Transmission Eigenvalues

via

Upper Triangular Compact Operators

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Elementary Scattering Theory
for Helmholtz

Incident Waves $\quad (\Delta + k^2) v^0 = 0 \quad \text{in } \mathbb{R}^n$

Radiated Waves $\quad (\Delta + k^2) u^+ = f$

! outgoing/radiating solution to
radiation condition $\quad \frac{\partial u^+}{\partial r} - iku^+ = o\left(\frac{1}{r^{(n-1)/2}}\right)$

Total Waves $\quad (\Delta + k^2(1+m))u^m = 0$

$m(x)$ is the scatterer

Spaces $\quad f \in L^2_\delta(\mathbb{R}^n) \quad v^0, u^+, u^m \in L^2_{-\delta}(\mathbb{R}^n) = (L^2_\delta)^*$

$\quad F \in B^{2,1}(\mathbb{R}^n) \quad v^0, u^+, u^m \in B^{2,\infty}(\mathbb{R}^n) = (B^{2,1})^*$

The Scattering Operator

$$(\Delta+k^2)v^0 = 0$$
$$(\Delta+k^2)u^+ = f$$
$$(\Delta+k^2(1+m))u^m = 0$$

Theorem

(a) Every $u^m \; ! = v^0 + u^+$

(b) Every $v^0 \; ! = u^m - u^+$

Proof of (a): ① Solve $(\Delta+k^2)u^+ = -k^2 m u^m$

② Set $v^0 = u^m - v^+$

outside

Definition

$$v^0 \xmapsto{\;S_m(k)\;} u^+\Big|_{(\mathbb{R}^n \setminus \operatorname{supp} m)^\infty}$$

If we describe v^0 and u^+ in terms of asymptotics, Rellich's lemma and unique continuation imply that S is equivalent to the standard relative scattering operator.

Transmission Eigenvalues

Every $u^n \overset{!}{=} v^0 + u^+$
Every $v^0 \overset{!}{=} u^n - u^+$

Is $v^0 \xrightarrow{S(k)} u^+$ 1-1 ?

$u^+ \equiv 0$ outside $\Longleftrightarrow$ $u^+|_{\partial D} = 0$ $\quad \frac{\partial u^+}{\partial \nu}\Big|_{\partial D} = 0$

Interior Transmission Eigenvalue

$$(\Delta + k^2(1+m))\, u^+ = -k^2 m\, v^0$$

$$(\Delta + k^2)\, v^0 = 0$$

$$u^+|_{\partial D} = 0 \qquad \frac{\partial u^+}{\partial \nu}\Big|_{\partial D} = 0$$

Interior Transmission Eigenvalue Problem

$$(\Delta + k^2(1+m))\, u^+ = -k^2 m\, v^0$$

$$(\Delta + k^2)\, v^0 = 0$$

$$u^+|_{\partial D} = 0$$

$$\frac{\partial u^+}{\partial \nu}\Big|_{\partial D} = 0$$

Note:

$$u^+ \equiv \begin{cases} u & \text{in } D \\ 0 & \text{in } \mathbb{R}^n \setminus D \end{cases}$$

is outgoing

v^0 may <u>not</u> extend to $\mathbb{R}^n$? ?

Back Ground

Inverse Scattering for Helmholtz

Linear Sampling and Factorization

Do all incident waves scatter?

Are scattered far fields dense? [in $L^2(S^{n-1})$]

Obstacle IF $k^2 \neq$ Dirichlet eigenvalue

Penetrable Medium IF $k^2 \neq$ Transmission Eigenvalue

$\boxed{m > 0}$

1988 Colton-Monk

1989 Colton-Kirsch-Päivärinta [Discrete]

2008 Päivarinta-S. [Exist if $m > \frac{\lambda^2}{\mu}$]

2009 Cakoni-Gintedes [Exist]

2009 Cakoni-Gintedes-Haddar [∞ many exist]

2010 Cakoni-Colton-Haddar [cavities]

Fourth Order Equation (Rynne – Sleeman 1991)

$$\boxed{m > 0}$$

$$\Delta + \tau(1+m)u = -mv$$
$$(\Delta+\tau)v = 0$$

$u = 0 \quad \frac{\partial u}{\partial \nu} = 0$

$$\Rightarrow \frac{1}{m}\left(\Delta + \tau(1+m)u\right) = -v$$

$$\Downarrow$$

$$(\Delta+\tau)\frac{1}{m}\left(\Delta+\tau(1+m)\right)u = 0$$

$u = 0 \quad \frac{\partial u}{\partial \nu} = 0$

$$\Downarrow$$

$$\left[\Delta \frac{1}{m} \Delta_{00} + F(\tau)\right] u = 0$$

self-adjoint positive definite

Relatively compact quadratic in τ

self-adjoint positive definite

Relatively compact quadratic in τ and $F(0)=0$

$$\left[\Delta \frac{1}{m} \Delta_{oo} + F(\tau)\right]^{-1} = \left[\left(\Delta \frac{1}{m} \Delta_{oo}\right)\left(I + \left(\Delta \frac{1}{m}\Delta_{oo}\right)^{-1} F(\tau)\right)\right]^{-1}$$

$$= \left[I + \left(\Delta \frac{1}{m} \Delta_{oo}\right)^{-1} F(\tau)\right]^{-1} \left(\Delta \frac{1}{m} \Delta_{oo}\right)^{-1}$$

Analytic Fredholm Theorem
invertible on $\mathbb{C} \setminus$ discrete set

Conclusion: ITE's at most discrete

Paivarinta - S. (2008) $\qquad (m>0)$

$$(\Delta+\tau)\frac{1}{m}(\Delta_{oo}+\tau(1+m))u=(\Delta+\tau(1+m))\frac{1}{m}(\Delta_{oo}+\tau)$$

For all real τ,

$T(\tau)$ is self-adjoint, bounded below, with compact resolvent

$T(0)=\Delta\frac{1}{m}\Delta_{oo}$ is positive definite

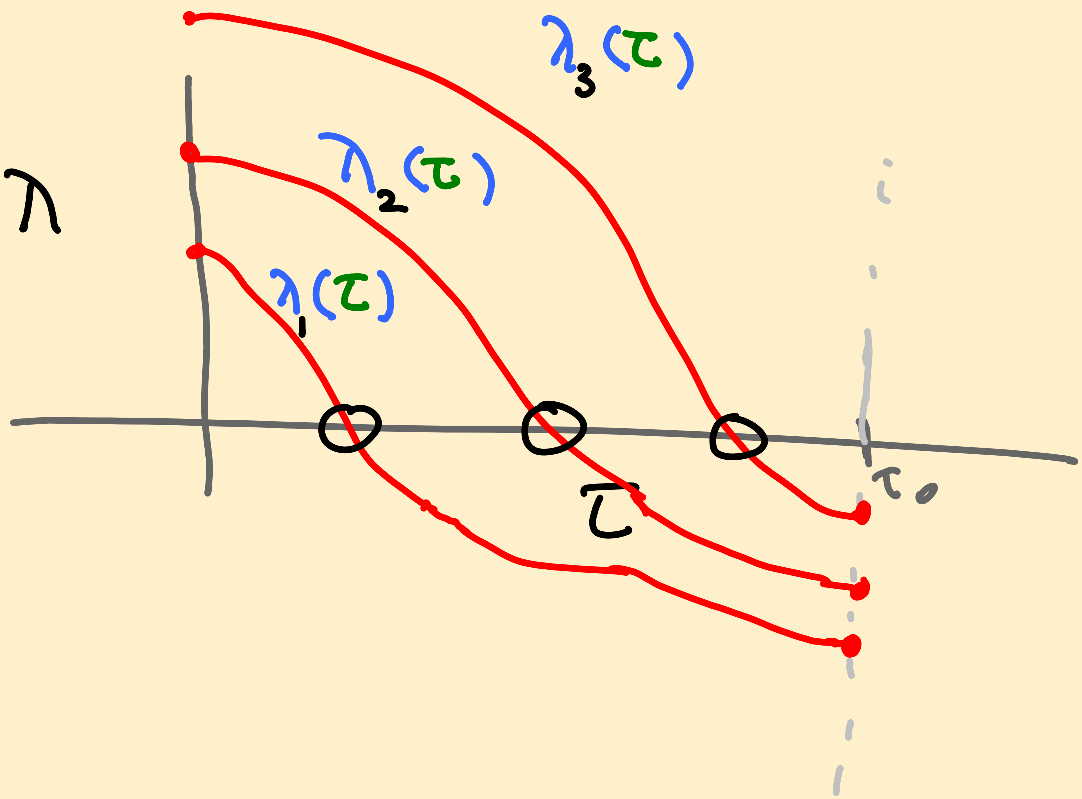
$$Q^{\tau}(u)=\int\frac{1}{m}|(\Delta+\tau)u|^2-\tau\int|\nabla u|^2+\tau^2\int|u|^2$$

is negative on an N-dimensional subspace [For m ~~big~~ enough and some τ]

$T(0)$ is positive definite

$T(\tau)$ has N negative eigenvalues.

Eigenvalues of $T(\tau)$

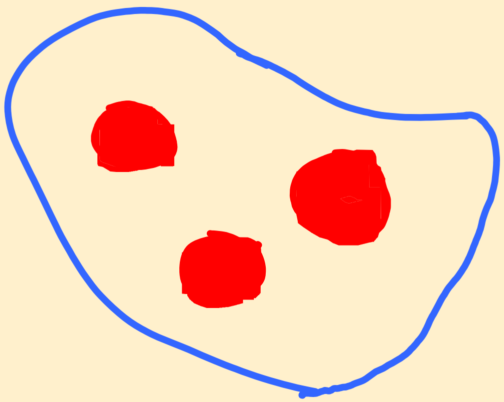


0's = ITE's

Cakoni - Gintedes   Cakoni - Gintedes - Haddar

$$Q_D^{\tau}(u) = \int \frac{1}{m} |(\Delta + \tau) u|^2 - \tau \int |\nabla u|^2 + \tau^2 \int |u|^2$$

monotone in m



= Translates of the same disk of radius R

$m_* < \min_{x \in D} m(x)$

τ_R = lowest TE of

u_R = eigenfunction $\in H_0^2(\bullet)$

Check on Span$\{u_j\}$

$$Q_D^{\tau_R}(u) \le Q_{\bullet}^{\tau_R}(u) = 0$$

$$u_j = \begin{cases} u_R(x + x_j) & \text{on } \bullet \\ 0 & D \setminus \bullet \end{cases}$$

Conclude $\exists$ N real TE's $\le \tau_R$

Count carefully (Serov - S) $N(\tau) \geq K(m)(|D|\tau)^{n/2}$

Eigenvalue Problem for 2x2 System

$$[\Delta + k^2(1+m)]u = mv$$

$$[\Delta + k^2]v = 0$$

$u = 0$ $\quad \frac{\partial u}{\partial \nu} = 0$

$$\Longrightarrow \begin{cases} \Delta_{oo} u + m v = -k^2(1+m) u \\ \Delta_{\_} v = -k^2 v \end{cases}$$

$$B := \begin{pmatrix} \Delta_{oo} & m \\ 0 & \Delta_{\_} \end{pmatrix} \quad I_m := \begin{pmatrix} 1+m & 0 \\ 0 & 1 \end{pmatrix} \quad \lambda = -k^2$$

$$(B - \lambda I_m)\begin{pmatrix} u \\ v \end{pmatrix} = 0$$

$$B = \begin{pmatrix} \Delta_{oo} & m \\ 0 & \Delta_{-} \end{pmatrix} \qquad I_m = \begin{pmatrix} 1+m & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left[(I_m^{-1} B) - \lambda I \right] \begin{pmatrix} u \\ v \end{pmatrix} = 0 \qquad \text{Transmission Eigenvalue} = -\lambda \; [= k^2]$$

λ = spectrum of $I_m^{-1} B$

Born Approximation

[set $I_m = I$]

λ_B = spectrum of B

Domain of B

$$B : H_0^2(D) \oplus \{ v \in L^2(D) \mid \Delta v \in L^2(D) \} \to L^2(D) \oplus L^2(D)$$

$$B = \begin{pmatrix} \Delta_{oo} & m \\ 0 & \Delta_{-} \end{pmatrix}$$ Born Approximation

Theorem Suppose $m(x) \in L^\infty(D)$ and $\exists\, \theta \in [0, 2\pi]$ such that $\mathrm{Re}\left(e^{i\theta} m(x)\right)\Big|_{N(\partial D)} > m_* > 0$

then B has Upper Triangular Compact resolvent [i.e. $B - \lambda_0 I$ is invertible for at least one $\lambda_0 \in \mathbb{C}$ and $(B - \lambda_0 I)^{-1}$ is UTC $= \begin{pmatrix} c & c \\ \mathbf{B} & c \end{pmatrix}$]

Corollary Spectrum of B is discrete and has finite multiplicity (? empty ?)

$$\begin{pmatrix} D_{oo} & m \\ 0 & D_- \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} 1+m & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

Transmission Eigenvalue Problem

Theorem Suppose ① $\mathrm{Re}(1+m) > \delta > 0$ in all of D

② $\exists\, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ s.t. $\mathrm{Re}\left(e^{i\theta} m(x)\right)\Big|_{N(\partial D)} > m_* > 0$

or

②' m is real in all of D and $m\Big|_{N(\partial D)} < -m_* < 0$

then $I_m^{-1} B$ has Upper Triangular Compact resolvent

Corollary spectrum of $I_m^{-1} B$ is discrete & finite multiplicity (? nonempty ?)

Interior Estimates λ large positive   No BC's

support of $(1-\varphi)$

$$\left.\begin{matrix} \|v_{\text{Inside}}\| \\ \|\nabla(v_{\text{Inside}})\| \end{matrix}\right\} \lesssim \frac{\|v_{\text{Outside}}\| + \|g\|}{\sqrt{\lambda}}$$

$$\|\nabla(\varphi v)\|^2 + \lambda\|\varphi v\|^2 \lesssim \|\varphi g\|^2 + \|v\|^2$$

$$(\lambda - 1)\|\varphi v\|^2 \lesssim \|(1-\varphi)v\|^2 + \|\varphi g\|^2$$

$+\ \|(1-\varphi)v\|^2$

$$\|v\|^2 \lesssim \|(1-\varphi)v\|^2 + \|\varphi g\|^2$$

$m > m_*$

$$\lesssim \left|\int m|(1-\varphi)v|^2\right| + \|\varphi g\|^2$$

$\int m|v|^2 \le m^* \|v\|^2$

$N(\varphi)$

$$\boxed{\|v\|^2 \lesssim \left|\int_D m|v|^2\right| + \|\varphi g\|^2}$$

Estimating v

$$(\Delta-\lambda)u \,\bar{v} + m v \bar{v} = f \bar{v}$$

$$u(\Delta-\lambda)\bar{v} = \bar{g} u$$

$$\int m|v|^2 = \int f\bar{v} - \bar{g}u$$

$$(\Delta-\lambda)u + mv = f$$

$$(\Delta-\lambda)v = g$$

$u=0$

$\frac{\partial u}{\partial \nu}=0$

$$\int_D \bar{v}(\Delta-\lambda)u_{oo} = \int_D (\Delta-\lambda)\bar{v}\, u_{oo}$$

For large positive λ

$$\|v\|^2 \lesssim \int m|v|^2 + \frac{\|g\|^2}{\lambda}$$

$$\boxed{\|v\|^2 \lesssim \left(\|f\|^2 + \frac{\|g\|^2}{\lambda} + \|g\|\,\|u\|\right)}$$

Estimate u in the standard way

$$(\Delta-\lambda)u + mv = f$$
$$(\Delta-\lambda)v = g$$

$u=0$

$\frac{\partial u}{\partial \nu}=0$

$$\bar{u}(\Delta-\lambda)u + mv\bar{u} = f\bar{u}$$

$$-\int|\nabla u|^2 - \lambda\int|u|^2 \lesssim \left(\|f\| + \|v\|\right)\|u\|$$

$$\boxed{\begin{aligned} \|u\| &\lesssim \frac{1}{\lambda}\left(\|f\|+\|v\|\right) \\ \|\nabla u\| &\lesssim \frac{\left(\|f\|+\|v\|\right)}{\sqrt{\lambda}} \end{aligned}}$$

$$\|v\|^2 \le \|f\|^2 + \frac{\|g\|^2}{\lambda}$$

$$\|u\|^2 \le \frac{1}{\lambda^2}\left(\|f\|^2 + \frac{\|g\|^2}{\lambda}\right)$$

$$\|\nabla u\|^2 \le \frac{1}{\lambda}\left(\|f\|^2 + \frac{\|g\|^2}{\lambda}\right)$$

$$(B - \lambda I)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

$$B = \begin{pmatrix} \Delta_{\infty} & m \\ 0 & \Delta_- \end{pmatrix}$$

$$B^* = \begin{pmatrix} \Delta_- & 0 \\ \overline{m} & \Delta_{\infty} \end{pmatrix}$$

[B^* = B with u & v switched & $m \longmapsto \overline{m}$]

1. $\ker(B - \lambda I) = \emptyset$
2. B has closed range
3. $\ker(B^* - \lambda I) = \emptyset$

$$\boxed{(B - \lambda I) \text{ invertible}}$$

$$\|v\|^2 \le \|f\|^2 + \frac{\|g\|^2}{\lambda^2}$$

$$\|u\|^2 \le \frac{1}{\lambda^2}\left(\|f\|^2 + \frac{\|g\|^2}{\lambda^2}\right)$$

$$\|\nabla u\|^2 \le \frac{1}{\lambda}\left(\|f\|^2 + \frac{\|g\|^2}{\lambda^2}\right)$$

$$\begin{pmatrix} \Delta_{00}-\lambda & m \\ 0 & \Delta_{\_}-\lambda \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

$$(B-\lambda I)^{-1} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

$$R_{11} : \begin{pmatrix} f \\ 0 \end{pmatrix} \xmapsto{\text{compact}} u \qquad R_{12} : \begin{pmatrix} 0 \\ g \end{pmatrix} \xmapsto{\text{compact}} u$$

$$R_{21} : \begin{pmatrix} f \\ 0 \end{pmatrix} \xmapsto{\underline{\text{Not}}\ \text{compact}} v \qquad R_{22} : \begin{pmatrix} 0 \\ g \end{pmatrix} \xmapsto{\text{compact}} v$$

Not so obvious

<u>Example</u> ($\lambda = 0$ $m = 1$)

f harmonic $\Rightarrow v = f$

$v_n = \sqrt{n+1}\, z^n$

$\|v_n\| = 1$

$R_{22}: g \mapsto v$

$$\int m|v|^2 = \int \cancel{f \bar{v}} - \bar{g} u$$

$$\cancel{f\bar v} = 0$$

Interior estimates $g \mapsto \varphi v$ compact

Suppose $g_n \longrightarrow 0$ then $u_n \longrightarrow 0$

$\varphi v_n \longrightarrow 0$

and

$$\int m|v_n|^2 = -\int g_n u_n \longrightarrow 0$$

$$\int m(1-\varphi^2)|v_n|^2 + \underbrace{\int m|\varphi v_n|^2}_{\to 0} = -\int g_n u_n \longrightarrow 0$$

$$m_* \int (1-\varphi^2)|v_n|^2 \longrightarrow 0$$

$$v_n \longrightarrow 0$$

Upper Triangular Compact (UTC)

Resolvent Identity.

$$(B-\mu I) = [(B-\lambda_0 I)-(\mu-\lambda_0) I] = \underbrace{(B-\lambda_0 I)}_{\text{invertible}} \underbrace{\left(I - (\mu-\lambda_0)(B-\lambda_0 I)^{-1}\right)}_{\text{Identity plus UTC}}$$

Apply UT analytic Fredholm theorem

Conclude spectrum of B is discrete [Born TE's]

Apply to TE's

$$(B-\lambda_0 I_m) = (B-\lambda_0 I)\left(I - \overbrace{(B-\lambda_0 I)^{-1}\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}}^{\text{UTC}}\right)$$

Conclude spectrum of $I_m^{-1} B$ is discrete or ~~all of $\mathbb{C}$~~

Analytic Fredholm Theorem

$C(z)$ compact analytic

If

$$\ker(I + C(z_0)) = \phi \quad \text{or} \quad \operatorname{coker}(I + C(z_0)) = \phi$$

then

$I + C(z)$ is invertible except for a disrete set

UT Analytic Fredholm Theorem

$$I + UTC = \begin{pmatrix} I + C_{11} & C_{12} \\ B & I + C_{22} \end{pmatrix} = \underbrace{\begin{pmatrix} I & 0 \\ B & I \end{pmatrix}}_{\text{Invertible}} \underbrace{\begin{pmatrix} I + C_{11} & C_{12} \\ -BC_{11} & I + C_{22} - BC_{12} \end{pmatrix}}_{\text{Identity + compact}}$$

Gaussian Elimination modulo compact operators coker $(B-\lambda I_m) = \emptyset$

$$\left.\begin{array}{l} (\Delta-\lambda)u = \lambda \bar{m} u \Rightarrow \int \bar{v}(\Delta-\lambda)u = \lambda \int \bar{m} u \bar{v} \\ (\Delta-\lambda)v_{oo} + \bar{m}\, u = 0 \Rightarrow \int u(\Delta-\lambda)\bar{v} = -\int m|u|^2 \end{array}\right\} \Rightarrow \int \bar{m} u \bar{v} = \frac{-\int m|u|^2}{\lambda}$$

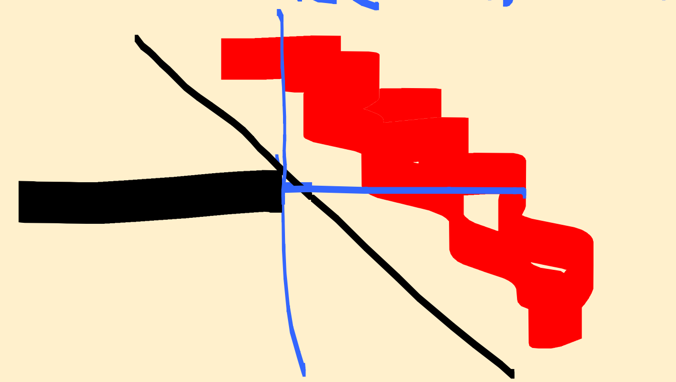
$$\Downarrow$$

$$-\int|\nabla v|^2 - \lambda \int |v|^2 = -\int \bar{m} u \bar{v}$$

$$\underbrace{-\int|\nabla v|^2 - \lambda\int|v|^2}_{\text{negative real}} = \frac{1}{\lambda}\underbrace{\int m|u|^2}$$

concentrates near ∂D as $\lambda \to +\infty$ where m takes values in a cone that excludes $-\mathbb{R}$

$\mathrm{Re}(e^{i\theta} m) > m_*$



What about Existence?

<u>Example</u>

$$\left(\frac{d}{dx} - \lambda\right) u = f$$

$$u(0) = \alpha\, u(1)$$

$$u(x) = \int_0^x e^{\lambda(x-y)} f(y)\, dy + e^{\lambda x} \cdot \left[\frac{\int_0^1 e^{\lambda(1-y)} f(y)\, dy}{1 - \alpha e^{-\lambda}} \right]$$

Compact Resolvent $\forall \alpha$

<u>Spectrum</u> $\{\lambda \mid e^{\lambda} = \alpha\}$

$$\lambda_n = \log \alpha + 2\pi i n$$

$$u_n = e^{\lambda_n x}$$

Spectrum runs away to ∞ as $\alpha \to 0$ (IVP)

$$\begin{pmatrix} \Delta & m \\ 0 & \Delta \end{pmatrix}$$

$u|_{\partial D} = 0$

$\alpha \frac{\partial u}{\partial \nu}\Big|_{\partial D} = (1-\alpha)\, v\Big|_{\partial D}$

UTC resolvent $\forall\ \alpha$

Double Dirichlet Spectrum at $\alpha = 0$

Does spectrum run away as $\alpha \to 1$?