The Support of a Far Field

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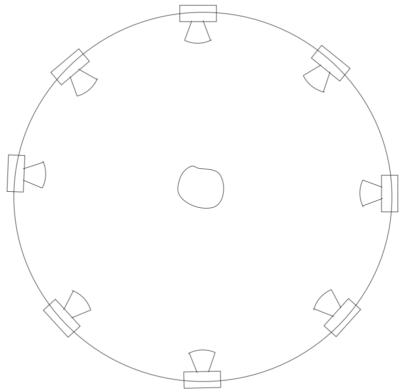
Inverse Source Problem



$$u_{tt} - \Delta u = F(x,t)$$
$$u(x,0) = 0$$
$$u_t(x,0) = 0$$

We measure the field u far away from the source (in the far field). The zero initial conditions guarantee that we are seeing only the field radiated by F, not by some other source that was present in the past.

Time Harmonic Model

$$F(x,t) = e^{i\omega t}F(x)$$
$$u(x,t) = e^{i\omega t}u(x)$$

The wave equation becomes the Helmholtz equation ($k = n_0\omega$).

$$\left(\Delta + k^2\right) u = F(x)$$

and the zero initial conditions tell us that the relevant solution is the *outgoing solution*. The outgoing far field is exactly the restricted Fourier transform of the source.

$$\widehat{u}(\xi) = \frac{\widehat{F}(\xi)}{-\xi^2 + (k - i0)^2}$$

$$u \sim \alpha(\Theta)\frac{e^{-ikr}}{ikr^{\frac{1}{2}}} + 0\frac{e^{ikr}}{ikr^{\frac{1}{2}}} = \widehat{F}(k\Theta)\frac{e^{-ikr}}{ikr^{\frac{1}{2}}}$$

Why the source problem?

- This is the relevant model for passive remote sensing.
- The source problem is the Born or linear approximation to many active remote sensing problems, including back-scattering.
- The inverse source problem is linear, therefore simpler.

Why work with the far field?

- Its mathematically simpler. Admits some exact formulas.
- You can (really) compute the far field from the near field and (theoretically) conversely.
- There's more information in the near field, but you have to be smarter to use it.

Why work at a fixed frequency?

- Need to work at fixed k when the radiation pattern of the source is time dependent or the medium is dispersive.
- A specific feature of a complicated medium may dominate the scattering at a particular frequency.
- Active scanner may only generate narrow bandwith.
- MUSIC and some Time Reversal/Phase conjugation algorithms utilize sensor arrays to locate point sources with this sort of data.
- I can do it now.

Non-uniqueness for the Inverse Problem

Gravitation

$$\Delta u = \rho(x)$$

Any two spherically symmetric planets with the same total mass have the same gravitational far field.

Helmholtz

$$\left(\Delta + k^2\right) u = F(x)$$

Any two radially symmetric sources radiate the same far field if $\int f(r) J_0(kr) r dr$ is the same for both.

Unique Properties to compute

To compute a property of the source, we must either:

- Know something special about F, i.e. constrain F's to obtain uniqueness (see e.g. Isakov's book, Inverse source Problems):
  - $F = \chi_K$ — the indicator (Heavyside) function of a convex set
  - $F = \sum a_i \delta_{p_i}$ — a sum of point sources
  - $F = \chi_{S_p}$ — S_p is star shaped about p
- Find something that all the F's have in common.

The Support of the Source

$$(\Delta + k^2)u = F$$
$$u \sim \widehat{F}(k\Theta)\frac{e^{-ikr}}{ikr^{\frac{1}{2}}}$$

For any distribution Φ with compact support, $F = (\Delta + k^2)\Phi$ is a non-radiating source because its far field is its restricted Fourier transform, $(|\xi|^2 - k^2)\widehat{\Phi}(\xi)$, which vanishes when $\xi = k\Theta$.

Because F and $F + (\Delta + k^2)\Phi$ have the same far field, we can always make the (support of the) source bigger without changing the far field.

We cannot find an upper bound on the support of F.

Support of the Far Field

The *support* of the far field will be the smallest set in a class (e.g. convex sets) that supports a source which radiates the far field. It will satisfy:

- It must have been in there. Any source or scatterer which radiates (has restricted Fourier transform equal to) the far field must *contain* the *scattering support* of the data.

- It could have been the entire source. There is a source which radiates the field, and is supported in (any neighborhood of) its scattering support.

Classes of scattering support

- convex sets
- unions of well-separated convex sets

  $B = \bigcup B_k \qquad \text{diam}(B_k) < \underset{j \neq k}{\text{dist}}(B_k, B_j)$

  – This condition is independent of wavelength.

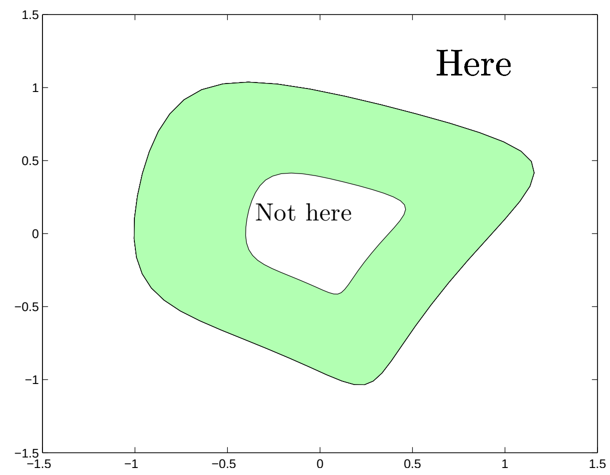
  – Point sources are always UWSCS
- sets which are star shaped about p

- Why does it have to be so complicated ?
- Examples of sources which radiate the same far fields.

Mathematical Tools

- Rellich's Lemma – You can compute the outgoing wave far away from the source using only the measured far field.

- Unique Continuation – You can (in theory) compute the outgoing wave using only the far field at exactly those points which can be connected to infinity (where you measured the far field) by a path which avoids the source.



Single and Double Layers Sources for Helmholtz

Multi-layer source

$$\gamma = \text{ a piecewise smooth curve}$$

$$\delta_\gamma = \text{monopole layer}$$

$$\delta_\gamma^{'} = \text{dipole layer}$$

$$\mathcal{C}_\gamma = f\delta_\gamma + g\delta_\gamma^{'}$$

Free Multi-layer Source

$$\mathcal{C}_\gamma^v = \frac{\partial v}{\partial \nu}\delta_\gamma + v\delta_\gamma^{'}$$

$$(\Delta + k^2)v = 0$$

e.g.

$$v = J_0(kr)$$

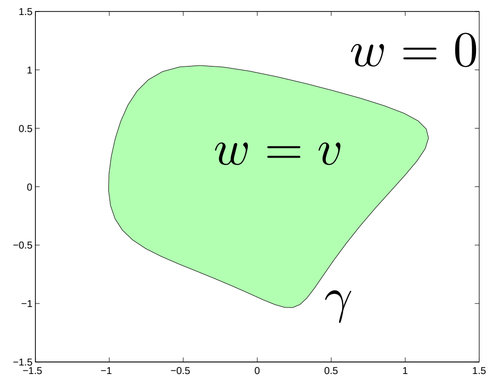
$$v = e^{ik\Theta \cdot x}$$

You can fatten up a multiple layer source without changing the far field by convolution with a normalized radially symmetric function.

Closed Curves support Non-radiating Layers

- If v is free and γ is closed (i.e $\gamma = \partial\Omega$) then $\mathcal{C}_\gamma^v$ is non-radiating.
- Define w as in the picture, and check the jump formulas for single and double layers.

$$(\Delta + k^2)w = \mathcal{C}_\gamma^v$$



Converse: All non-radiating layers are sums of free $\mathcal{C}_{\partial\Omega}^v$'s

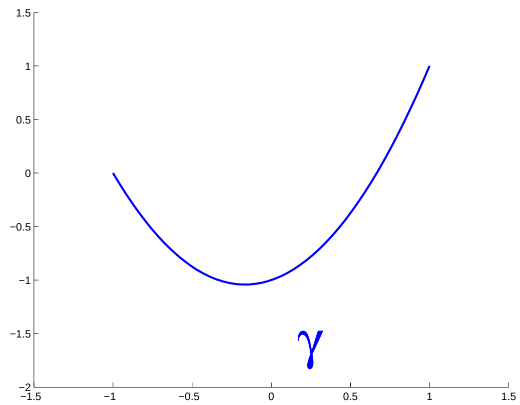
Arcs are Minimal

Hypothesis:

- $\mathcal{C}_\gamma = f\delta_\gamma + g\delta_\gamma'$
- No fake parts of γ — $\operatorname{supp} \mathcal{C}_\gamma = \gamma$
- γ contains no closed curve (no boundary) — $\mathbb{R}^2 \setminus \gamma$ connected.

Conclusion:

- f and g are unique.
- γ is minimal – you can't shrink it.

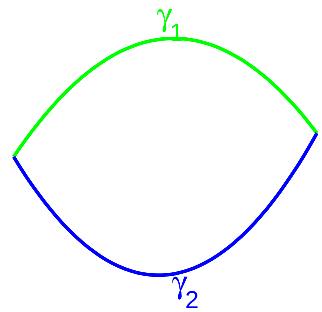


Minimal vs. Smallest

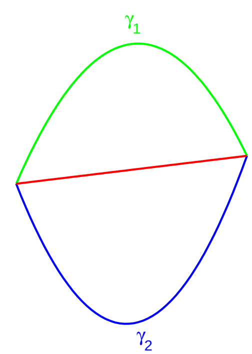
- Minimal sources can't be made strictly smaller. You can't build a source on a subset that radiates the same far field.
- The smalllest source (the support of the far field) — you can't build any source which doesn't contain this one, that radiates its far field.
- If there's a smallest source, its the only minimal source.

Multiple Minimal Sources

- If γ_1 and γ_2 have the same endpoints, and v is free, then $\mathcal{C}^v_{\gamma_1}$ and $\mathcal{C}^v_{-\gamma_2}$ radiate the same far field — because their sum is a $\mathcal{C}^v_\gamma$ with γ closed.



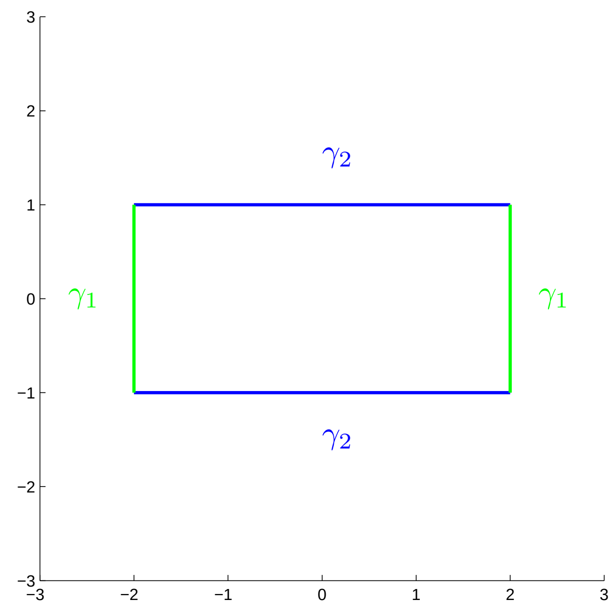
- Both $\mathcal{C}^v_{\gamma_1}$ and $\mathcal{C}^v_{-\gamma_2}$ are minimal, so there can be no smalllest source.



- However, there is a smallest convex source – the straight line connecting the endpoints.

Unions of Disjoint Convex Sources

- If v is free, then $\mathcal{C}^v_{\gamma_1}$ and $\mathcal{C}^v_{-\gamma_2}$ radiate the same far field.
- Both $\mathcal{C}^v_{\gamma_1}$ and $\mathcal{C}^v_{-\gamma_2}$ are minimal, so there can be no smalllest disjoint union of convex sets that radiates this far field.
- However, only γ_1 is a well-separated union of convex sets.



Intesection Lemma

1. supp $F_1 \subset \Omega_1$
2. supp $F_2 \subset \Omega_2$
3. $\widehat{F}_1(k\Theta) = \widehat{F}_2(k\Theta)$
4. $\mathbb{R}^n \setminus (\Omega_1 \bigcup \Omega_2)$ connected.

$\Longrightarrow$

- there exists F_3
- supp$F_3 \subset N_\varepsilon(\Omega_1 \cap \Omega_2)$
- $\widehat{F}_3(k\Theta) = \widehat{F}_1(k\Theta)$

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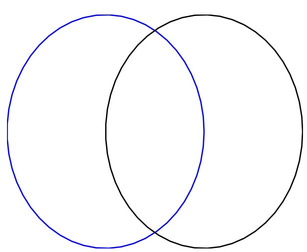
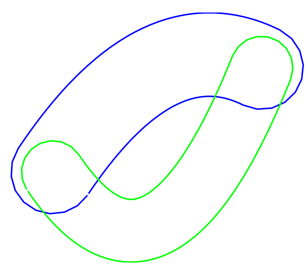
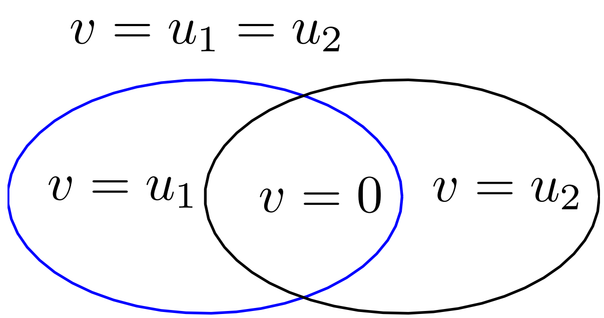
4.  and not 

Class of scattering support must be closed under intersection and have property 4.

Proof of lemma

$$\begin{aligned}(\Delta + k^2)u_1 &= F_1\\ (\Delta + k^2)u_2 &= F_2\end{aligned}$$

$$v = \begin{cases} \phi u_1, & x \in \mathbb{R}^n \backslash \Omega_1 \\ \phi u_2, & x \in \mathbb{R}^n \backslash \Omega_2 \\ 0, & x \in \Omega_1 \cap \Omega_2 \end{cases}$$



$$F_3 = (\Delta + k^2)v$$

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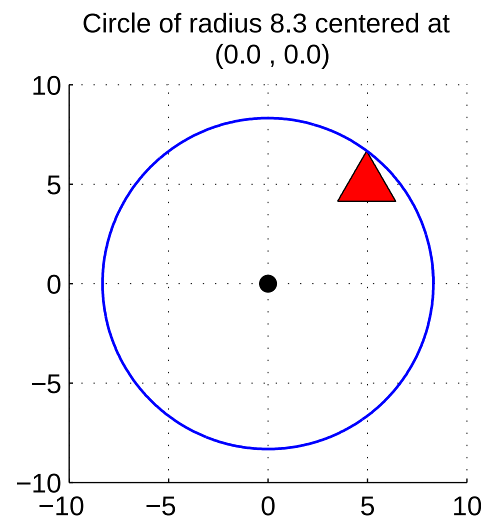
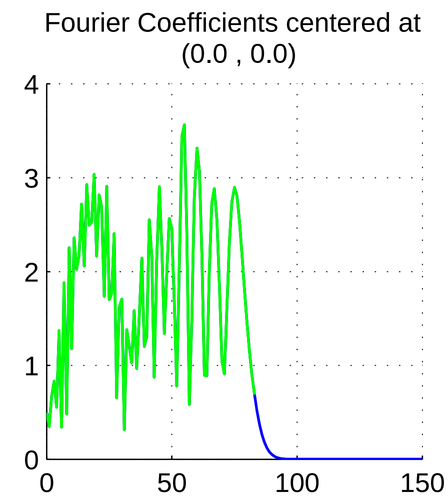
v is a well-defined function as long as $u_1 = u_2$ on $\mathbb{R}^n \setminus (\Omega_1 \bigcup \Omega_2)$, which follows from Rellich's lemma and unique continuation.

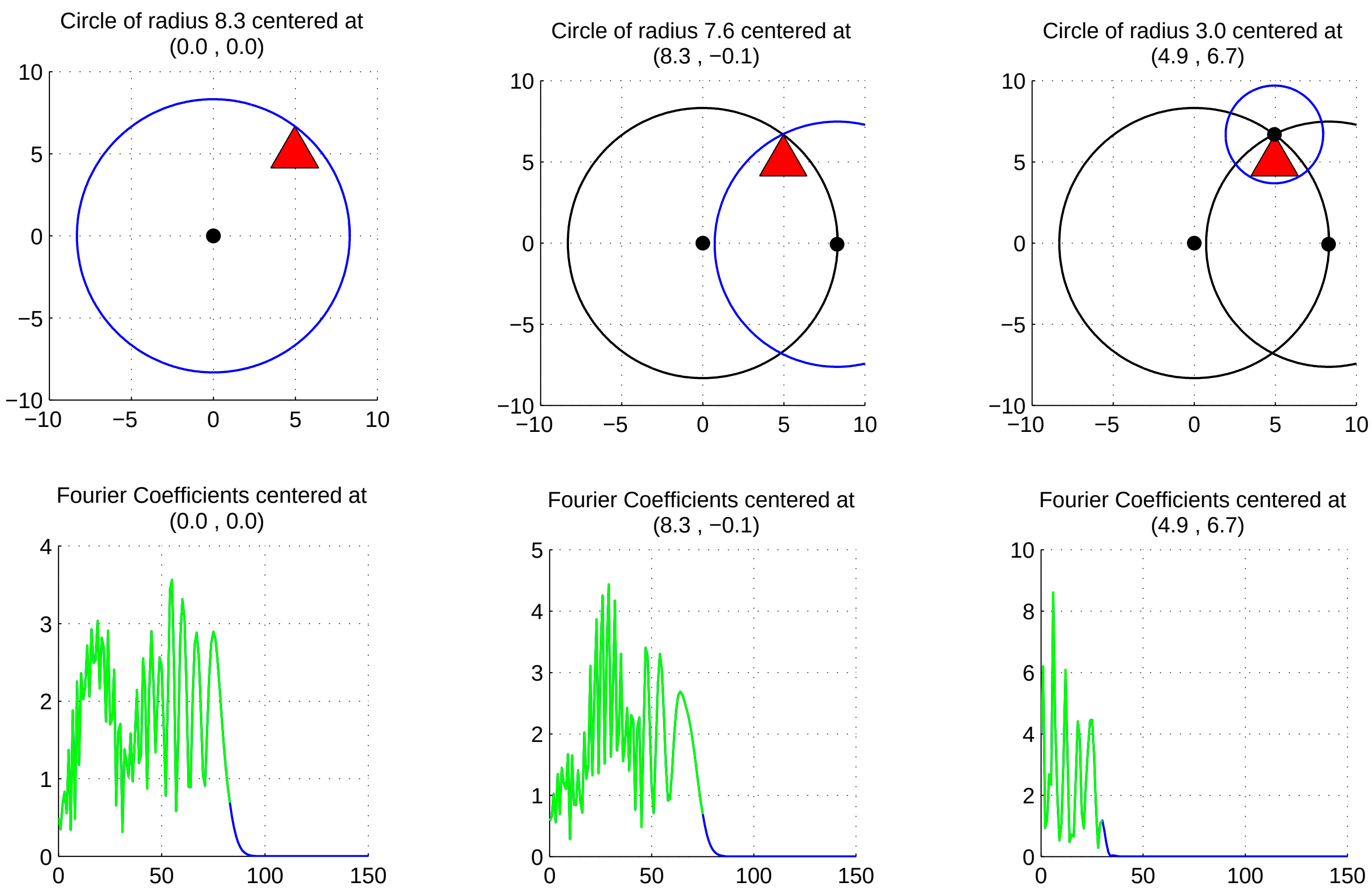
Computing the Convex Support of a Far Field (k = 10)

- Expand the far field $f(\theta)$ in a Fourier series
- Plot the modulus of the Fourier coefficients.
- Find the value of n where they become effectively zero.
- The convex support of the far field is contained in the ball of radius $R = \frac{n}{k}$ centered at zero.





- Replace $f(\theta)$ by $e^{ik\Theta \cdot c} f = e^{ik|c|\cos(\theta - \phi_c)} f(\theta)$ and repeat.



Three Part Algorithm

1. Test — find the radius of the smallest ball, centered at zero, that can radiate the field.

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2. Translation formula — The far field has no natural origin. In the far field, you can translate with a formula.

$$F_c = F(x - c)$$

$$\widehat{F}_c(k\Theta) = e^{ik\Theta \cdot c}\widehat{F}(k\Theta)$$

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3. Intersect — It would be difficult to draw conclusions from the tests if we couldn't intersect. That's why the special classes.

The Test

$$\widehat{F}(k\Theta) = \sum f_n e^{in\theta}$$

There exists $F \in L^2(B_R)$ $\iff$ $\left\{ \frac{f_n}{\left(\int_0^R J_n^2(ks)sds\right)^{\frac{1}{2}}} \right\} \in l^2$

$$\Longrightarrow \quad f_n = \int F(r\Phi) J_n(kr) e^{in\phi} dV \quad \leq \quad ||F||_{L^2} \cdot ||J_n||_{L^2(\mathrm{supp}F)}$$

$\Longleftarrow$ Extend $\sum f_n e^{in\theta}$ to $\sum f_n e^{in\theta} \frac{J_n(R\rho)}{J_n(Rk)}$

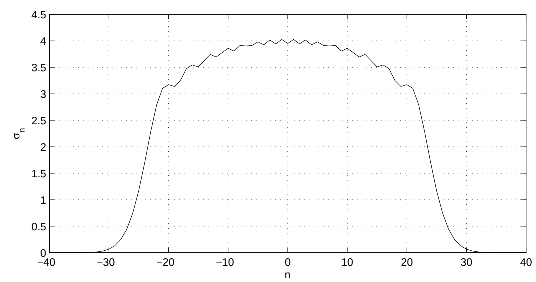
$$J_n(R\rho)e^{in\theta} = (\widehat{e^{in\phi}\delta_{r=R}})$$

Rapid Transition to Evanescence

$$|f_n| \leq \frac{C}{k^2}\sigma_n(kR)$$

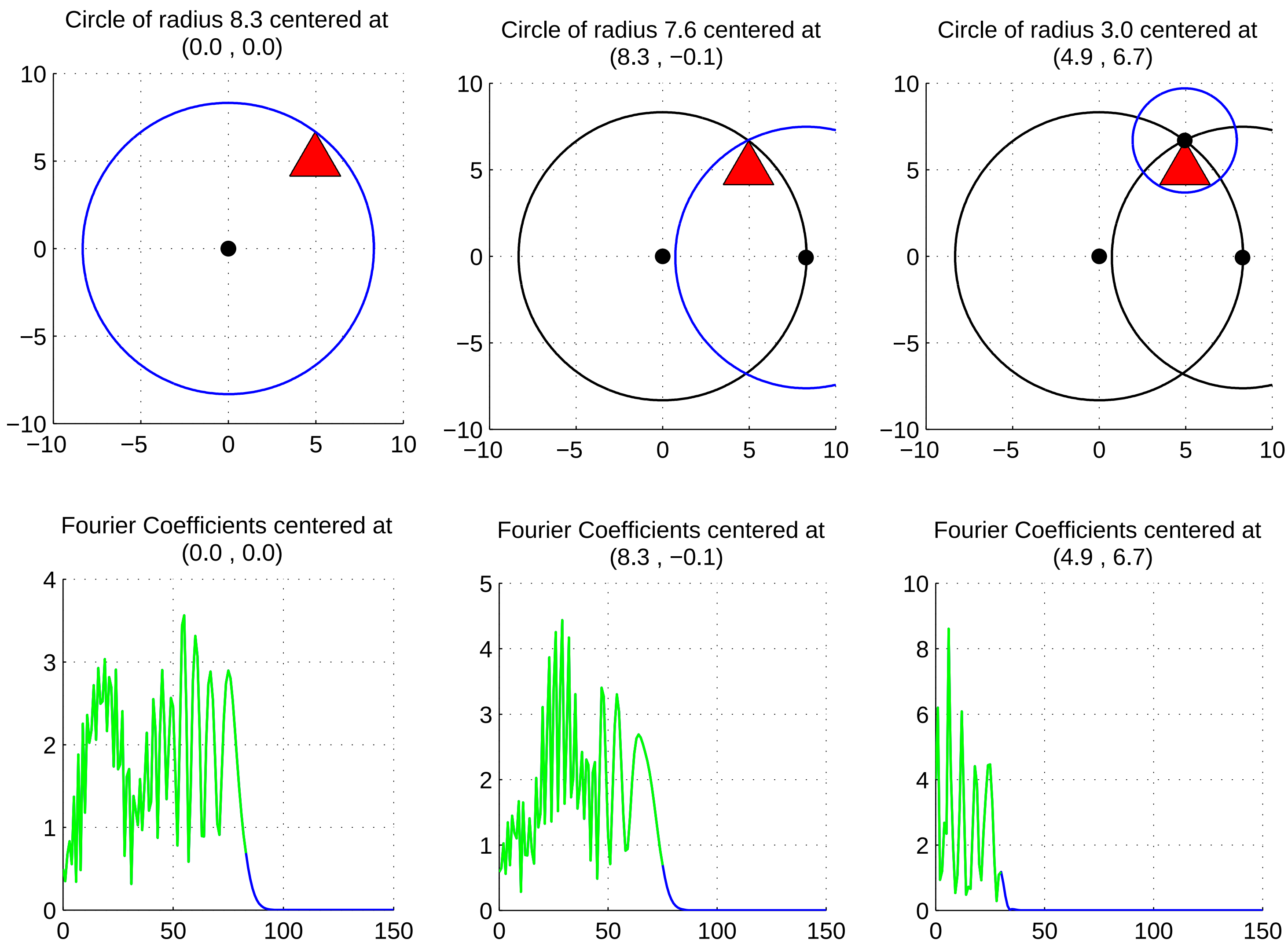
$$\sigma_n(R) := ||J_n||_{L^2(B_R)}$$

- $\sigma_n(R)$ uniformly big if $n < R$
- $\sigma_n(R)$ uniformly small if $n > R$

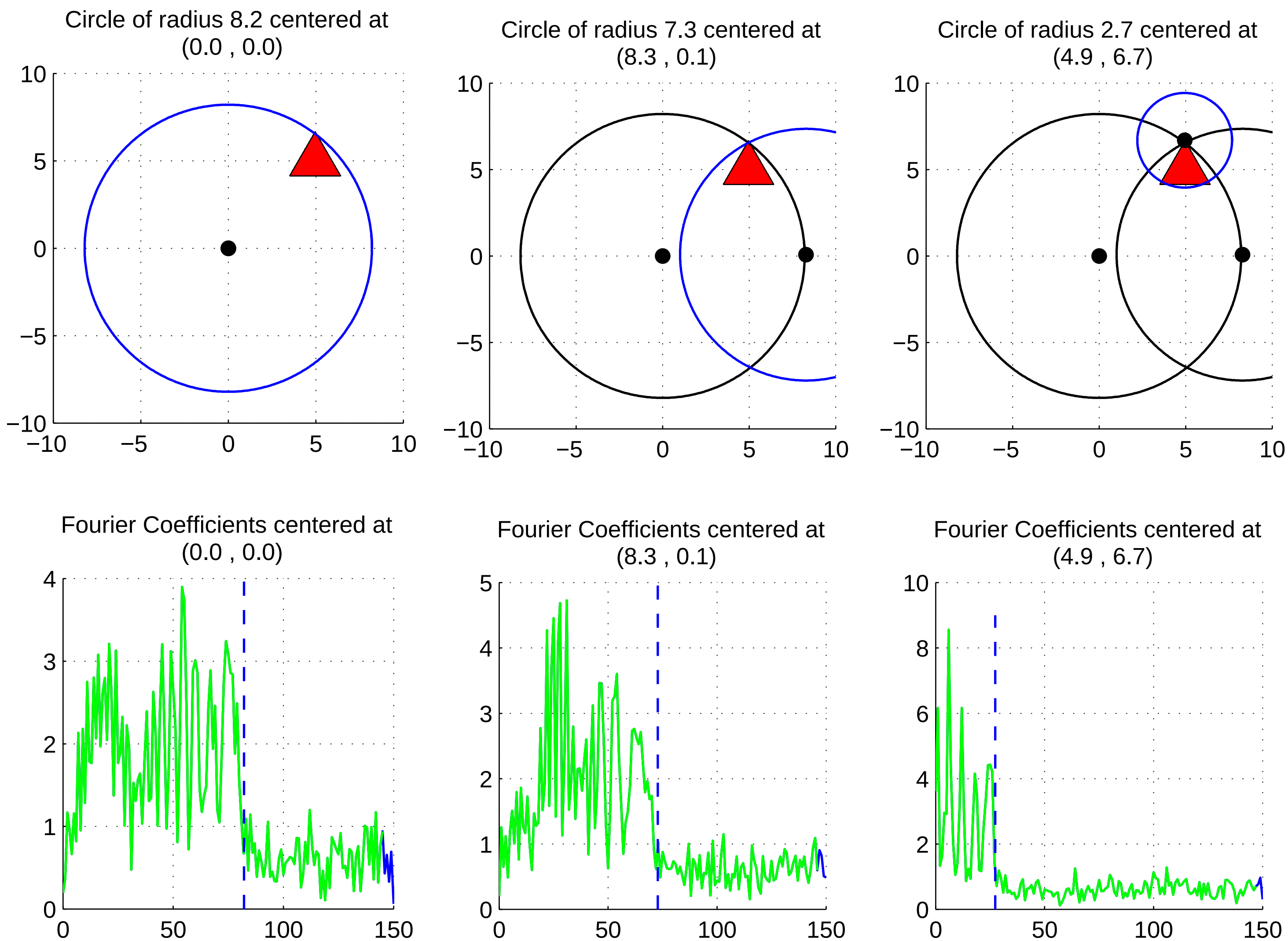


$$\sigma_n(R) \sim \begin{cases} R^{\frac{1}{2}}(R^2 - n^2)^{\frac{1}{4}} & |n| < R \\ \frac{1}{\sqrt{n+\frac{1}{2}}}\left(\frac{eR}{2(n+\frac{1}{2})}\right)^{(n+\frac{1}{2})} \sim 0 & |n| > R \end{cases}$$

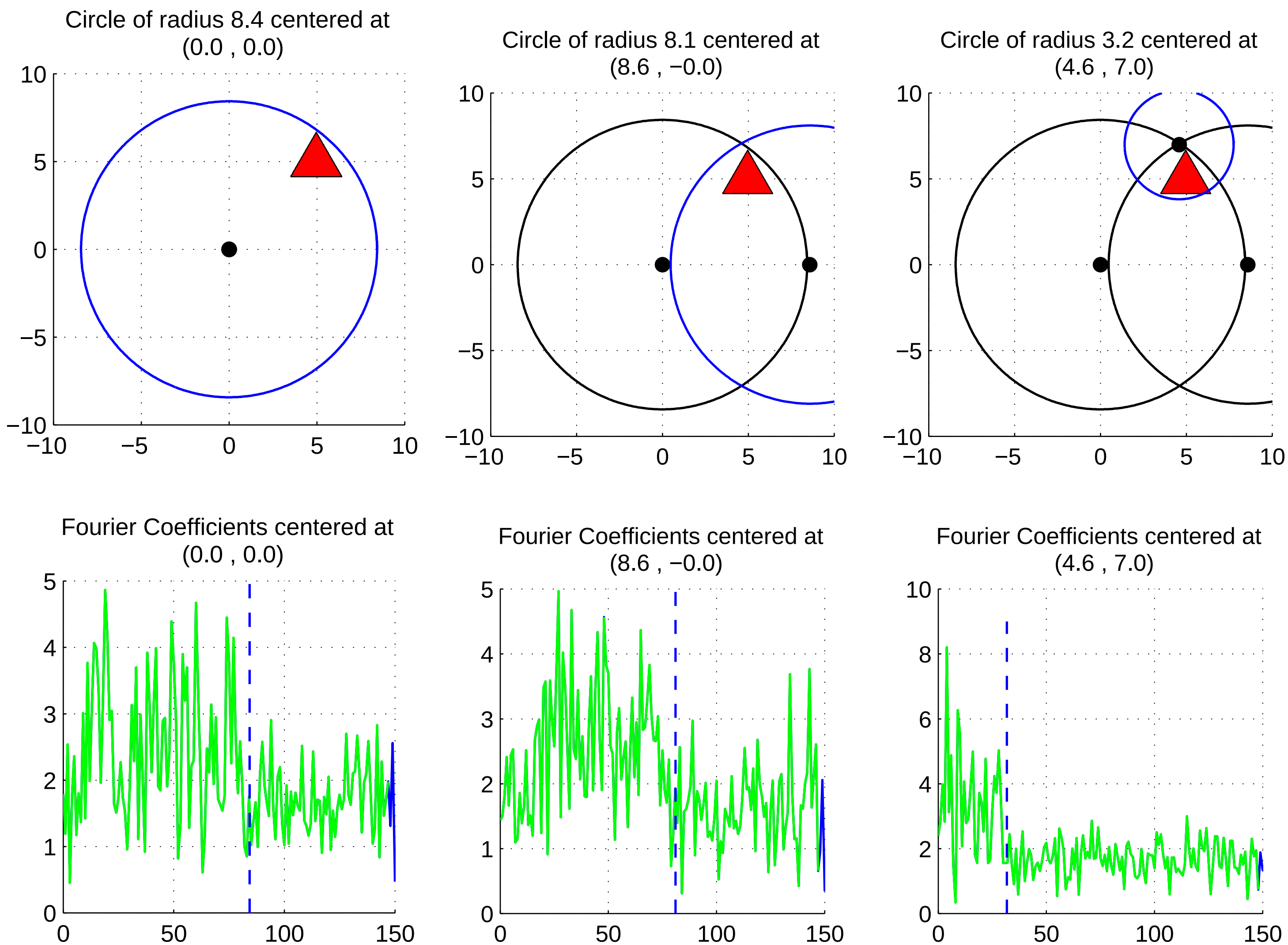
Locating the Scattering Support (k = 10)



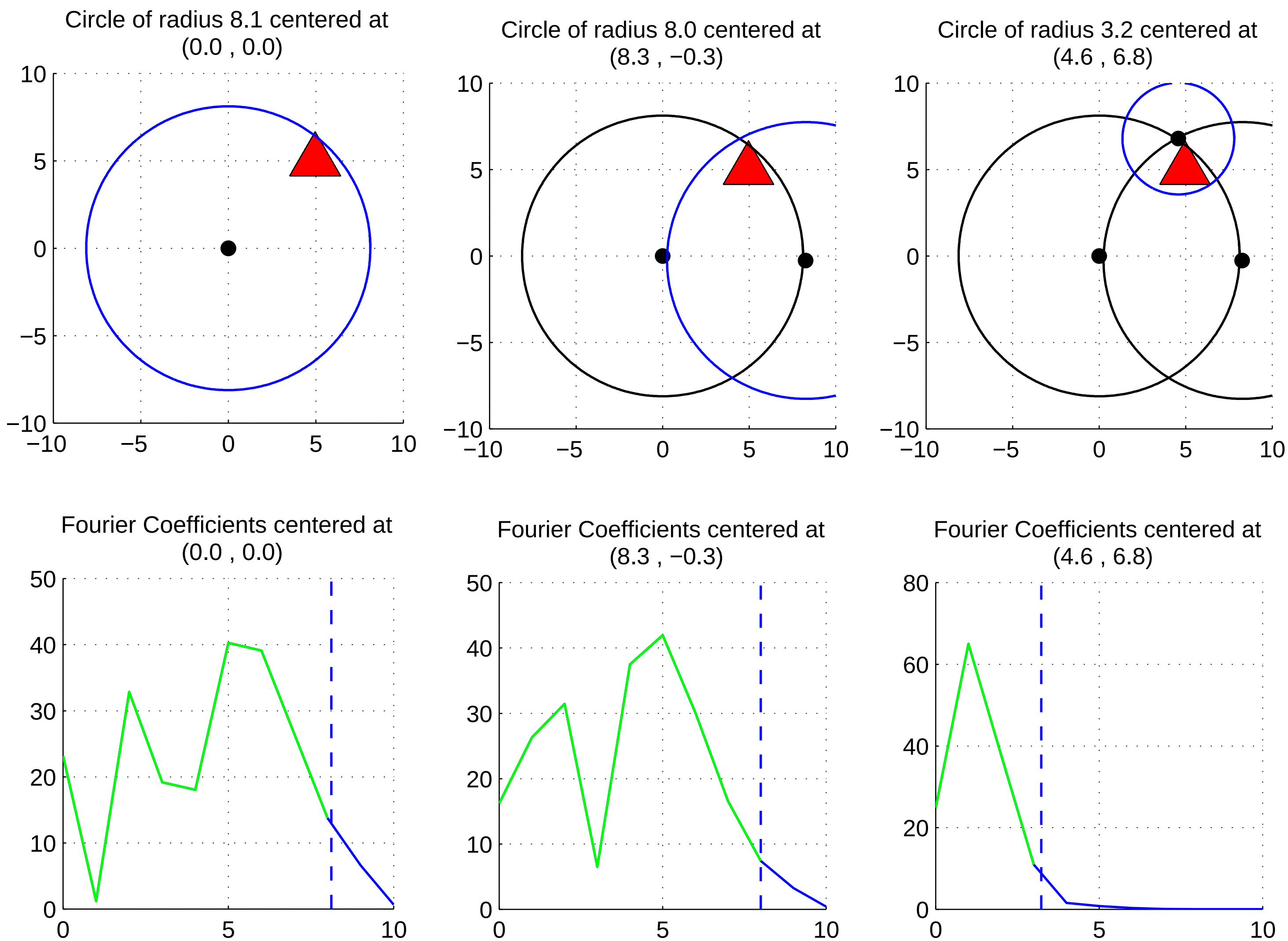
20% additive gaussian noise (k = 10)



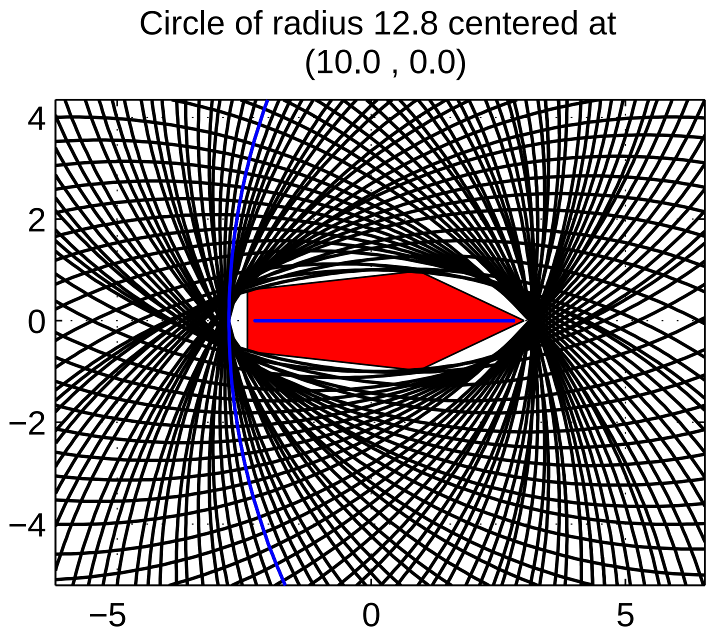
50% additive gaussian noise (k = 10)

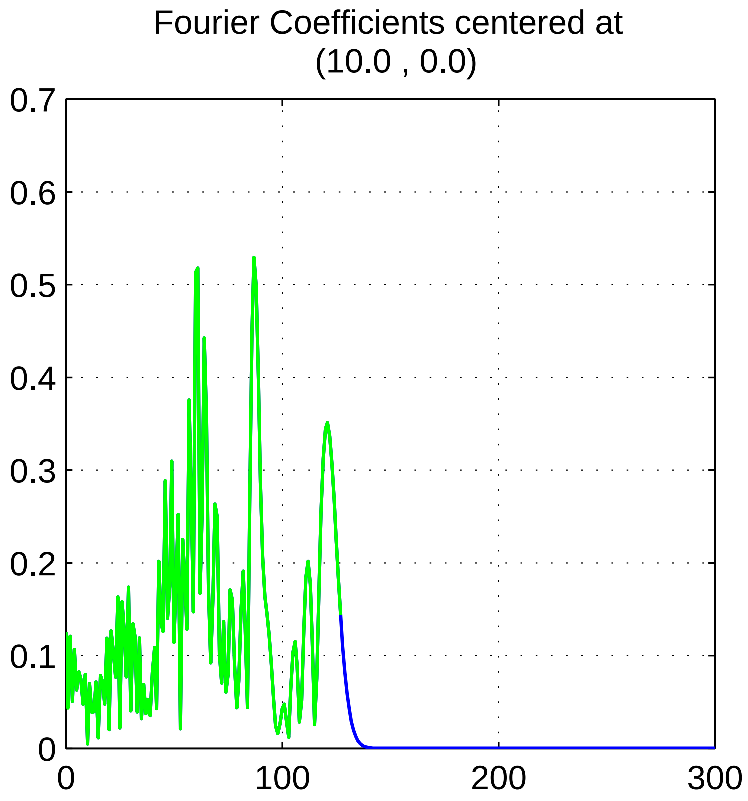


20 nodes (k = 1)


Circle of radius 8.1 centered at
(0.0 , 0.0)
Circle of radius 8.0 centered at
(8.3 , −0.3)
Circle of radius 3.2 centered at
(4.6 , 6.8)
Fourier Coefficients centered at
(0.0 , 0.0)
Fourier Coefficients centered at
(8.3 , −0.3)
Fourier Coefficients centered at
(4.6 , 6.8)

A Pentagon and 100 circles





Credits

Joint work with Steve Kusiak

Related work by and with Roland Potthast

Motivated by the Linear Sampling method of Colton Kirsch and the the Factorization Method of Kirsch

Future Directions

- Use Partial Bandwidth
- Partial Aperture
- Compute UWSCS when possible
- Estimate number of WSCS directly