An Estimate for the Free Helmholtz equation that Scales *

John Sylvester †
Department of Mathematics
University of Washington
Seattle, Washington 98195
U.S.A.

June 2, 2008

Abstract

Wavelength plays a distinguished role in classical electromagnetic and acoustic scattering. Most significant features of the far field patterns radiated by a collection of sources or scatterers are related to their sizes and relative distances, measured in wavelengths. These significant features are reflected in the invariance of the Helmholtz equation with respect to translation, and its homogeneous scaling with respect to dilations. The weighted norms that were first developed to capture the correct decay properties of waves in $\mathbb{R}^n$ do not scale homogeneously and are not invariant with respect to translation. L^p estimates scale homogeneously and commute with translations and rotations. However, their scaling properties give estimates with a weaker dependence on wavenumber (for bounded sources and scatterers with

*Dedicated to David Colton and Rainer Kress. It is a pleasure to express my gratitude for your many interesting and practical insights into inverse scattering, which have played a big role in directing my research, and that of many others. More importantly, I applaud the warm and welcoming mathematical community that you have shaped around you. It is privilege to have become part of that community.

†This research was supported in part by NSF grant DMS-0355455, ONR grant N00014-05-1-0716

support that extends over many wavelengths). We introduce some norms and estimates that commute with translations and scale homogeneously under dilations, while retaining the same sharp dependence on wavelength for extended sources as that of the weighted estimates.

1 Introduction

This paper is about estimates for the resolvent of the Laplacian in $\mathbb{R}^n$. The Laplacian $(-\Delta)$ is an unbounded self-adjoint operator on $L^2(\mathbb{R}^n)$, with domain $H^2(\mathbb{R}^n)$. Consequently, for every $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$, and $f \in L^2$

$$(\Delta + \lambda)u = f \tag{1.1}$$

has a unique solution $u \in L^2(\mathbb{R}^n)$, and

$$||u||_{L^2} \leq \frac{1}{\Im(\lambda)}||f||_{L^2} \tag{1.2}$$

$$||u||_{H^2} \leq \frac{C_n}{\Im(\lambda)}||f||_{L^2}$$

where $\Im(\lambda)$ means the imaginary part of λ, the constant C_n depends only on the dimension n, and the H^m-norm is defined as

$$||f||_{H^m} = \left(\sum_{|\alpha|<m} ||D^\alpha f||^2_{L^2} \right)^{\frac{1}{2}}$$

with α denoting a multi-index and D^α a partial derivative.

If we substitute k^2 for λ, then we call (1.1) the Helmholtz equation. Real k denotes wavenumber, and wavelength is $\frac{2\pi}{k}$.

$$(\Delta + k^2)u = f \tag{1.3}$$

Whenever we consider (1.3), we shall consider $k \in \overline{\mathbb{C}^+}$, the closed upper half plane. By the square root, $k = \sqrt{\lambda}$, we will always mean the holomorphic map from $\mathbb{C}\setminus\mathbb{R}^+$ to $\mathbb{C}^+$. In particular, when k tends to the positive real axis, $\mathbb{R}^+$, λ tends to $\mathbb{R}^+$ from above (i.e. with positive imaginary part). When k tends to $\mathbb{R}^-$, λ tends to $\mathbb{R}^+$ from below. If we denote the solution operator to (1.3) as $R(k)$, then (1.2) becomes

$$||R(k)|| \leq \frac{1}{\Im(k^2)}$$

Although $R(k)$ is a holomorphic function for $k \in \mathbb{C}^+$, with values in the space of bounded maps from L^2 to itself, it does not have a continuous extension to the real axis. However, $R(k)$ can be extended continuously to the real axis as a function whose values are bounded linear operators between other Hilbert spaces. For this purpose, Agmon introduced the L^2_δ spaces.

$$||f||_{L^2_\delta} := ||(1+|x|^2)^{\frac{\delta}{2}} f||_{L^2}$$
$$||f||_{H_{m,\delta}} = \left(\sum_{|\alpha|<m} ||D^\alpha f||^2_{L^2_\delta} \right)^{\frac{1}{2}}$$

and proved the following (see Lemma 4.1 of [1]).

Theorem 1. *Let $K > 1$, $\delta > \frac{1}{2}$, and $k \in \mathbb{C}^+$ with $K^{-1} < |k| < K$. There exists a constant $C(K,\delta)$, such that, for all $u \in H^2(\mathbb{R}^n)$,*

$$||u||_{L^2_{-\delta}} \leq C(K,\delta) || \left(\Delta + k^2\right) u||_{L^2_\delta} \tag{1.4}$$

He also remarked that, for k uniformly close to the real axis, the dependence of the constant $C(K,\delta)$ could be described more explicitly, i.e.

$$||u||_{L^2_{-\delta}} \leq \frac{C(n,\delta)}{(1+|k|^2)^{\frac{1}{2}}} || \left(\Delta + k^2\right) u||_{L^2_\delta} \tag{1.5}$$

Agmon used this inequality, along with the Holder continuity of $R(k)$ to show a *limiting absorption principle*, which extended $R(k)$ (Agmon actually considered $R(k^2)$) by continuity to the real axis as a bounded operator from L^2_δ to $L^2_{-\delta}$. We refer to $u^+ = R(k)f$ (for real, as well as complex k) as the unique outgoing solution to (1.3).

For $k \in \mathbb{C}^+$ and $f \in L^2_\delta$, $u^+ := R(k)f$ satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{n-1}{2}} \left(\frac{du^+}{dr} - iku^+ \right) = 0$$

This property extends (at least for $f \in C_0^\infty$) by continuity to $k \in \overline{\mathbb{C}^+}$, and provides an alternate characterization of the unique outgoing solution to (1.3).

Corollary 2. *For $k \in \overline{\mathbb{C}^+}$, and $f \in L^2_\delta$, there exists a unique outgoing $u \in L^2_{-\delta}$ satisfying*

$$\left(\Delta + k^2\right) u = f \tag{1.6}$$

Moreover,

$$||u||_{L^2_{-\delta}} \leq C(K,\delta)||f||_{L^2_\delta} \tag{1.7}$$

Agmon and Hormander [2] showed that estimate (1.4) held with the L^2_δ norms replaced by the norms below, i.e.

Definition 3. *Let A_0 denote the unit ball and A_j the annulus $\{2^{j-1} \leq |x| \leq 2^j\}$ for $j \geq 1$. the norms*

$$\begin{aligned} ||f||_B &= \sum_{j=0}^{\infty} \sqrt{2^j}\ ||f||_{L^2(A_j)} \\ ||u||_{B^*} &= \sup_j \frac{1}{\sqrt{2^j}}\ ||u||_{L^2(A_j)} \end{aligned}$$

Theorem 4. *There exists a constant $C(n,K)$, such that, for all $u \in H^2(\mathbb{R}^n)$,*

$$||u||_{B^*} \leq C(n,K)|| \left(\Delta + k^2\right) u||_B \tag{1.8}$$

The large k dependence of the constant in (1.8) was made more explicit in [8], i.e.

$$||u||_{B^*} \leq \frac{C(n)}{(1+k^2)^{\frac{1}{2}}}|| \left(\Delta + k^2\right) u||_B$$

L^p estimates,derived as a combination of the arguments in [16] and [9], and presented in [14] and [15], show that

Theorem 5. *Let $k > 0$ and $\frac{2}{n} \geq \frac{1}{q} - \frac{1}{p} \geq \frac{2}{n+1}$ for $n \geq 3$ and $1 > \frac{1}{q} - \frac{1}{p} \geq \frac{2}{3}$ for $n = 2$, where $\frac{1}{q} + \frac{1}{p} = 1$. There exists a constant $C(n,p)$, independent of k, such that, for u solving (1.6)*

$$||u||_{L^p(\mathbb{R}^n)} \leq C(n,p)k^{n(\frac{1}{q}-\frac{1}{p})-2}||f||_{L^q(\mathbb{R}^n)} \tag{1.9}$$

Because they scale homogeneously, the dependence of the L^p estimates on wavenumber follows simply once one has proved the estimate for a single nonzero value of k. An important restatement of this homogeneity in k, is that L^p norms have units[1], so that criteria based on the size of such norms can be directly related to physical parameters. To see this more clearly, it is convenient to multiply the right hand side of the free Helmholtz equation by k^2 — this gives both u and f the same units, as is the case in acoustic scattering.

$$\left(\Delta + k^2\right) u \quad = \quad k^2 f \tag{1.10}$$

Equation (1.10) is invariant with respect to scaling (change of units). Specifically, if

$$\begin{aligned} u(x) &= v(\alpha x) \\ f(x) &= g(\alpha x) \end{aligned}$$

then (v, g) satisfy (1.10) with at a different wavenumber:

$$(\Delta + \left(\frac{k}{\alpha}\right)^2)v \quad = \quad \left(\frac{k}{\alpha}\right)^2 g \tag{1.11}$$

Once we have an estimate at one wavenumber, say $k = 1$,

$$||v||_{L^p} \quad \leq \quad C(n,p)\ ||g||_{L^q}$$

implies

$$||u(\frac{x}{k})||_{L^p} \quad \leq \quad C(n,p)||f(\frac{x}{k})||_{L^q}$$

so that the equivalent of (1.9)

$$||u||_{L^p(\mathbb{R}^n)} \leq C(n,p) k^{n(\frac{1}{q}-\frac{1}{p})} ||f||_{L^q(\mathbb{R}^n)} \tag{1.12}$$

simply by changing variables in both of the integrals above. The difference in the power of k is simply because we have included a factor of k^2 in the right

[1] If x has units of length and u is unitless, then $\int_{\mathbb{R}^n} u(x)^p dx$ has units of length^n and its pth root has units of length$^{\frac{n}{p}}$.

hand side of (1.10). The homogeneous scaling is more than a convenience. It implies that the constant $C(n,p)$ is unitless, which means that criteria that depend on its size can be physically meaningful. Notice also that L^p norms, and therefore estimates in terms of those norms, are translation invariant. The properties of the scattering operator with respect to translation play an important role in inverse scattering, so norms that transform simply with translation are likely to be more useful in understanding this operator.

Estimates in weighted norms share neither the homogeneous scaling nor the translation invariance. This means that the constants in the corresponding estimates depend on a choice of units, so a criterion based on these constants will be, at the least, difficult to interpret physically. Translating a source or scatterer can make it larger in weighted norms, which is inconsistent with the underlying physics. Weighted estimates involve the choice of an origin. For a source which is supported in a ball, it is sensible to choose the origin at the center of that ball, but for a source supported in a union of balls, which are far apart, there is no appropriate choice of center. In weighted norms, translating a source, can change its norm, which is inconsistent with what we know about the Helmholtz equation and scattering theory.

The L^p estimates are stronger than the weighted estimates for sources which are supported on small sets (i.e. sets with diameters that are less than a wavelength in diameter). However, most inverse scattering/remote sensing applications involve sources (f's) that are supported on sets which are many wavelengths in diameter. In this regime, the weighted estimates are stronger. As we move through the allowable ranges of p's and q's in theorem 5, the dependence of the constant in (1.9) ranges between k^0 and $k^{\frac{-2}{n+1}}$, while the weighted estimates hold with a constant proportional to k^{-1}.

We will introduce some norms that scale homogeneously and are invariant with respect to translations. The corresponding estimates will scale as k^{-1}, so they exhibit comparable behavior to the weighted norms at large wavenumbers. Moreover, we will show that all the estimates for weighted spaces are immediate corollaries of the estimates we will prove below.

Definition 6. *For each unit vector $\Theta \in S^{n-1}$, and domain $D \subset \mathbb{R}^n$, let*

$$||u||_{\Theta^{\infty,2}(D)} = \sup_{\tau\in\mathbb{R}} \left(\int\limits_{D\cap\{x\cdot\Theta=\tau\}} |u|^2 dS \right)^{\frac{1}{2}}$$

$$||f||_{\Theta^{1,2}(D)} = \int\limits_{\tau\in\mathbb{R}} \left(\int\limits_{D\cap\{x\cdot\Theta=\tau\}} |f|^2 dS \right)^{\frac{1}{2}} d\tau$$

In most cases we will choose $D = \mathbb{R}^n$, and write simply $||f||_{\Theta^{1,2}}$ or $||u||_{\Theta^{\infty,2}}$. We let $\Theta^{1,2}$ and $\Theta^{\infty,2}$ denote the Banach spaces consisting of measurable functions for which the corresponding norms are finite. C_0^∞ is dense in $\Theta^{1,2}$, but not in $\Theta^{\infty,2}$. It is easy to check that these norms scale simply.

$$\begin{aligned} ||u(\frac{x}{k})||_{\Theta^{\infty,2}} &= |k|^{\frac{n-1}{2}} ||u||_{\Theta^{\infty,2}} \\ ||f(\frac{x}{k})||_{\Theta^{1,2}} &= |k|^{\frac{n+1}{2}} ||f||_{\Theta^{1,2}} \end{aligned}$$

Our estimates will make use of $n+1$ different Θ's.

Definition 7. *Let $\{\Theta_j\}_{j=1}^n$ be an orthonormal basis for $\mathbb{R}^n$, and $\Theta_0 = \frac{1}{\sqrt{2}}(\Theta_1 + \Theta_2)$. We will denote this collection of $n+1$ vectors by*

$$\mathcal{J}_n = \{\Theta_j\}_{j=0}^n$$

Our main theorem, which we will prove in section 2, is

Theorem 8. *Let $k \in \overline{\mathbb{C}^+}$, $f \in C_0^\infty$, and let u be the unique outgoing solution to the Helmholtz equation (1.6). There is a constant $C(n)$, such that*

$$u = \sum_{i=0}^n u_i$$

$$||u_i||_{\Theta_i^{\infty,2}} \leq \frac{C(n)}{|k|} ||f||_{\Theta_i^{1,2}} \tag{1.13}$$

In addition,

$$\begin{aligned}\nabla u &= \sum_{i=1}^{n} w_i & (1.14)\\ ||w_i||_{\Theta_i^{\infty,2}} &\leq C(n)\ ||f||_{\Theta_i^{1,2}} & (1.15)\\ ||\Delta u||_{\Theta_i^{\infty,2}} &\leq C(n)\ \left(||f||_{\Theta_i^{\infty,2}} + |k|\ ||f||_{\Theta_i^{1,2}}\right) & (1.16)\end{aligned}$$

Theorem 8 decomposes u as a sum and estimates each summand u_i in a different norm. the paragraph below is a brief, and admittedly imprecise, motivation for this decomposition.

If f were a function of the single variable, $\tau = x \cdot \Theta$, we would expect u to also be a function of a single variable, and an easy estimate of the outgoing solution to the one dimensional Helmholtz equation would give $||u||_{L^\infty} \leq \frac{C}{k}||f||_{L^1}$, where the L^1 norm means integration in τ. If $f = F(\tau)e^{iky}$, where $y = x \cdot \xi$ with ξ a unit vector perpendicular to Θ, separating variables reveals that the same estimate cannot hold. The subspace spanned by these functions f is the kernel of the $(n-1)$ dimensional Helmholtz operator in the $\Theta^\perp$ plane. The proof of theorem 8 decomposes f into a sum of $n+1$ functions, each of which is orthogonal to this subspace (for different Θ's). It turns out that this is enough to give the estimates (1.13) for each u_i.

The next few lemmas show that, for every Θ, the $\Theta^{1,2}$ norm is dominated by, and $\Theta^{\infty,2}$ norm dominates, the corresponding weighted norms as well as L^2 norms on compact sets. Thus (1.13) and (1.15) directly imply estimates that do not require a decomposition.

Lemma 9. *Let $\delta > \frac{1}{2}$ and $\Theta \in S^{n-1}$, for every $f \in L^2_\delta$, and every $u \in \Theta^{\infty,2}$,*

$$\begin{aligned}||f||_{\Theta^{1,2}} &\leq \left(\frac{2\delta}{2\delta-1}\right)^{\frac{1}{2}} ||f||_{L^2_\delta} & (1.17)\\ ||u||_{L^2_{-\delta}} &\leq \left(\frac{2\delta}{2\delta-1}\right)^{\frac{1}{2}} ||u||_{\Theta^{\infty,2}} & (1.18)\end{aligned}$$

Proof. For $\Theta \in S^{n-1}$, we let

$$x = t\Theta + y$$

decompose $x \in \mathbb{R}^n$ into components parallel to and orthogonal to Θ.

$$
\begin{aligned}
||f||_{\Theta^{1,2}} &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} |f(t,y)|^2 dy \right)^{\frac{1}{2}} dt \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} |f(t,y)|^2 dy \right)^{\frac{1}{2}} (1+t^2)^{\frac{\delta}{2}} (1+t^2)^{\frac{-\delta}{2}} dt \\
&\leq \left(\int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |f(t,y)|^2 dy (1+t^2)^{\delta} dt \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} (1+t^2)^{-\delta} dt \right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |f(t,y)|^2 (1+t^2+|y|^2)^{\delta} dy dt \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} (1+t^2)^{-\delta} dt \right)^{\frac{1}{2}} \\
&\leq ||f||_{L^2_\delta} \left(\frac{2\delta}{2\delta - 1} \right)^{\frac{1}{2}}
\end{aligned}
$$

The verification of (1.18) by calculation is similar and we omit it. In fact, (1.13) follows from (1.17) by duality because $L^2_{-\delta}$ is the dual of L^2_δ and $\Theta^{\infty,2}$ is the dual of $\Theta^{1,2}$. $\square$

Lemma 10. *Let D be a bounded domain in $\mathbb{R}^n$. Let μ_1 denote one-dimensional Lebesgue measure and define*

$$
\begin{aligned}
d(D,\Theta) &= \sup_{x \in D} \mu_1 \left(\left\{ t \;\middle|\; x + t\Theta \in D \right\} \right) \\
d(D) &= \sup_{\Theta} d(D,\Theta)
\end{aligned}
$$

Then, for very Θ,

$$
\frac{1}{\sqrt{d(D,\Theta)}} ||f||_{\Theta^{1,2}(D)} \leq ||f||_{L^2(D)} \leq \sqrt{d(D,\Theta)} ||f||_{\Theta^{\infty,2}(D)} \tag{1.19}
$$

The constant $d(D)$ satisfies

1. *$d(D)$ is less than or equal to the diameter of D.*
2. *If $D = \bigcup D_i$, then $d(D) \leq \sum d(D_i)$.*

In particular, $d(D)$ is less than or equal to the sum of the diameters of the connected components of D.

Proof.

$$\int\limits_D |f|^2 dx \quad = \int\limits_{-\infty}^{\infty} \left[\int\limits_{D \bigcap \{x \cdot \Theta = \tau\}} |f|^2 dS_{\Theta^\perp} \right] d\tau \quad \leq d(D,\Theta)||f||^2_{\Theta^{\infty,2}(D)}$$

Because $d(D,\Theta)$ is the measure of the intersection of a line (in the Θ direction) with D, it is certainly less than the diameter of D. The intersection of a line with the union $D = \bigcup D_i$ is equal to the union of the latter intersections, so the measure of the former is less than or equal to the sum of the measures of the intersections, which shows that $d(D) \leq \sum d(D_i)$.

The left hand inequality in (1.19) follows from the right hand one by duality, i.e.,

$$\begin{aligned} ||f||_{\Theta^{1,2}(D)} &= \sup_{w \in \Theta^{\infty,2}(D)} \frac{\langle w, u \rangle}{||w||_{\Theta^{\infty,2}(D)}} \\ &\leq \sup_{w \in \Theta^{\infty,2}(D)} \frac{\langle w, u \rangle \sqrt{d(D,\Theta)}}{||w||_{L^2(D)}} \\ &= \sqrt{d(D,\Theta)}||u||_{L^2(D)} \end{aligned} \tag{1.20}$$

□

As is the case with the L^2_δ norms, the B and B^* norms do not transform simply under dilations and translations. However, we have the analog of lemma 9 for these norms as well.

Lemma 11.

$$\begin{aligned} ||f||_{\Theta^{1,2}} &\leq ||f||_B \\ ||u||_{B^*} &\leq ||u||_{\Theta^{\infty,2}} \end{aligned}$$

Proof.

$$||f||_{\Theta^{1,2}} \quad = \quad \sum_{j=0}^{\infty} ||f||_{\Theta^{1,2}(A_j)}$$

Because 2^j is the diameter of A_j, we may apply (1.19) to each summand

$$\begin{aligned} &\leq \sum_{j=0}^{\infty} \sqrt{2^j}\ ||f||_{L^2(A_j)} \\ &= ||f||_B \end{aligned}$$

Similarly,

$$\begin{aligned} ||u||_{B^*} &= \sup_j \frac{1}{\sqrt{2^j}}\ ||f||_{L^2(A_j)} \\ &\leq \sup_j ||u||_{\Theta^{\infty,2}(A_j)} \\ &\leq ||u||_{\Theta^{\infty,2}} \end{aligned}$$

□

Corollary 12. *Theorem 1 holds with* $C(K,\delta) = \frac{C(n)}{|k|}\frac{2\delta}{2\delta-1}$ *and theorem 4 holds with constant* $C(n,k) = \frac{C(n)}{|k|}$. *If* f *is supported in the bounded domain* D, *then, with* $d = d(D)$,

$$||u^+||_{L^2(D)} \leq C(n)\frac{d}{|k|}||f||_{L^2(D)} \tag{1.21}$$

Proof. We will prove (1.21). The proofs that theorems 1 and 4 hold with the new constants are exactly analogous.

According to theorem 8, we may write $u^+ = \sum u_i$,

$$||u^+||_{L^2(D)} \leq \sum ||u_i||_{L^2(D)}$$

which, according to (1.19)

$$\leq \sum \sqrt{d}||u_i||_{\Theta^{\infty,2}(D)}$$

Applying (1.13) gives

$$\leq \frac{C(n)}{|k|}\sum \sqrt{d}||f||_{\Theta^{1,2}(D)}$$

and then (1.19) again

$$\begin{aligned} &\leq \frac{C(n)}{|k|}\sum d||f||_{L^2(D)} \\ &\leq \frac{(n+1)C(n)d}{|k|}||f||_{L^2(D)} \end{aligned}$$

□

The applications we will give in section 3 could depend only on this corollary. Nevertheless, for the sake of completeness, we define some spaces:

Definition 13.

$$
\begin{aligned}
\mathcal{J}_{\cap}^{1,2} &= \bigcap_{j=0}^{n} \Theta_j^{1,2} \\
||f||_{\mathcal{J}_{\cap}^{1,2}} &= \max_{0\le j\le n} ||f||_{\Theta_j^{1,2}} \\
\mathcal{J}_{+}^{\infty,2} &= \overset{n}{\underset{j=0}{+}} \Theta_j^{\infty,2} \\
||u||_{\mathcal{J}_{+}^{\infty,2}} &= \inf_{u=\sum u_j} \sum ||u_j||_{\Theta_j^{\infty,2}}
\end{aligned}
$$

With these definitions, we may restate (1.13) as

$$
||u||_{\mathcal{J}_{+}^{\infty,2}} \le \frac{C(n)}{|k|} ||f||_{\mathcal{J}_{\cap}^{1,2}} \tag{1.22}
$$

2 Proof of Theorem 8

Because the norms on both sides of (1.13) scale naturally with dilations, it would be enough to prove the theorem for $|k| = 1$. We will nevertheless carry out the proof for general $k \in \mathbb{C}^{+}$, because we feel that it emphasizes the role that translations and dilations play in scattering problems. Let $\Theta \in \mathbb{R}^n$ be a unit vector, we let

$$
\begin{aligned}
x &= t\Theta + y \\
\xi &= \tau\Theta + \eta
\end{aligned} \tag{2.1}
$$

denote the unique decompositions of $x \in \mathbb{R}^n$, and $\xi \in \mathbb{R}^n$, into vectors parallel to Θ and one perpendicular to Θ. We denote the Fourier transform as

$$\widehat{f}(\xi) = \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi} f(x)dx$$

which we may write as

$$\widehat{f}(\tau,\eta) = \int\limits_{\mathbb{R}^n} e^{it\tau} e^{iy\cdot\eta} f(t,y)dy$$

We denote the partial Fourier transform, in the $\Theta^\perp$ hyperplane, by

$$\tilde{f}(t,\eta) = \mathcal{F}_\Theta f = \int\limits_{x\cdot\Theta=t} e^{i\eta\cdot y} f(t,y)dy$$

We emphasize that this partial transform depends on the choice of Θ (we will make several different choices), and that η denotes a vector in the $(n-1)$-dimensional subspace $\Theta^\perp$. When we use the notation $\tilde{f}$ below, we are referring to a single, fixed, but arbitrary choice of Θ. We begin by applying the partial Fourier transform to the Helmholtz equation,

$$\left(\Delta + k^2\right) u = f \tag{2.2}$$

$$\frac{d^2\tilde{u}}{dt^2} + (k^2 - |\eta|^2)\tilde{u} = \tilde{f}$$

For $k \in \mathbb{C}^+$, there is a unique $u \in H^2(\mathbb{R}^n)$ solving (2.2), for every $f \in L^2$, and it is straightforward to check that

$$\tilde{u}(t,\eta) = \int\limits_{\mathbb{R}} \frac{e^{i\sqrt{k^2-|\eta|^2}|t-s|}}{2i\sqrt{k^2-|\eta|^2}} \tilde{f}(s,\eta)ds \tag{2.3}$$

The formula (2.3) represents many different formulas, one for each choice of Θ. We intend to decompose every $f \in C_0^\infty$ into a sum of f's, using different choices of Θ to avoid the zeros of the denominator in (2.3). We need to be sure that they all represent the same solution u to (1.3). Our estimates below will verify that the right hand side of (2.3) defines an L^2 function for $k \in \mathbb{C}^+$, so that we can invoke uniqueness to guarantee that all the representations are equal. Once we have avoided the zeros of the denominator, we will see that k times the right hand side of (2.3) is continuous on $\overline{\mathbb{C}^+}$, which then guarantees that for nonzero real k, (2.3) represents the unique outgoing solution to (1.3).

In order to construct our decomposition, we define:

$$W_\Theta^\varepsilon = \left\{\xi \in \mathbb{R}^n \middle| \, |1-|\eta|^2| \le \varepsilon^2\right\} \tag{2.4}$$

which is the same as

$$= \left\{\xi \in \mathbb{R}^n \middle| \, |1-|\xi|^2+(\Theta\cdot\xi)^2| \le \varepsilon^2\right\}$$

because $\eta = \xi - (\Theta\cdot\xi)\Theta$. We also note that, for $\alpha > 0$

$$\alpha W_\Theta^\varepsilon = \left\{\xi \in \mathbb{R}^n \middle| \, |\alpha^2-|\eta|^2| \le \varepsilon^2\alpha^2\right\} \tag{2.5}$$

Proposition 14. *Let* $\varepsilon \le \sqrt{\frac{n-1}{4(n+1)n}}$, $\alpha > 0$, $\{\Theta_j\}_{j=1}^n$ *be an orthonormal basis for* $\mathbb{R}^n$, *and* $\Theta_0 = \frac{1}{\sqrt{2}}(\Theta_1+\Theta_2)$. *There is a constant* $C(n)$, *such that every* $f \in C_0^\infty$ *can be decomposed as*

$$f = \sum_{j=0}^n f_j \tag{2.6}$$

with

$$||f_j||_{\Theta_j^{1,2}} \le C(n)||f||_{\Theta_j^{1,2}} \tag{2.7}$$

and

$$\text{supp } \widehat{f_j} \bigcap \alpha W_{\Theta_j}^\varepsilon = \emptyset \tag{2.8}$$

or, equivalently,

$$\text{supp } (\mathcal{F}_{\Theta_j} f_j) \bigcap \left\{(t,\eta)\middle| \, \eta \in \alpha W_{\Theta_j}^\varepsilon\right\} = \emptyset \tag{2.9}$$

The proof of Theorem 8 is a direct consequence of this proposition.

Proof of Theorem 8. Let u_i denote the solution to (2.2) with $f = f_i$, and apply the partial Fourier transform as in (2.3) with $\Theta = \Theta_i$.

$$\begin{aligned}
\tilde{u_i}(t,\eta) &= \int\limits_{\mathbb{R}} \frac{e^{i\sqrt{k^2-|\eta|^2}|t-s|}}{2i\sqrt{k^2-|\eta|^2}}\tilde{f}_i(s,\eta)ds \\
\int |\tilde{u_i}|^2 d\eta &= \int\limits_{\mathbb{R}^{n-1}} \overline{\left(\int\limits_{\mathbb{R}} \frac{e^{i\sqrt{k^2-|\eta|^2}|t-s|}}{2i\sqrt{k^2-|\eta|^2}}\tilde{f}_i(s,\eta)ds\right)} \left(\int\limits_{\mathbb{R}} \frac{e^{i\sqrt{k^2-|\eta|^2}|t-r|}}{2i\sqrt{k^2-|\eta|^2}}\tilde{f}_i(r,\eta)dr\right) d\eta
\end{aligned}$$

Because $\Im(\sqrt{k^2-|\eta|^2}) \geq 0$, the exponential is bounded by one. We can find an upper bound for the denominator by noting that $|\sqrt{k^2-|\eta|^2}| \geq |\sqrt{|k|^2-|\eta|^2}|$, and combining (2.9) and (2.5) with $\alpha = |k|$, so that

$$\begin{aligned}
|\int |\tilde{u_i}|^2 d\eta| &\leq \frac{1}{(\varepsilon|k|)^2} \int\limits_{\mathbb{R}^{n-1}} \left(\int\limits_{\mathbb{R}} |\tilde{f}_i(s,\eta)|ds\right)\left(\int\limits_{\mathbb{R}} |\tilde{f}_i(r,\eta)|dr\right) d\eta \\
&= \frac{1}{(\varepsilon|k|)^2} \int\limits_{\mathbb{R}\times\mathbb{R}} \int\limits_{\mathbb{R}^{n-1}} |\tilde{f}_i(s,\eta)|\ |\tilde{f}_i(r,\eta)| d\eta ds dr \\
&\leq \frac{1}{(\varepsilon|k|)^2} \int\limits_{\mathbb{R}\times\mathbb{R}} ||\tilde{f}_i(s,\cdot)||_{L^2} ||\tilde{f}_i(r,\cdot)||_{L^2} dr dt
\end{aligned}$$

which, according to the Plancherel equality,

$$\begin{aligned}
&= \frac{1}{(\varepsilon|k|)^2} \int\limits_{\mathbb{R}\times\mathbb{R}} ||f_i(s,\cdot)||_{L^2} ||f_i(r,\cdot)||_{L^2} dr dt \\
&\leq \frac{1}{(\varepsilon|k|)^2} ||f_i||^2_{\Theta_{1,2}}
\end{aligned}$$

We don't need the decomposition to estimate the first derivatives. Instead, we note that

$$v \cdot \nabla u = \sum_{i=1}^{n} (v \cdot \Theta_i)\Theta_i \cdot \nabla u$$

We can estimate the derivative in the Θ_i direction by applying the partial Fourier transform, $\mathcal{F}_{\Theta_i}$, and differentiating (2.3) to obtain

$$\frac{d\tilde{u}(t,\eta)}{dt} = \int_{\mathbb{R}} e^{i\sqrt{k^2-|\eta|^2}|t-s|}\tilde{f}(s,\eta)ds$$

so that

$$\left|\left|\frac{d\tilde{u}(t,\eta)}{dt}\right|\right|_{\Theta_i^{\infty,2}} \leq ||f||_{\Theta_i^{1,2}}$$

We need only note that

$$||\Theta_i \cdot \nabla u||_{\Theta_i^{\infty,2}} = \left|\left|\frac{d\tilde{u}(t,\eta)}{dt}\right|\right|_{\Theta_i^{\infty,2}}$$

to prove (1.15). Finally, we verify (1.16) by writing

$$\begin{aligned} \Delta u &= f - k^2 u \\ ||\Delta u||_{\Theta_i^{\infty,2}} &\leq ||f||_{\Theta_i^{\infty,2}} - |k|^2||u||_{\Theta_i^{\infty,2}} \\ &\leq ||f||_{\Theta_i^{\infty,2}} - C(n)|k|\ ||f||_{\Theta_i^{1,2}} \end{aligned}$$

which completes the proof of theorem 8. □

Proof of Proposition 14. We define

$$\sigma_\Theta(\xi) = \xi \cdot \xi - (\Theta \cdot \xi)^2 \tag{2.10}$$

and, as in (2.4),

$$W_\Theta^\varepsilon := \left\{\xi \in \mathbb{R}^n \ \middle| |\sigma_\Theta(\xi) - 1| < \varepsilon^2\right\}$$

Our first task is to show that the complements of the W_Θ^ε cover $\mathbb{R}^n$.

Lemma 15. *Let $\varepsilon \leq \sqrt{\frac{n-1}{4(n+1)n}}$, $\{\Theta_j\}_{j=1}^n$ be an orthonormal basis for $\mathbb{R}^n$, and $\Theta_0 = \frac{1}{\sqrt{2}}(\Theta_1 + \Theta_2)$. Then*

$$\bigcap_{k=0}^{n} W_\Theta^\varepsilon = \emptyset \tag{2.11}$$

Proof. For $\xi \in \mathbb{R}^n$, we let the ξ_k denote its coordinates in the basis formed by the $\{\Theta_j\}_{j=1}^n$. We let τ denote the vector with components $\tau_k = \xi_k^2$ and

$\mathbf{1}$ denote the vector with all components equal to 1. It is straightforward to check that $\xi \in \bigcap_{k=1}^{n} W_{\Theta}^{0}$ if and only if τ satisfies the linear equation

$$M\tau = \mathbf{1} \tag{2.12}$$

where $M$is the matrix with 0's on the diagonal and ones elsewhere, i.e.

$$M = \mathbf{1} \otimes \mathbf{1} - I$$
$$M^{-1} = \frac{1}{n-1}\mathbf{1} \otimes \mathbf{1} - I$$

and

$$||M^{-1}||_{\infty,\infty} = 2 - \frac{1}{n+1} \tag{2.13}$$

The norm in (2.13) views M as a linear map on $\mathbb{R}^n$ with the l^∞ norm.

The unique solution to (2.12) is

$$\tau = \frac{1}{n-1}\mathbf{1}$$

so that if $\xi \in \bigcap_{k=1}^{n} W_{\Theta}^{\varepsilon}$ then

$$||\tau - \frac{1}{n-1}\mathbf{1}||_{l^\infty} \leq (2 - \frac{1}{n+1})\varepsilon^2$$

and, in particular, for each $1 \leq k \leq n$

$$|\xi_k| \geq \sqrt{\frac{1}{n-1} - (2 - \frac{1}{n+1})\varepsilon^2} \tag{2.14}$$

We can express σ_{Θ_0} as

$$\sigma_{\Theta_0} - 1 \quad = \quad \frac{1}{2}(\sigma_{\Theta_1} - 1) + \frac{1}{2}(\sigma_{\Theta_2} - 1) - \xi_1\xi_2$$

which implies that, for $\xi \in \bigcap_{k=1}^{n} W_{\Theta}^{\varepsilon}$

$$|\sigma_{\Theta_0} - 1| \quad \geq \quad |\xi_1\xi_2| - \varepsilon^2$$

Using (2.14) gives,

$$|\sigma_{\Theta_0} - 1| \geq \frac{1}{n-1} - (3 - \frac{1}{n+1})\varepsilon^2$$

so that (2.11) follows as long as

$$\varepsilon^2 \leq \frac{1}{n-1} - (3 - \frac{1}{n+1})\varepsilon^2$$

or, equivalently,

$$\varepsilon^2 \leq \frac{n-1}{4(n+1)n}$$

□

Our next task is to construct a partition of unity, subordinate to the W_{Θ_k}'s, and prove that the associated Fourier multipliers are bounded operators in the $\Theta^{1,2}$ norms. Let $\Phi \in C_0^\infty(\mathbb{R})$ satisfy

$$\Phi(s) = \begin{cases} 1 & |1-s| < \frac{\varepsilon}{2} \\ 0 & |1-s| > \varepsilon \end{cases} \tag{2.15}$$

As before, for each fixed Θ, we let

$$\xi = \tau\Theta + \eta$$

denote the unique decomposition of $\xi \in \mathbb{R}^n$ into a vector parallel to Θ and one perpendicular to Θ. We define

$$\begin{aligned} \phi_\Theta(\xi) &= \phi_\Theta(\tau, \eta) \\ &= \Phi(|\eta|^2) \\ &= \Phi(\sigma_\Theta(\xi)) \end{aligned} \tag{2.16}$$

where $\sigma_\Theta(\xi)$ is defined in (2.10).

Lemma 16. *1.*

$$\begin{aligned} \text{supp } \phi_\Theta &\subset W_\Theta^\varepsilon \\ \text{supp } (1-\phi_\Theta) &\bigcap W_\Theta^{\frac{\varepsilon}{2}} = \emptyset \end{aligned}$$

2. The Fourier transform of ϕ_Θ, which we denote by $\widehat{\phi_\Theta}$, is a bounded measure with norm

$$|||\widehat{\phi_\Theta}|||_1 := \int\limits_{\mathbb{R}^n} |\widehat{\phi_\Theta}|$$

3. *For any $\alpha > 0$, the $||| \; |||_1$-norm of the Fourier transform of $\phi_\Theta(\frac{\xi}{\alpha})$is independent of α.*

4. *For $\varepsilon \leq \sqrt{\frac{n-1}{4(n+1)n}}$, $\{\Theta_j\}_{j=0}^n$ defined as in lemma 15, and using ϕ_j to denote ϕ_{θ_j}*

$$\sum_{m=0}^{n} \left((1 - \phi_m(\xi)) \prod_{j<m} \phi_j(\xi) \right) = 1 \qquad (2.17)$$

Proof. The two statements in item 1 follow directly from the definitions (2.15) and (2.16).

Because $\Phi \in \mathcal{S}(\mathbb{R})$, $\phi(\eta) := \Phi(|\eta|^2) \in \mathcal{S}(\Theta^\perp)$, so its Fourier transform, $\widehat{\phi}(y)$ is also a Schwartz class function of $(n-1)$ variables. Because ϕ_Θ is the product of $\phi(\eta)$ and a (constant) function of τ, its Fourier transform is the product of the two Fourier transforms.

$$\begin{aligned} \phi_\Theta(\tau, \eta) &= \phi(\eta) \otimes \mathbf{1} \\ \widehat{\phi_\Theta} &= \widehat{\phi}(y) \otimes \delta(t) \end{aligned}$$

from which we see that $\widehat{\phi_\Theta}$ is a measure supported on the plane $x \cdot \Theta = 0$ with a Schwartz class density $\widehat{\phi}$, and therefore

$$|||\widehat{\phi_\Theta}|||_1 = \int_{\mathbb{R}^{n-1}} |\widehat{\phi}(y)| dy$$

Item 3 follows from the scaling properties of the Fourier transform (it's homogeneous of degree $(-n)$), but we make a simpler, more explicit, calculation below

$$\begin{aligned} \widehat{\phi_\Theta(\frac{\xi}{\alpha})} &= \widehat{\phi(\frac{\eta}{\alpha})} \otimes \widehat{1} \\ &= \alpha^{n-1} \widehat{\phi}(\alpha y) \otimes \delta(t) \end{aligned}$$

so that the norm of the measure is the $L^1(\mathbb{R}^{n-1})$-norm of $\alpha^{n-1}\widehat{\phi}(\alpha y)$,which is independent of α.

To establish item 4, we expand the sum in (2.17) to see that

$$\sum_{m=0}^{n}\left((1-\phi_m(\xi))\prod_{j<m}\phi_j(\xi)\right)=1-\prod_{j=0}^{n}\phi_j(\xi)$$

and note that the final term on the left can only be nonzero on $\bigcap_{k=0}^{n} W_{\Theta}^{\varepsilon}$, which is empty according of lemma 15. $\square$

As a consequence of (2.17), (2.6) and (2.8) both hold if we define

$$\widehat{f_m}=\left((1-\phi_m(\frac{\xi}{\alpha}))\prod_{j<m}\phi_j(\frac{\xi}{\alpha})\right)\widehat{f}$$

Because $\widehat{f_m}$ is the product of $\widehat{f}$ with at most n functions, and the Fourier transform of each of these is a bounded measure (with a bound that is independent of α, according to item 3 of lemma 16); the estimate (2.7) will follow once we show that convolution with a bounded measure maps $\Theta^{1,2}(\mathbb{R}^n)$ to itself. This is the content of lemma 17 below, which we state and prove in a slightly more general context.

Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, V a subspace of $\mathbb{R}^n$ and $V^{\perp}$ its orthogonal complement, i.e.

$$\mathbb{R}^n = V \oplus V^{\perp}$$

We define the function space $V^{p,q}(\mathbb{R}^n)$ to be the measurable functions with finite $V^{p,q}$ norm,i.e.

$$||f||_{V^{p,q}} := \left[\int\limits_{V}\left(\int\limits_{V^{\perp}}|f|^q\right)^{\frac{p}{q}}\right]^{\frac{1}{p}} = ||\ ||f(v,\cdot)||_{L^q(V^{\perp})}\ ||_{L^p(V)}$$

Lemma 17. *Convolution with a bounded measure induces a bounded map from $V^{p,q}$ to itself, i.e.,*

$$||f*\mu||_{V^{p,q}} \leq |||\mu|||_1\ ||f||_{V^{p,q}}$$

Proof.

$$
\begin{aligned}
||f*\mu||^p_{V^{p,q}} &= \int_V \left(\int_{V^\perp}\left[\int_{\mathbb{R}^{n-1}} f(x-x',y-y')\mu(x',y')\right]^q dy\right)^{\frac{p}{q}} dx \\
&\leq \int_V \left(\int_{V^\perp}\left[\left(\int_{\mathbb{R}^n} |f(x-x',y-y')|^q\mu(x',y')\right)\left(\int_{\mathbb{R}^n}\mu(x',y')\right)^{\frac{q}{q'}}\right] dy\right)^{\frac{p}{q}} dx \\
&= \int_V \left[\left(\int_{V^\perp} |f(x-x',y-y')|^q dy\right)\left(\int_{\mathbb{R}^n}\mu(x',y')\right)^{1+\frac{q}{q'}}\right]^{\frac{p}{q}} dx \\
&= \int_V \left(\int_{V^\perp} |f(x-x',y-y')|^q dy\right)^{\frac{p}{q}} dx \left(\int_{\mathbb{R}^n}\mu(x',y')\right)^p \\
&= ||f||^p_{V^{p,q}}\ |||\mu|||^p_1
\end{aligned}
$$

□

This completes the proof of Proposition 14. □

3 Unitless criteria related to Born Series and Transmission Eigenvalues

We will use the estimates we have just derived to give a criterion for judging the applicability of the Born approximation for a compactly supported scatterer. By this we mean a medium with index of refraction $n(x)$ which is identically equal to one outside a compact set. It is common to write $n^2(x) = 1 + m(x)$ with $-1 < m(x)$ referred to as the contrast. Note that $n(x)$ is the ratio of the wave speed in the vacuum to the wave speed in the medium, and therefore unitless.

The following theorem describes the Born approximation, and contains a summary of a very minimal scattering theory for a compactly supported scatterer. Equation (3.1) below is satisfied by the total wave, which is expressed as the sum of an incident wave, which satisfies the free equation (3.3) below, and the scattered wave, which is required to be outgoing. We will allow two different possibilities for describing incident and total waves:

1. $A = L^2(D)$ – Incident waves need only be defined and satisfy (3.3) in D. Total waves are defined and satisfy (3.1) only in D. Scattered waves

are still outgoing, defined in all of $\mathbb{R}^n$, and have far fields (asymptotics at infinity).

2. $A = \mathcal{J}_+^{\infty,2}$ – Incident waves, total waves, and outgoing scattered waves are all defined and satisfy partial differential equations in $\mathbb{R}^n$.

See [10], [13] or [5] for more complete descriptions of scattering theories, and [12] for a discussion of how the very minimal theory described in item 1 can be viewed as an extension of the classical one.

Theorem 18. *Suppose that* supp $m \subset D$ *and let* $d(D)$ *be the constant in lemma 10 (which is less than the diameter of* D*, and less than the sum of the diameters of its components). Let* $u^m \in A$ *and solve*

$$\left(\Delta + k^2(1+m)\right) u^m = 0 \tag{3.1}$$

Then u^m *has a unique decomposition into an incident plus an outgoing wave*

$$u^m = u^0 + u^+ \tag{3.2}$$

where

$$\left(\Delta + k^2\right) u^0 = 0 \tag{3.3}$$

$u^0 \in A$ *and* $u^+ \in \mathcal{J}_+^{\infty,2}$.
If

$$C(n)|k|d \; ||m||_{L^\infty(D)} < 1 \tag{3.4}$$

then the Born series

$$u^+ = \sum_{j=1}^{\infty} [k^2 R(k) m]^j u^0 \tag{3.5}$$

converges, we can estimate the scattered wave with

$$||u^+||_A \leq \frac{C(n)|k|d \; ||m||_{L^\infty(D)}}{1 - C(n)|k|d \; ||m||_{L^\infty(D)}} ||u_0||_A \tag{3.6}$$

and the Born approximation, $u_B^+ = k^2 R(k) m u^0$ *satisfies*

$$||u^+ - u_B^+||_A \leq \frac{C(n)|k|d \; ||m||_{L^\infty(D)}}{1 - C(n)|k|d \; ||m||_{L^\infty(D)}} ||u_B^+||_A \tag{3.7}$$

Remark 19. *Notice that the criterion (3.4), and the constants in (3.6) and (3.7) are unitless. The index of refraction (a ratio of wavespeeds) , and hence the contrast, is unitless. The unitless quantity kd must be less than or equal to the diameter of D measured in wavelengths. A similar result, based on weighted estimates, appears in [11].*

We can also base our convergence criterion on the L^p estimates of theorem 5. In this case we obtain a criterion which is stronger for $kd < 1$ and weaker for $kd > 1$.

Theorem 20. *Under the hypotheses of theorem 18, the Born series converges if*

$$C(n)(kd)^s < 1 \tag{3.8}$$

where $\frac{2n}{n+1} \leq s \leq 2$ *for* $n \geq 3$ *and* $\frac{2n}{n+1} \leq s < 2$ *for* $n = 2$.

The proof of theorem 18 will be based on a lemma.

Lemma 21. *Let* $m \in L^\infty(D)$ *(i.e.* $m \in L^\infty(\mathbb{R}^n)$ *and* $\operatorname{supp} m \subset D$*), and* A *be either* $L^2(D)$*, or* $\mathcal{J}_+^{\infty,2}$*. Then*

$$||k^2 R(k)mv||_A \ \leq \ C(n)||m||_{L^\infty(D)}|k| \ d(D) \ ||v||_A \tag{3.9}$$

Proof. According to (1.21),

$$||k^2 R(k)mv||_{L^2(D)} \leq |k|d||mv||_{L^2(D)} \leq |k|d||m||_{L^\infty}||v||_{L^2(D)}$$

Similarly, it follows from (1.22) that

$$||k^2 R(k)mv||_{\mathcal{J}_+^{\infty,2}} \ \leq \ C(n)|k| \ ||mv||_{\mathcal{J}_\cap^{1,2}}$$

which, for some j

$$= \ C(n)|k| \ ||mv||_{\Theta_j^{1,2}}$$

Now apply (1.19),

$$\begin{aligned} &\leq \ \tilde{C}(n)|k|\sqrt{d}||mv||_{L^2(D)} \\ &\leq \ \tilde{C}(n)|k|||m||_{L^\infty(D)}\sqrt{d}||v||_{L^2(D)} \\ &\leq \ \tilde{C}(n)|k|||m||_{L^\infty(D)}\sqrt{d} \inf_{v=\sum v_i} \sum ||v_i||_{L^2(D)} \end{aligned}$$

and then apply (1.19) again

$$
\begin{aligned}
&\leq \tilde{C}(n)|k|||m||_{L^\infty(D)}\sqrt{d} \inf_{v=\sum v_i} \sum \sqrt{d}||v_i||_{\Theta_i^{\infty,2}} \\
&\leq \tilde{C}(n)|k|||m||_{L^\infty(D)} d||v||_{\mathcal{J}_+^{\infty,2}}
\end{aligned}
$$

□

Proof of theorem 18. Note that

$$\left(\Delta + k^2\right) u^m \;=\; -k^2 m u^m \tag{3.10}$$

Define u^+ to be the unique outgoing solution to

$$\left(\Delta + k^2\right) u^+ \;=\; -k^2 m u^m \tag{3.11}$$

and define

$$u^0 \;=\; u^m - u^+$$

As $u^m \in A$, the right hand side of (3.11) belongs to $\mathcal{J}_\cap^{1,2}$, so $u^+ \in \mathcal{J}_+^{\infty,2}$, which is contained in A. As u^+ is unique and belongs to A, u^0 is also unique and belongs to A. Subtracting (3.11) from (3.10) shows that u^0 satisfies (3.1). Moreover,

$$
\begin{aligned}
u^+ &= k^2 R(k) m u^m \\
&= k^2 R(k) m (u^0 + u^+)
\end{aligned}
$$

from which we see that u^+ is given by the Born series (3.5), when it converges. The fact that (3.4) implies convergence, and the two estimates, (3.6) and (3.7), then follow from (3.9) of lemma 21.

□

Theorem 18 tells us that the Born series converges, and the Born approximation is accurate, for a single small scatterer or for a collection of small scatterers as long as the sum of the diameters is small enough.

Remark 22. *In treating a collection of scatterers theorem 18 makes essential use of the translation invariance of our new norms. In the L^2_δ or B^* norms, the norm of operator of multiplication by m as a map from $L^2_{-\delta}$ to L^2_δ, or from B^* to B, changes as we translate m, which makes any estimate of this sort difficult, if not impossible, to obtain with these norms.*

A wavenumber k is called a transmission eigenvalue (see [5, 6, 3]) for m in D if there exists a $u^m \in L^2(D)$, such that the corresponding (as in (3.2)) scattered wave u^+ is supported in D. That is, $u^m = u^0$ is an incident wave that doesn't scatter. As another application, we show that small scatterers, or collections of small scatterers, don't have small transmission eigenvalues. A similar result, with dk replaced by $(dk)^2 \log(dk)$, appears in [3].

Theorem 23. *If k is a transmission eigenvalue for $m \geq 0$ in D, then*

$$1 \leq C(n) dk \sup |m|$$

Proof. Let v^0 be any incident wave (i.e. solution to (3.3)). If we multiply (3.11) by v^0,

$$\int_D v^0 \left(\Delta + k^2\right) u^+ = -\int_D k^2 m u^m v^0 \tag{3.12}$$

If k is a transmission eigenvalue

$$0 = \int_D k^2 m u^m v^0$$

where the last line follows by integration by parts, together with the fact that $u^+ \in H^2_{loc}$ vanishes outside D. We recall (3.2), multiply both sides by $m\overline{u^m}$,

$$\begin{aligned} u^m &= u^0 + u^+ \\ \int m|u^m|^2 &= \int m\overline{u^m}u^0 + \int m\overline{u^m}u^+ \end{aligned} \tag{3.13}$$

and note that, because of (3.12), the first term on the right is zero. Therefore,

$$\begin{aligned} \int m|u^m|^2 &= \int m u^m u^+ \\ &\leq ||mu^m||_{L^2(D)} ||u^+||_{L^2(D)} \\ &\leq ||mu^m||_{L^2(D)} \frac{C(n)d}{k} ||k^2 m u^m||_{L^2(D)} \end{aligned}$$

Making use of (1.21) gives

$$\begin{aligned}
&\leq C(n)dk||mu^m||^2_{L^2(D)} \\
\int m|u^m|^2 &\leq C(n)dk \int m^2|u^m|^2 \\
1 &\leq C(n)dk \sup |m|
\end{aligned}$$

□

We give one last corollary, which is an immediate consequence of (1.22). The Hardy space properties of the the resolvent play an important role in one dimensional inverse scattering. The theorem below shows that the free resolvent in higher dimensions shares one of these properties.

Theorem 24. *As an operator valued function of k, taking values in the space of bounded maps from $\mathcal{J}^{1,2}_{\cap}$ to $\mathcal{J}^{\infty,2}_{+}$, $kR(k)$ belongs to the Hardy class $H^\infty(\mathbb{C}^+)$.*

4 Conclusions

We have introduced some new norms and used them to study the dependence of the free Helmholtz equation (the resolvent of the Laplacian) in $\mathbb{R}^n$ on wavenumber. The new norms are translation invariant and scale homogeneously with dilations, which allows them to exploit the natural invariance of the Helmholtz equation. We utilized these invariance properties to prove some estimates for the free Helmholtz equation with exactly the same dependence on wavelength as in the one dimensional case.

We gave some simple corollaries, which included a rigorous mathematical justification of the well known physical principle that classical scatterers that are small compared to the wavelength of an incident wave scatter weakly. We gave unitless criteria for the convergence of the Born series, the absence of transmission eigenvalues, and established a Hardy space property of the resolvent.

Our main point, however, is simply to introduce a different estimate for the constant coefficient Laplacian on $\mathbb{R}^n$, the most central operator in mathematical physics. This estimate improves on the weighted estimates of Agmon and Agmon-Hormander, primarily because the norms have units (scale homogeneously) and commute with translation. The dependence on wavenumber (they scale differently) shows that these estimates are different than known L^p estimates. We note with regret that our norms are not invariant under rotation; they rely on a choice of orthogonal frame, and a different choice of frame gives a possibly inequivalent norm.

Our proof relies primarily on the simplicity of the characteristic variety of the associated multiplier ($|\xi|^2 - k^2$), making it likely that similar estimates hold for other operators with simply characteristic varieties.

Finally, we remark that related estimates, in bounded domains with different norms, but with a similar dependence on wavenumber, play a role in the finite element analysis of scattering (see [4] and [7]).

References

[1] S. Agmon. Spectral properties of Schrödinger operators and scattering theory. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 2(2):151–218, 1975.

[2] S. Agmon and L. Hörmander. Asymptotic properties of solutions of differential equations with simple characteristics. *J. Analyse Math.*, 30:1–38, 1976.

[3] F. Cakoni, D. Colton, and P. Monk. On the use of transmission eigenvalues to estimate the index of refraction from far field data. *Inverse Problems*, 23:507–522, 2007.

[4] S. Chandler-Wilde and P. Monk. Wave-number-explicit bounds in time-harmonic scattering. *SIAM J. Math. Anal.*, 39(5):1428–1455 (electronic), 2008.

[5] D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, second edition, 1998.

[6] D. Colton, L. Päivärinta, and J. Sylvester. The interior transmission problem. *Inverse Probl. Imaging*, 1(1):13–28, 2007.

[7] F. Ihlenburg. *Finite element analysis of acoustic scattering*, volume 132 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1998.

[8] C. E. Kenig, G. Ponce, and L. Vega. Small solutions to nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(3):255–288, 1993.

[9] C. E. Kenig, A. Ruiz, and C. D. Sogge. Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators. *Duke Math. J.*, 55(2):329–347, 1987.

[10] S. T. Kuroda. *An introduction to scattering theory*, volume 51 of *Lecture Notes Series*. Aarhus Universitet Matematisk Institut, Aarhus, 1978.

[11] F. Natterer. An error bound for the Born approximation. *Inverse Problems*, 20(2):447–452, 2004.

[12] L. Päivärinta and J. Sylvester. Transmission eigenvalues. *SIAM J. Math. Anal.*, To appear, 2008.

[13] M. Reed and B. Simon. *Methods of modern mathematical physics. III.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1979. Scattering theory.

[14] A. Ruiz. Lectures at the summer school at Oulu, June 2002.

[15] V. Serov. Fourier transforms and distributions, with applications to schrödeinger operators, 2007.

[16] P. A. Tomas. A restriction theorem for the Fourier transform. *Bull. Amer. Math. Soc.*, 81:477–478, 1975.