applying the operator $\Delta + k^2$ to (1.1). We will seek values of k^2 (we replace k^2 with the letter τ below) for which there is a nontrivial solution u to the fourth order boundary value problem.

$$(\Delta + k^2)\tfrac{1}{m}(\Delta + \tau(1+m))u = 0 \tag{2.1}$$

$$u(x) = 0, \quad \tfrac{\partial u}{\partial \nu}(x) = 0, \quad x \in \partial D. \tag{2.2}$$

We will work in different weighted Sobolev spaces, depending on how m decays or grows as we approach the boundary. In all cases, when we say that

τ is a transmission eigenvalue, we will mean the existence of a nontrivial weak solution to (2.1) which belongs to the standard Sobolev spaces $H^2_{loc}(D)$ and $H^s_0(D)$ for some $s > \frac{3}{2}$, so that (2.2) is satisfied. The multiplicity of a transmission eigenvalue is the dimension of the solution space.

We let $N_{m,D}(\tau)$ denote the counting function, the number of real transmission eigenvalues (counting multiplicity) less than or equal to τ and let $\rho(x)$ denote the distance from x to the boundary of D. Our main result is the following:

Theorem 1. *Let D be a bounded domain in $\mathbb{R}^d$, and suppose that there are two positive constants m_α and m_β such that*

$$m_\beta \rho^\beta \leq m(x) \leq m_\alpha \rho^\alpha \tag{2.3}$$

with

$$-1 < \alpha \leq \beta < \alpha + 2$$

Then there is a constant K_0 small enough, depending on both $m(x)$ and D, such that

$$N_{m,D}(\tau) \geq K_0 \tau^{\frac{d}{2}} - 1 \tag{2.4}$$

A particular consequence of (2.4) is that there are infinitely many transmission eigenvalues. We will give a more explicit description of the constant K_0 in proposition 11.

Because

$$L_\tau = (\Delta + \tau)\frac{1}{m}(\Delta + \tau(1+m)) = (\Delta + \tau(1+m))\frac{1}{m}(\Delta + \tau) \tag{2.5}$$

with a domain that includes the boundary conditions (2.2), is self-adjoint, we may analyze its spectrum by considering the quadratic form

$$Q_\tau(u) = \int_D \overline{(\Delta + \tau)\, u}\ \frac{1}{m}\ (\Delta + \tau(1+m))\, u \tag{2.6}$$

We will consider Q_τ as an unbounded quadratic form on the Hilbert space $H = L^2_{-\delta}(D)$, the closure of $C_0^\infty(D)$ in the norm

$$||f||_{-\delta} := ||\rho^{-\delta} f||_{L^2(D)} \tag{2.7}$$

We will show that, if we choose δ satisfying

$$\max(\beta - \tfrac{\alpha}{2}, -\tfrac{\alpha}{2}) \leq \delta < \tfrac{\alpha}{2} + 2$$

then the domain of Q_τ satisfies

$$M^2_{-(\frac{\beta}{2}+2)} \subset \mathrm{Dom}(Q_\tau) \subset M^2_{-(\frac{\alpha}{2}+2)}$$

where M^p_σ is the closure of $C^\infty_0(D)$ in the norm

$$||f||^2_{p,\sigma} := \sum_{|\eta|=0}^{p} ||D^\eta f||^2_{\sigma+|\eta|} \tag{2.8}$$

We will make use of a Hardy inequality and a few simple propositions:

Proposition 2. *If $\delta > \frac{1}{2}$, there is a positive constant K, depending on D and δ, such that*

$$||u||_{1,-\delta} \leq K||\nabla u||_{-(\delta-1)} \tag{2.9}$$

If $\delta > \frac{3}{2}$, there is a positive constant K, depending on D and δ, such that

$$||u||_{2,-\delta} \leq K||\Delta u||_{-(\delta-2)} \tag{2.10}$$

Proposition 3. *If $u \in M^p_{-\delta}$ with $p > \frac{1}{2}$ and $\delta > \frac{1}{2}$, then u vanishes on ∂D. If $p > \frac{3}{2}$ and $\delta > \frac{3}{2}$, then ∇u vanishes on ∂D as well.*

Proposition 4. *If $D_1 \subset\subset D_2$ and $\delta_1 \geq p$, then $M^p_{-\delta_1}(D_1) \subset M^p_{-\delta_2}(D_2)$.*

Proposition 5. *If $p_1 > p_2$, and $\delta_1 > \delta_2$, then the embedding of $M^{p_1}_{-\delta_1}$ into $M^{p_2}_{-\delta_2}$ is compact.*

Proposition 3 gives a convenient way to remember the inclusions among the weighted spaces. A function in $M^p_{-\delta}$ *vanishes to order* δ at the boundary of D as long as p is large enough to ensure that the restriction makes sense. Proposition 4 says that compactly supported functions vanish to all orders at the boundary, and proposition 5 says that spaces with more regularity and more decay are compactly embedded in spaces with less decay and regularity.

A proof of proposition 2 can be found in [16] or [21]. Note that a consequence of proposition 2 is that the unweighted Sobolev space $H_0^2(D)$ is equivalent to $M^2_{-2}(D)$. Thus proposition 3 follows from the corresponding statements for $H_0^s(D)$ with $s > \frac{1}{2}$ and $s > \frac{3}{2}$. Proposition 4 is a consequence of the fact that an $L^2(D)$ function with compact support belongs to $L^2_\delta(D)$ for any δ.

Proof of Proposition 5. Suppose u_n converges weakly to zero in $M^{p_1}_{-\delta_1}$. We need to show that $||u_n||_{p_2,-\delta_2} \to 0$. Fix $\epsilon > 0$ and define $D_k = \{\rho(x) \geq \frac{1}{k}\}$. Since the $||u_n||_{p_1,-\delta_1}$ are uniformly bounded ,

$$\int_{D\setminus D_k} |\rho^{|\alpha|-\delta_2} D^\alpha u_n| \leq \left(\frac{1}{k}\right)^{\delta_1-\delta_2} \int_{D\setminus D_k} |\rho^{|\alpha|-\delta_1} D^\alpha u_n| \tag{2.11}$$

$$\leq \left(\frac{1}{k}\right)^{\delta_1-\delta_2} ||u_n||_{p_1,-\delta_1} \leq \epsilon \tag{2.12}$$

if we choose k large enough. For each compact set D_k,

$$\int_{D_k} |\rho^{-\delta_2} D^\alpha u_n| \leq k^{\delta_2} ||u_n||_{H^{p_2}(D_k)} \tag{2.13}$$

and

$$\int_{D_k} |\rho^{-\delta_1} D^\alpha u_n| \geq (\mathrm{diam(D)})^{-\delta_1} ||u_n||_{H^{p_1}(D_k)}$$

The compact embedding of $H^{p_1}(D_k)$ in $H^{p_2}(D_k)$ guarantees that choosing $n = n(k)$ large enough will ensure that the left hand side of (2.13) is less than ϵ. $\square$

We will choose a Hilbert space H for which the quadratic forms Q_τ are bounded from below and have form domains which are compactly embedded in H. To see what this requires,we note that

$$Q_0(u) = \int \overline{(\Delta u)} \frac{1}{m} (\Delta u)$$

and expand

$$Q_\tau(u) = Q_0(u) + 2\tau\Re\int \overline{u}\frac{1}{m}(\Delta u) + \tau^2\int\frac{1}{m}|u|^2 + \tau\int\overline{u}\Delta u + \tau^2\int|u|^2$$

We estimate each term,

$$K_\alpha ||u||_{2,-(\frac{\alpha}{2}+2)} \le \frac{1}{m_\alpha}||\Delta u||_{-\frac{\alpha}{2}} \le \ Q_0(u) \le \ \frac{1}{m_\beta}||\Delta u||_{-\frac{\beta}{2}} \tag{2.14}$$

where the inequality on the far left made use of the Hardy inequality (and requires $\alpha > -1$).

$$\begin{aligned}
\left|\int \overline{u}\frac{1}{m}(\Delta u)\right| &\le \frac{1}{m_\beta}||u||_{-(\beta-\frac{\alpha}{2})}||\Delta u||_{-\frac{\alpha}{2}} \le \epsilon Q_0 + \frac{m_\alpha}{m_\beta^2\epsilon}||u||^2_{-(\beta-\frac{\alpha}{2})} \\
\int\frac{1}{m}|u|^2 &\le \frac{1}{m_\beta}||u||^2_{-\frac{\beta}{2}} \\
\left|\int\overline{u}(\Delta u)\right| &\le ||u||_{\frac{\alpha}{2}}||\Delta u||_{-\frac{\alpha}{2}} \le \epsilon Q_0(u) + \frac{m_\alpha}{\epsilon}||u||^2_{\frac{\alpha}{2}} \\
\int|u|^2 &\le ||u||_0^2
\end{aligned}$$

If we choose $\delta \ge \max(\beta - \frac{\alpha}{2}, \frac{\beta}{2}, -\frac{\alpha}{2}, 0)$, which is the same as $\max(\beta-\frac{\alpha}{2}, -\frac{\alpha}{2})$, then

$$(1-\epsilon)Q_0(u) - K_\epsilon||u||^2_{-\delta} \le Q_\tau(u) \le (1+\epsilon)Q_0(u) + K_\epsilon||u||^2_{-\delta} \tag{2.15}$$

and

$$|Q_{\tau_1}(u) - Q_{\tau_2}(u)| \le K|\tau_1-\tau_2|\max(1,\tau_1,\tau_2)\left(Q_0(u) + ||u||^2_{-\delta}\right) \tag{2.16}$$

The inequality (2.15) tells us that, as unbounded quadratic forms on the Hilbert space $L^2_{-\delta}(D)$, the domain of Q_τ is independent of τ, and

$$M^2_{-(\frac{\beta}{2}+2)}(D) \subset \mathrm{Dom}(Q_\tau) = \mathrm{Dom}(Q_0) \subset M^2_{-(\frac{\alpha}{2}+2)}(D) \tag{2.17}$$

which is compactly embedded in $L^2_{-\delta}(D)$ as long as $\delta < \frac{\alpha}{2}+2$. The inequality (2.16) illustrates the (uniform on compact sets) continuous dependence of Q_τ on τ. A direct consequence is

Proposition 6. *If $-1 < \alpha \le \beta < \alpha + 2$ and $\max(\beta - \frac{\alpha}{2}, -\frac{\alpha}{2}) \le \delta < \frac{\alpha}{2} + 2$, then the spectrum of the self adjoint generalized eigenvalue problem*

$$L_\tau u_n = \lambda_n(\tau)\rho^{2\delta}u_n \tag{2.18}$$

is real, discrete, and infinite. Each $\lambda_n(\tau)$ depends continuously on τ. Each $u_n \in M^2_{-(\frac{\alpha}{2}+2)}(D)$ and therefore must vanish, together with its first derivatives, on ∂D. The eigenvalues may be characterized by the min-max principle

$$\lambda_n(\tau) := \max_{V_n} \min_{\substack{u \in V_n^{\perp} \cap Dom(Q_\tau) \\ ||u||_{-\delta}=1}} Q_\tau(u) \tag{2.19}$$

where V_n denotes any n-dimensional subspace of $L^2_{-\delta}$.

Proof. As a consequence of (2.15) and (2.17), Q_τ is bounded below with $Dom(Q_\tau)$ compactly embedded in $H = L^2_{-\delta}$, so has purely discrete spectrum, accumulating only at infinity and obeying (2.19) (see [19], page 780). We may use (2.16) in combination with (2.19) to establish their continuous dependence on τ. □

Proposition 7 below is a slight generalization of the technique introduced in [2] to show the existence of infinitely many transmission eigenvalues for contrasts that are bounded from above and below. Recall that $N_{m,D}(\tau)$ is the number of transmission eigenvalues less than or equal to τ.

Proposition 7. *Suppose $N_{m_0,D_0}(\tau) \geq 1$, and suppose that $\{m_j, D_j\}_{j=1}^N$ represent translations of (m_0, D_0) (i.e. $m_j(x) = m_0(x + x_j)$ on $D_j = D_0 + x_j$). If the D_j are disjoint, each $D_j \subset D$, and $m(x) \geq m_j(x)$ on each D_j, then*

$$N_{m,D}(\tau) \geq N \tag{2.20}$$

Proof. Let $u_{\tau_0}(x)$ denote the transmission eigenfunction for (m_0, D_0) and u_j the translated eigenfunction (with the same τ_0) for (m_j, D_j) Because $u_j(x) \in H^2_0(D_j) = M^2_{-2}(D_j)$, the extension of u_j, defined to be zero in the rest of D, belongs to $M^2_{-\delta}$ for any δ. We have N such eigenfunctions, each supported on disjoint sets, so they are linearly independent.

Because $m(x) \geq m_j(x)$, the quadratic form Q_{τ_0}, is non-positive on this N-dimensional subspace, so, according to the min-max characterization of the eigenvalues, Q_{τ_0} has N non-positive eigenvalues. Because Q_0 has all positive eigenvalues, and they are continuous functions of τ, there must be N transmission eigenvalues between $\tau = 0$ and $\tau = \tau_0$. □

We know that the unit disk in $\mathbb{R}^d$, with any constant contrast, has transmission eigenvalues [14] [15]. Because the unit disk is a subset of the unit cube, proposition 7 assures us that the cube has one too. We will let $\tau_0(M,1)$ denote the lowest transmission eigenvalue of the cube with side 1 with constant contrast M. We do not discuss its dependence on M, or on the dimension d, here. Its dependence on the side length of the cube R is easy to see by scaling. If $u(x)$ is a transmission eigenfunction for the cube with side length 1, then $u(\frac{x}{R})$ is a transmission eigenfunction for the cube with side length R, and the transmission eigenvalue decreases by a factor of R^2, i.e.

Proposition 8. *The lowest transmission eigenvalue of the the cube with side R and constant contrast M is $\frac{\tau_0(M,1)}{R^2}$.*

An immediate consequence of proposition 7 is then

Proposition 9. *Suppose $m(x) > 0$ in D and $m(x) \geq M$ on the disjoint union of P cubes, all with identical side length R and all contained in D. Then*

$$N_{m,D}(\frac{\tau_0(M,1)}{R^2}) \geq P \tag{2.21}$$

Proposition 10. *For any open set $O \subset \mathbb{R}^d$, let $P(R)$ denote the maximum number of disjoint cubes of radius R contained in O. $R^d P(R)$ increases to $\mu(O)$, the Lebesgue measure of O, as R decreases to zero.*

Proof. The limsup of $R^d P(R)$ is the inner measure of the Borel set O. For Borel sets, the inner measure equals the Lebesgue measure. □

Proposition 11. *Let O be an open subset of D on which $m(x) > M$. Then*

$$\lim_{\tau\to\infty} \tau^{-\frac{d}{2}} N_{m,D}(\tau) \geq \tau_0(M,1)^{-\frac{d}{2}} \mu(O) \tag{2.22}$$

If $m(x)$ is continuous, then we may choose $O = \{m(x) > M\}$.

Proof. If we choose $R = \sqrt{\frac{\tau_0(M,1)}{\tau}}$, (2.21) becomes

$$N_{m,D}(\tau) \geq P\left(\sqrt{\frac{\tau_0(M,1)}{\tau}}\right)$$

$$\left(\sqrt{\frac{\tau_0(M,1)}{\tau}}\right)^d N_{m,D}(\tau) \geq \left(\sqrt{\frac{\tau_0(M,1)}{\tau}}\right)^d P\left(\sqrt{\frac{\tau_0(M,1)}{\tau}}\right)$$

and proposition 10 tells us that the right hand side increases to the measure of O as τ approaches infinity. $\square$

Proposition 11 tells us that because $N_{m,D}(\tau) + 1$ is bounded from below by a constant times $\tau^{\frac{d}{2}}$ for large enough τ. It is also bounded from below by 1, for all τ, so the conclusion (2.4) of theorem 1 follows for a small enough choice of constant K_0.

3 Conclusions

We have shown the existence of real transmission eigenvalues for degenerate contrasts, which may vanish at the boundary to an arbitrarily high order, but we require some uniformity in the boundary behavior. The upper and lower bounds must involve powers of ρ which differ by no more than two. A similar result, with a more general background but a more restricted contrast was proved by Hickman [9]

With similar restrictions, we also allowed singular contrasts which grow at the boundary more slowly than $\frac{1}{\rho}$. A scattering theory for singular contrasts can be found in [17].

We have also proved a lower bound on the counting function, similar to that for Dirichlet eigenvalues. If ω_d is the volume of the unit disk in $\mathbb{R}^d$, the number of Dirichlet eigenvalues less than or equal to λ is asymptotic to $\frac{\omega_d}{(2\pi)^d}\mu(D)\lambda^{\frac{d}{2}}$. Proposition 11 tells us that the number of transmission eigenvalues less than or equal to τ is greater than or equal to a constant, which depends on a lower bound for m, times $\mu(D)\tau^{\frac{d}{2}}$. Unfortunately, we have no upper bound. The dependence of $\tau_0(M, 1)$ on the constant contrast M is monotonic decreasing, but we know little else at present.

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