Transmission Eigenvalues and Non-Radiating Sources

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Time Harmonic Waves and Far Fields

Free Waves $\left(\Delta + k^2\right) u^0 = 0$ $\mathcal{H} : \mu^0 \mapsto u^0$

$$u^0 \sim \mu^0(\Theta)\frac{e^{ikr}}{(ikr)^{\frac{n-1}{2}}} + \mu^0(-\Theta)\frac{e^{-ikr}}{(-ikr)^{\frac{n-1}{2}}}$$

Total Waves $\left(\Delta + k^2(1+m)\right) u^m = 0$ $\mathcal{S} : \mu^m \mapsto \gamma$

$$u^m \sim \left(\mu^m(\Theta) + \gamma(\Theta)\right)\frac{e^{ikr}}{(ikr)^{\frac{n-1}{2}}} + \mu^m(-\Theta)\frac{e^{-ikr}}{(-ikr)^{\frac{n-1}{2}}}$$

Outgoing Waves $\left(\Delta + k^2\right) u^+ = f$ $\mathcal{H}^* : f \mapsto \mu^+$

$$u^+ \sim \mu^+(\Theta)\frac{e^{ikr}}{(ikr)^{\frac{n-1}{2}}}$$

Linear Sampling and the Factorization Method

If $\mathbb{R}^n \setminus D$ is connected, $m > 0$, and k is not a TE, the ranges of ${\mathcal{H}_D}^*\mathcal{H}_D$, and $(\mathcal{S}^*\mathcal{S})^{\frac{1}{2}}$ coincide. In particular, the range of $(\mathcal{S}^*\mathcal{S})^{\frac{1}{2}}$ is independent of m, and provides a means for calculating D.

$$\mathcal{S} = {\mathcal{H}_D}^*\left[k^2 m \left(I - G_k^+ k^2 m\right)^{-1}\right]\mathcal{H}_D$$

Interior Transmission Problem

A wavenumber k is called a *TE* if there exists a non-trivial pair (u^0, u^m) solving:

$$\begin{aligned}
\left(\Delta + k^2(1+m)\right) u^m &= 0 && \text{in} && D\\
\left(\Delta + k^2\right) u^0 &= 0 && \text{in} && D\\
u^0 = u^m,\ \frac{\partial u^0}{\partial \nu} &= \frac{\partial u^m}{\partial \nu} && \text{on} && \partial D
\end{aligned}$$

Lots of Previous Work

- Colton and Monk (1988) — spherically stratified medium – there exist infinitely many TE's
- McLaughlin, Polyakov, and Sacks (1994)– spherically stratified medium — transmission eigenvalues determine $n(r)$
- Colton, Kirsch and Päivärinta (1989) — If $m > 0$, transmission eigenvalues form a discrete set
- Rynne and Sleeman (1991) — if $m > 0$, TE's discrete via 4th order operator.
- Colton-Päivärinta-S. (2006) if $m > 0$ No Born TE's, $k^2 > \lambda_0(D)$
- Colton, Cakoni, Haddar — Maxwell's, anisotropic
- Päivärinta - S. — $m > C_D$, the discrete set is not empty.

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Estimates, Spaces, and Duality

Estimate

$$\left(\Delta + k^2\right) u^+ = f$$
$$||u^+||_{B^*} \leq C_0 \tfrac{1}{k} ||f||_B$$

Spaces

$$B = L^2_\delta(\mathbb{R}^n) \quad \delta > \frac{1}{2} \qquad ||f||_{L^2_\delta} = ||(1+|x|^2)^{\frac{\delta}{2}} f||_{L^2}$$
$$B = B_2^{\frac{1}{2},1} \qquad ||f||_B = ||\sqrt{2^j}||f||_{L^2(|x|\in[2^j,2^{j+1}])}||_{l^1}$$

$B_2^{\frac{1}{2},1}$, far fields exhaust $L^2(S^{n-1})$.

If $m* : B^* \longrightarrow B$ and $1+m>0$

$$\left(\Delta + k^2(1+m)\right) u^+ = f$$
$$||u^+||_{B^*} \leq C_m(k)||f||_B$$

Start with a total wave $u^m \in B^*$:

$$\left(\Delta + k^2\right) u^m = -k^2 m u^m$$

Find the unique u^+ with source $-k^2 m u^m$

$$\left(\Delta + k^2\right) u^+_m = -k^2 m u^m$$

Define

$$u^0 = u^m - u^+_m$$

Unique Decomposition theorems

$$u^m \quad ! = u^0 + u^+$$

$$u^0 \quad ! = u^m - u^+$$

The missing decomposition

$$u^+ \quad ! \stackrel{?}{=} u^m - u^0$$

Outside the support of the scatterer – Radiated Waves

Defintions – All sources and scatterers compactly supported in D

- The **radiated wave** is the outgoing wave u^+ outside the support of the source.

Is every radiated wave an m-scattered wave? Does every incident wave u^0radiate?

$$\begin{aligned} u^+\Big|_{\mathbb{R}^n\setminus D} &\stackrel{?}{=} u^m - u^0 \\ 0 &\stackrel{?}{=} (u^m - u^0)\Big|_{\mathbb{R}^n\setminus D} \end{aligned}$$

Outside D, u^+, u^0, u^m, all satisfy the same equation, $P^0 v = 0$.

Notation

$$P^0 := \left(\Delta + k^2\right) \qquad P^m := \left(\Delta + k^2(1+m)\right)$$

Outgoing Waves inside D $\quad u^+ \stackrel{?}{=} u^m - u^0$

Most functions inside D extend to be outgoing

Any $\phi_{00} \in H_0^2(D)$ is an outgoing wave.

$$P^0 \phi_{00} = f$$
$$\left.\frac{\partial \phi_{00}}{\partial \nu}\right|_{S^\infty} = ik\phi_{00}\Big|_{S^\infty}$$

More generally, any ϕ that satisfies $\left.\frac{\partial \phi}{\partial \nu}\right|_{\partial D} = ik\Lambda^+\phi\Big|_{\partial D}$ is outgoing.
Λ^+ is the exterior outgoing DN map.

m-scattered waves inside D are special

An m-scattered wave is an outgoing wave that is the difference of solutions to 2nd order PDE's.

m-scattered waves inside D

Theorem: For $m > 0$ in D

An outgoing wave, u^+, is an m-scattered wave iff

$$P^m \frac{1}{m} P^0 u^+ = 0 \qquad \text{in } D$$

Proof

u^+ is m-scattered if $u^+ = u^m - u^0$. If m is constant, the two operators commute, so the kernel of the product contains the sum of the kernels. Because the characteristic varieties are disjoint, we can change **contains** to **equals**. If m is not constant, P^0 and P^m don't commute, but

$$P^0 \frac{1}{m} P^m = P^m \frac{1}{m} P^0$$

$(P^m \frac{1}{m} P^0)$ is formally self-adoint

Consequences $\quad u^+ \quad ! \stackrel{?}{=} u^m - u^0$

Corollary: The following are equivalent:

- $0 \notin \sigma\left((P^m \frac{1}{m} P^0)_{00}\right)$
- Every radiated wave (i.e. u^+ outside) is m-scattered.
- Every incident wave radiates, i.e. $k^2 \neq \mathrm{TE}$

More Definitions

- $(P^m \frac{1}{m} P^0)_{00}$ is the unbounded self-adjoint operator on $L^2(D)$ with domain $H_0^2(D) \bigcap H^4(D)$.
- Two sources are **equivalent** if they radiate the same wave.
- A source is **non-radiating** if its radiated wave is zero.

Corollary: These are equivalent too:

- Every source is equivalent to one of the form $f = mv^m$.
- No nonzero $f = mv^m$ is non-radiating.

The quadratic form with form domain $H_0^2(D)$

$$T_\tau = \left(\Delta + k^2(1+m)\right)\left(\Delta + k^2\right)$$

$$t_\tau(u) = \tau^2(1+m)||u||^2 \quad - 2\tau\left((1+\tfrac{m}{2})||\nabla u||^2\right) + \quad ||\Delta u||^2$$

$$t_0(u) = ||\Delta u||^2 \geq \mu^0(D)||u||^2$$

Continuity of eigenvalues

If $t_{\tau^*}(u^*) < 0$, for some (τ^*, u^*), then t_{τ^*} has a negative eigenvalue, and, for some $0 < \tau < \tau^*$, t_τ has a zero eigenvalue.

Completing the Square

$$t_\tau(u) = A(u)\Big(\tau - B(u)\Big)^2 + C(u)$$

If $C(u^*) < 0$ and $B(u^*) > 0$, then $t_{B(u^*)}$ has a negative eigenvalue.

$A(u)(\tau - B(u))^2 + C(u)$ Restricted to $||u||^2 = 1$

$$A(u) = (1+m)$$

$$\frac{1+\frac{m}{2}}{1+m}\lambda_0(D) \leq B(u) = \frac{1+\frac{m}{2}}{1+m}||\nabla u||^2 \leq \frac{1+\frac{m}{2}}{1+m}||\Delta u||$$

$$C(u) = ||\Delta u||^2 - \frac{(1+\frac{m}{2})^2}{1+m}||\nabla u||^4$$

With u^*= lowest clamped plate eigenfunction $\Delta^2 u^* = \mu_0 u^*$

$$\begin{aligned} 0 < \quad B(u^*) &\leq \frac{1+\frac{m}{2}}{1+m}\mu_0^{\frac{1}{2}} \\ C(u^*) &\leq \mu_0 - \frac{(1+\frac{m}{2})^2}{1+m}\lambda_0^2 \end{aligned}$$

Conclusion

If $\frac{(1+\frac{m}{2})^2}{1+m} > \frac{\mu_0}{\lambda_0^2}$, then there is a TE with $k^2 \leq \frac{1+\frac{m}{2}}{1+m}\mu_0^{\frac{1}{2}}$

More TE's if m i s bigger

$$
\begin{aligned}
t_\tau(u) &= \tau^2(1+m) \quad -2\tau\left((1+\tfrac{m}{2})||\nabla u||^2\right) + \quad ||\Delta u||^2 \\
&\le \tau^2(1+m) \quad -2\tau(1+\tfrac{m}{2})\lambda_0 + \quad ||\Delta u||^2
\end{aligned}
$$

On the span of the lowest n clamped plate eigenfunctions.

$$
\begin{aligned}
&\le \tau^2(1+m) \quad -2\tau(1+\tfrac{m}{2})\lambda_0 + \quad \mu_n \\
&= (m+1)\left(\tau - \frac{1+\frac{m}{2}}{1+m}\lambda_0\right)^2 + \quad \mu_n - \frac{(1+\frac{m}{2})^2}{1+m}\lambda_0^2
\end{aligned}
$$

Conclusion

If $\frac{(1+\frac{m}{2})^2}{1+m} > \frac{\mu_n}{\lambda_0^2}$, then $t_{\left(\frac{1+\frac{m}{2}}{1+m}\lambda_0(D)\right)}$ has n negative eigenvalues, so there must be n TE's (counting multiplicity) with $k^2 \le \frac{1+\frac{m}{2}}{1+m}\lambda_0$

$T_\tau = \left(\Delta + k^2(1+m)\right)\left(\Delta + k^2\right)$

$$\begin{aligned} T_\tau &= (\Delta + \tau(1+m))^2 - m\tau\,(\Delta + \tau(1+m)) \\ t_\tau(u) &= ||\,(\Delta + \tau(1+m))\,u||^2 + m\tau\left(||\nabla u||^2 - (1+m)\tau\right) \\ &\geq m\tau\left(||\nabla u||^2 - (1+m)\tau\right) \\ &\geq 0 \qquad \text{if} \quad \tau < \frac{\lambda_0}{1+m} \end{aligned}$$

No Born TE's either

$$B_\tau = (\Delta + \tau)^2 \quad \text{with domain } H_0^2(D)$$
$$b_\tau(u) = ||\,(\Delta + \tau)\,u||^2$$

which is strictly positive for every τ.

Relation to Factorization

Non-Radiating Sources equal (Free Waves)$^{\perp}$

$$\int f v^0 = \int_D \left(\Delta + k^2\right) u^+ v^0 = \int_{\partial D} \frac{\partial u^+}{\partial \nu} v^0 - u^+ \frac{\partial v^0}{\partial \nu}$$

- RHS is zero for all v^0 iff $u^+ \equiv 0$ outside i.e. $u^+ \in H_0^2(D)$
- $f^{NR_D} = \left(\Delta + k^2\right) \phi_{00}$

Every D-source has a unique equivalent D-free source

$$\begin{aligned} f^{NR} + f^0 &= f \\ \left(\Delta + k^2\right) \phi_{00} + f^0 &= f \\ \left(\Delta + k^2\right)^2 \phi_{00} &= \left(\Delta + k^2\right) f \end{aligned}$$

Solve, and set

$$f^0 := f - \left(\Delta + k^2\right) \phi_{00}$$

Relation to Factorization

$S_D : B_D^0 \longrightarrow B_D^0$ $\quad$ $B_D^0 = L^2(D) \bigcap \{P^0 v^0 = 0\}$ in D

$$S_D : v^0 \mapsto v^m \mapsto [f^0 \sim -k^2 m v^m]$$

$$\mathcal{H}_D : L^2(S^{n-1}) \xrightarrow{1-1} B_D^0$$

$$\mathcal{H}_D^* : B_D^0 \xrightarrow{1-1} L^2(S^{n-1})$$

$$\mathcal{S} = \mathcal{H}_D^* S_D \mathcal{H}_D$$

TE's are the wavenumbers for which $0 \in \sigma(S_D)$