Lecture 1 - Modelling

Note Title 1/3/2020

Class Web Page

https://sites.math.washington.edu/~sylvest/courses/math307/

HomeWork - WebWork
not WebAssign

https://courses1.webwork.maa.org/webwork2/uw-math307ab/

Access etext through Canvas link.

https://canvas.uw.edu/courses/1356495/modules/items/10020079

This is the only way I will use Canvas.

Grades will be posted on Catalyst

https://catalyst.uw.edu/

Differential Equations

1. Modelling (i.e. Word Problems $F = ma$)
2. Calculus ← Easiest Part
3. Algebra
4. Interpretation

First Order Differential Equation

is a formula for the derivative of a function $\frac{dy}{dt}$ in terms of y and t.

Examples

$$\frac{dy}{dt} = y + t$$

$$\frac{dy}{dt} = \sin t$$

$$\frac{dy}{dt} = y$$

or

$$\frac{dy}{dx} = y + x$$

$$\frac{dy}{dx} = \sin x$$

$$\frac{dy}{dx} = y$$

Problem Suppose an small heavy object is thrown upwards with an initial velocity of 44.7 meters/sec.

(a) write a differential equation for the velocity as a function of time [before it hits the ground] [neglecting air resistance]

(b) write the Initial Value Problem.

Solution Newton - "Force = mass · acceleration"

acceleration = derivative of velocity

$$m\frac{dv}{dt} = \text{gravitational force}$$

$$\cancel{m}\,\dot{v} = -\cancel{m}g \quad ; \quad g = -9.8 \text{ m/s}$$

(a) DE is $\frac{dv}{dt} = -9.8$

(b) IVP is DE plus Initial Condition

$$\frac{dv}{dt} = -9.8 \qquad v(0) = 44.7 \text{ m/s}$$

Problem part (c) - Solve the IVP

Solution $v(t) = -9.8t + C$

$$44.7 = v(0) = -\cancel{9.8}\cdot 0 + C$$

$$v(t) = 44.7 - 9.8t$$

A newly constructed fish pond contains 2000 liters of water. Unfortunately the pond has been contaminated with 5 kg of a toxic chemical during the construction process. The pond's filtering system removes water from the pond at a rate of 200 liters/minute, removes 40% of the chemical, and returns the same volume of water to the pond. Write a differential equation for the time (measured in minutes) evolution of:

The total mass (in kilograms) of the chemical in the pond:

$$\frac{dm}{dt} = ?$$

I want $\frac{kg}{min}$ removed

I know $200 \frac{liters}{min}$ pass thru filter

How many kg in each of those liters?
(this depends on m)

$\frac{m}{2000}$ kg/liter

How many of those kg's are removed each minute?

$$\underbrace{0.4}_{\text{unitless}} \cdot \underbrace{\frac{m}{2000}}_{kg/\cancel{liter}} \cdot \underbrace{200}_{\cancel{liters}/minute}$$

$$\boxed{\frac{dm}{dt} = -\frac{0.4 \cdot 200}{2000} m}$$

The concentration (in kg/liter) of the chemical in the pond:

write c in terms of m

$$c = \frac{m}{2000}$$

Relate $\frac{dc}{dt}$ to $\frac{dm}{dt}$

$$\frac{dc}{dt} = \frac{1}{2000}\frac{dm}{dt} = \frac{-1}{2000}\,0.04\,m$$

Now eliminate m from the equation.

$$\frac{dc}{dt} = -0.04\left(\frac{m}{2000}\right)$$

$$\boxed{\frac{dc}{dt} = -0.04\,c}$$

Let s = time in hours, write a DE for the concentration as a function of time in hours

$$\boxed{\frac{dc}{dt} = -0.04\,c}$$

Let s = time in hours, write a DE for the concentration as a function of time in hours

Solution - Relate s and t

$t = 60s$ minutes = hours $\cdot$ 60

$$\frac{dc}{ds} = \frac{dc}{dt}\cdot\frac{dt}{ds} = (-0.04\,c)\cdot 60$$

$$\boxed{\frac{dc}{ds} = -2.4\,C}$$

Initial Value Problem

Example

(DE) $\dot{y} \quad f(t, y)$ $\qquad \dot{y} = t(4-y)$

(IC) $y(t_0) = y_0$ $\qquad y(0) = 0$

DE = Differential Equation

IC = Initial Condition

Theorem - There is exactly one solution to the IVP.

The solution is a function $y(t)$

To be discussed later

① There are conditions:

$f(t, y)$ = differentiable function

② Solution may not last forever.

" There is a unique solution defined in some interval about t_0. "

3 Topics

Direction Fields - Graphical representation of a Differential Equation that helps us to understand the behavior of solutions.

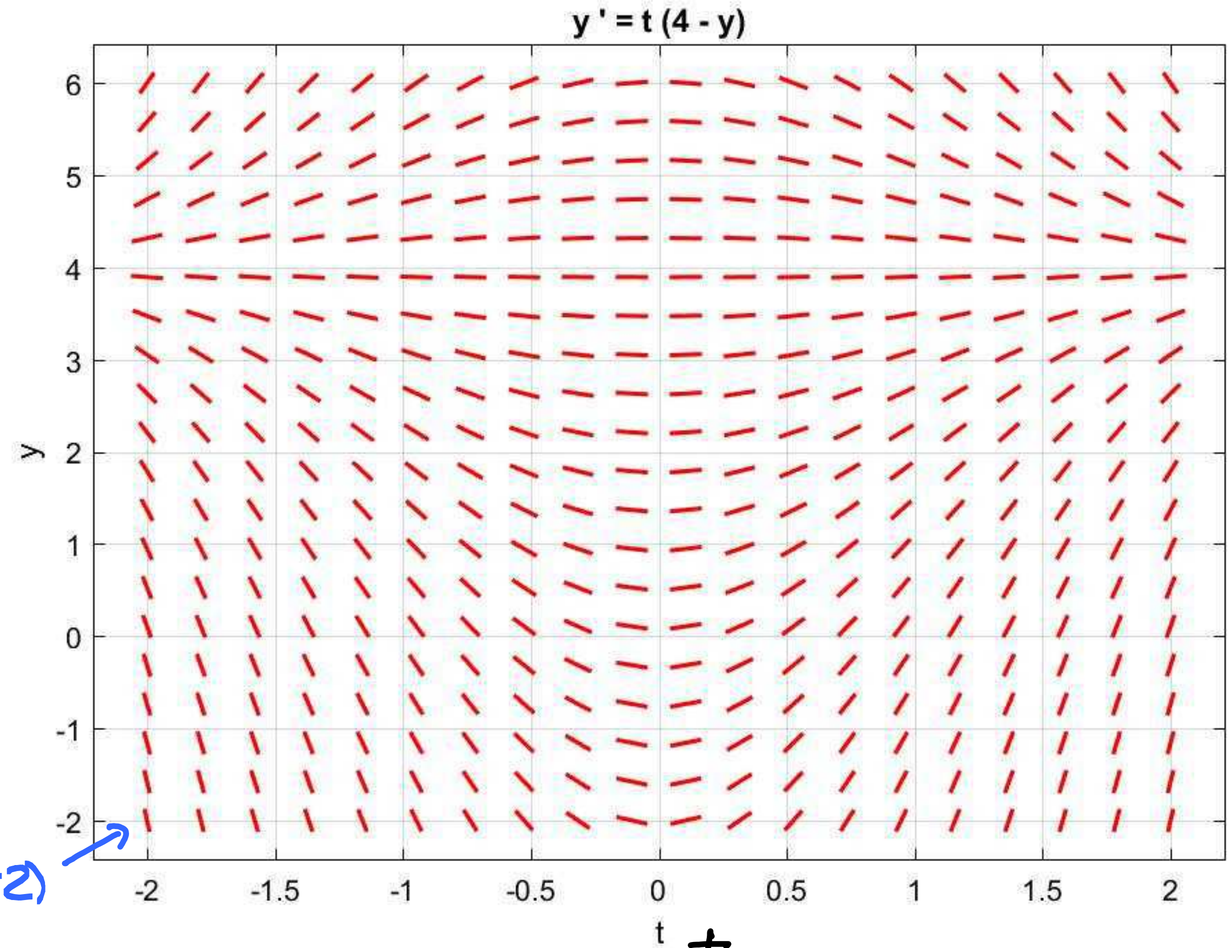
Separable Differential Equations

method for deriving a formula for the solution to one type of IVP.

Euler's Method

Method for writing a computer program for solving an IVP.

Direction Field $\dot{y} = t(4-y)$



y

$(-2,-2)$

t

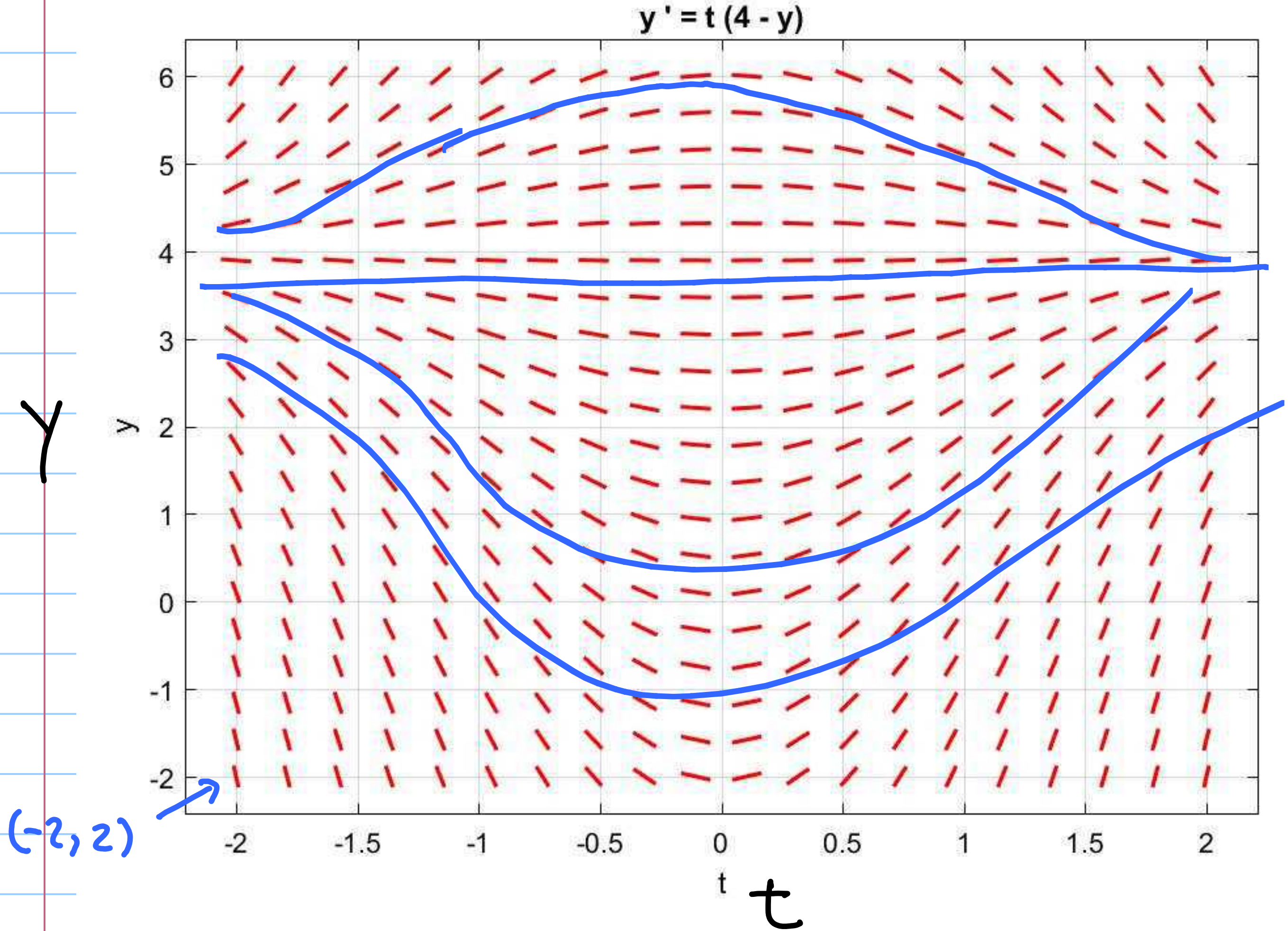
The red line at the point (t,y)

has slope $t(4-y)$

		Ans
•	All the lines at $(t,4)$ have slope ?	0
•	All the lines at $(0,y)$ have slope ?	0

• The line at $(-2,-2)$ has slope $-2(4+2) = -12$

Direction Field $\dot{y} = t(4-y)$

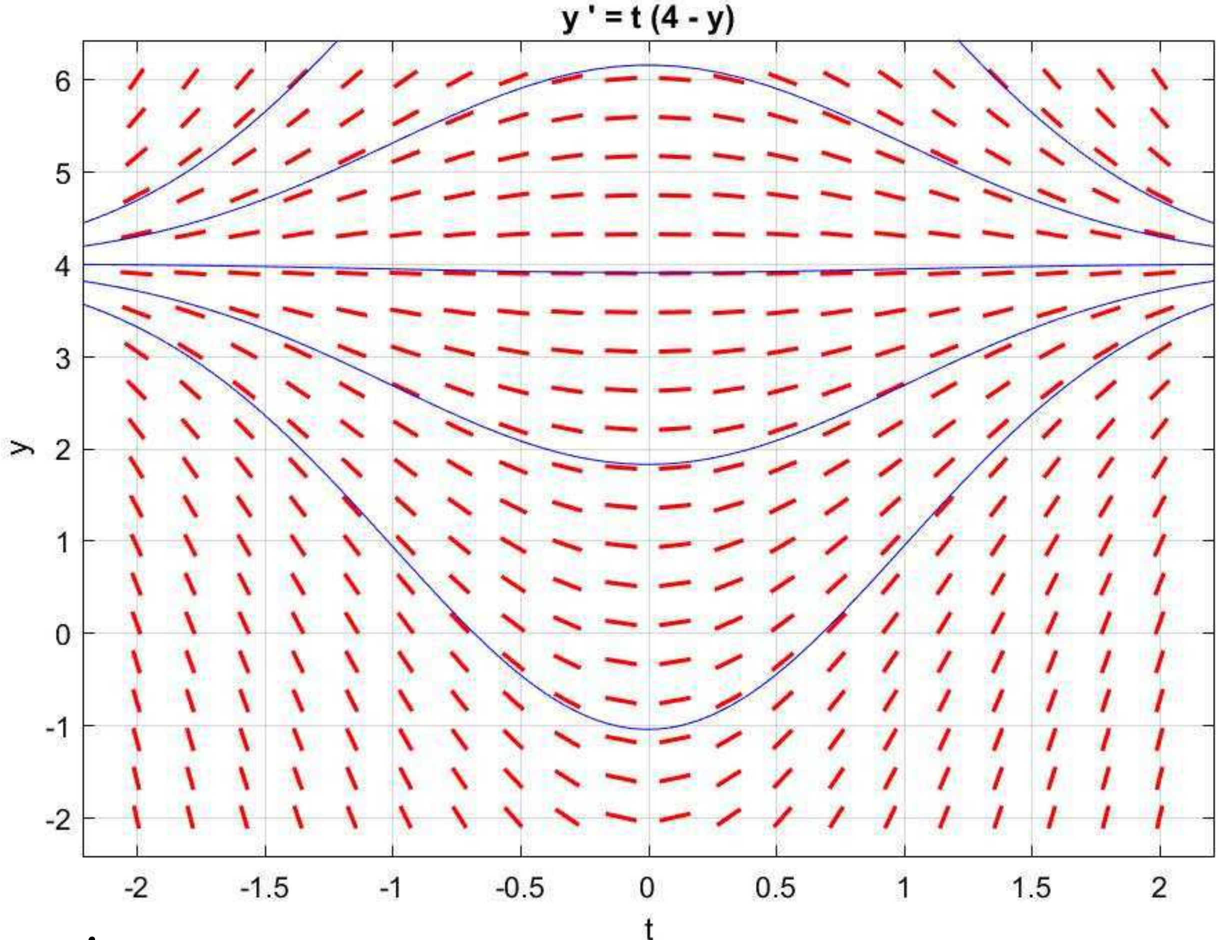


The red line at the point (t, y)

has slope $t(4-y)$

Curves that are tangent to the lines are solutions to the DE.

Draw Some Solutions



What important properties of solutions to the DE can we see from this picture?

Solutions that start below 4, ? stay below 4

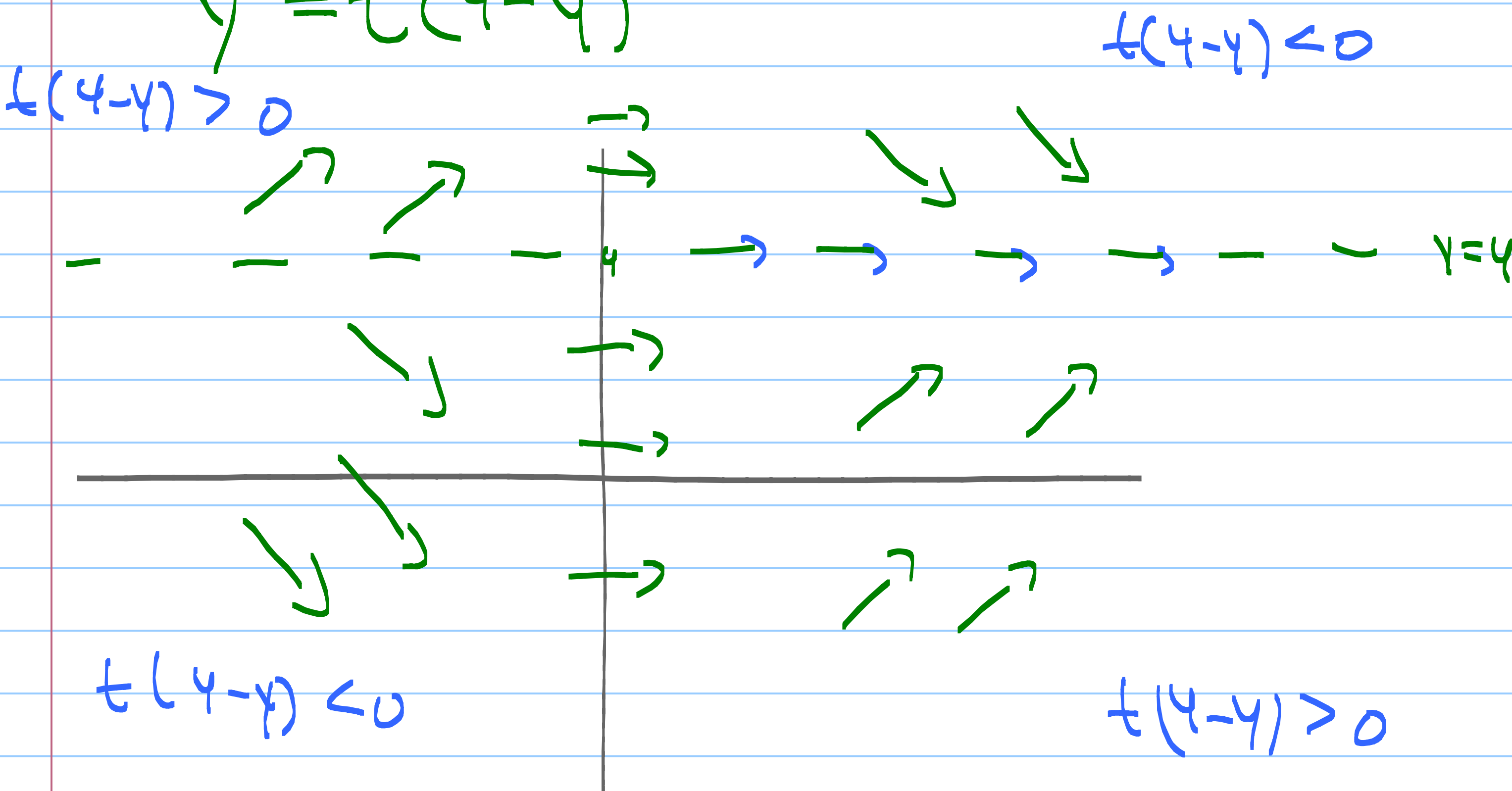
Solutions that start above 4, ? stay above 4

$\lim_{t \to \infty} y(t) = ?\ 4$ $\lim_{t \to -\infty} y(t) = ?\ 4$

The solution $y=4$ is a "stable equilibrium".

Draw DField by hand

$$\dot{y} = t(4-y)$$



Euler's Method

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$$\dot{y} = f(t, y)$$
$$y(0) = 0$$

Example

$$\dot{y} = t(4-y)$$
$$y(0) = 0$$

Problem Approximate y on the interval $[0, 2]$ using a step size of $h = \frac{1}{2}$.

Linear Approximation

For small h

$$y(t+h) \approx y(t) + h\,\dot{y}(t)$$
$$\approx y(t) + h\, f(t, y(t))$$

For $h = 1/2$

$$y(0) = 0$$
$$y(0+\tfrac{1}{2}) \approx 0 + \tfrac{1}{2} f(0, y(0))$$
$$y(1) \approx y(\tfrac{1}{2}) + \tfrac{1}{2} f(\tfrac{1}{2}, y(\tfrac{1}{2}))$$

Example

$$f(t, y) = t(4-y)$$

$$y(t) + h\, t(4-y)$$

$$y(0) = 0$$

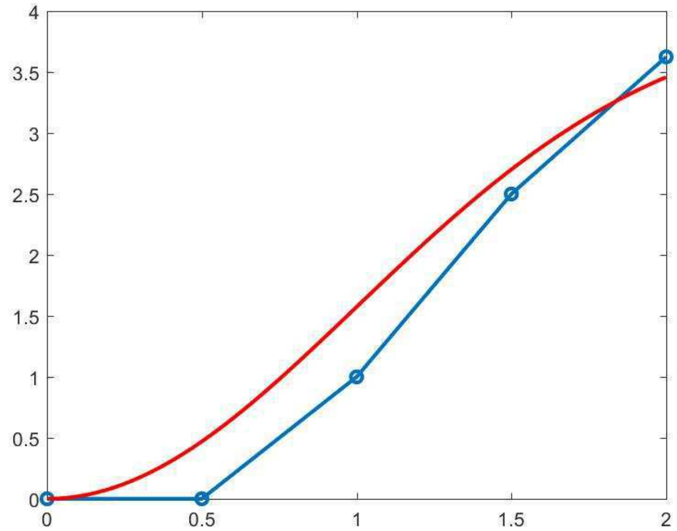
$$y(\tfrac{1}{2}) \approx 0 + \tfrac{1}{2}(0 \cdot (4-0)) = 0$$

$$y(1) \approx 0 + \tfrac{1}{2}(\tfrac{1}{2}(4-0)) = 1$$

Problem Approximate y on the interval $[0,2]$ using a step size of $h=\frac{1}{2}$

$$\dot{y} = t(4-y)$$

t	0	$1/2$	1	$3/2$	2
y					
$f(t,y)$					



$$\boxed{\begin{aligned} t_{New} &= t_{old} + h \\ Y_{New} &= Y_{old} + f(t_{old}, Y_{old})\, h \end{aligned}}$$

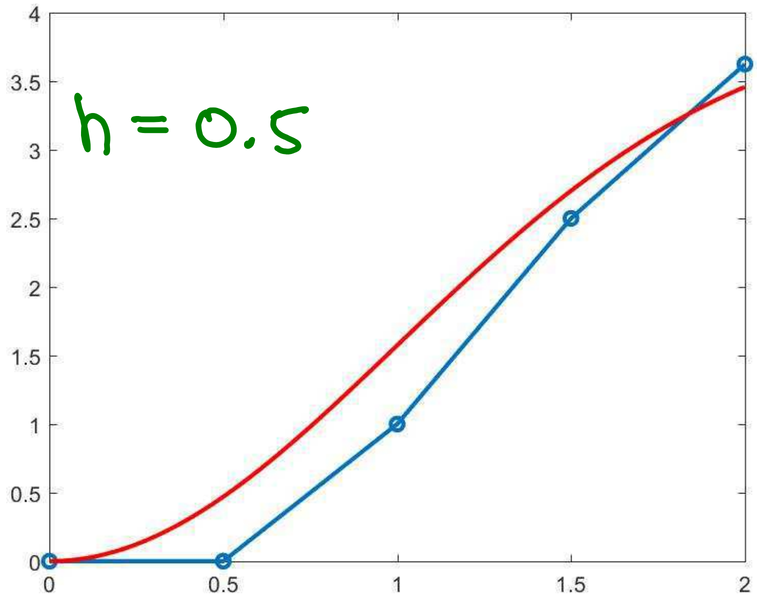
$$y(0) = 0$$

$$y\left(\tfrac{1}{2}\right) = 0 + \tfrac{1}{2}\left[0(4-0)\right] = 0$$

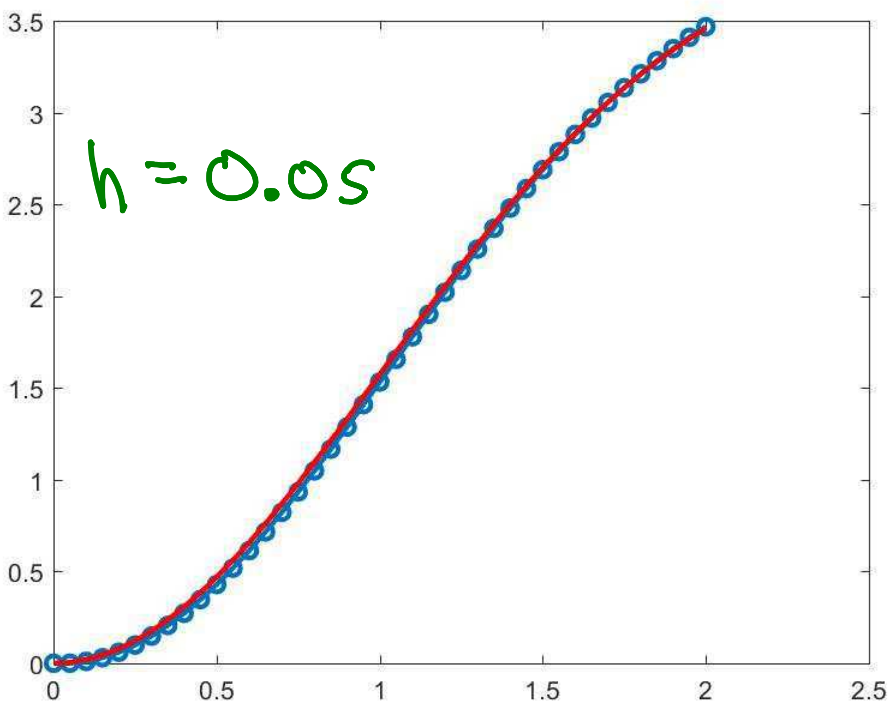
$$y(1) = 0 + \tfrac{1}{2}\left[\underline{\tfrac{1}{2}(4-0)}\right] = 1$$

$$y\left(\tfrac{3}{2}\right) = 1 + \tfrac{1}{2}\left[1(4-1)\right] = \tfrac{5}{2}$$

$$y(2) = \tfrac{5}{2} + \tfrac{1}{2}\left[\tfrac{3}{2}\left(4-\tfrac{5}{2}\right)\right] = \tfrac{29}{8}$$



t	0	$1/2$	1	$3/2$	2	
y	0	0	1	$5/2$	$\frac{29}{8}$	
$f(t,y)$	0	2	3	$9/4$		



We usually choose h much smaller than in the example above and write a program to calculate y and $f(t,y)$

```
%% Euler's method

% define the DE
f = @(t,y) t*(4-y);

%%
t(1) = 0; %initial time
y(1) = 0; % initial condition

% define h
h = 0.05
% how many steps
numsteps = round(2/h);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% This is Euler's method
for m = 1:numsteps
    t(m+1) = t(m)+h;
    y(m+1) = y(m) + h*f(t(m),y(m));
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% plot the results
figure(2)
plot(t,y,'marker','o')

t = linspace(0,2);
ytrue = 4.*(1-exp(-t.^2/2));
line(t,ytrue,'color','r')
```

$t_{new} = t_{old} + h$

$y_{new} = y_{old} + hf(t_{old}, y(t_{old}))$

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Finding A Formula

$\dot{y} = t(4-y)$

$y(0) = 0$

$$\frac{dy}{dt} = t(4-y)$$

$$\frac{dy}{4-y} = t\,dt$$

$$\frac{dy}{y-4} = -t\,dt$$

$$\ln|y-4| = -\frac{t^2}{2} + c$$

$$|y-4| = e^{-t^2/2} \cdot e^{c}$$

$$y-4 = \pm e^{c} e^{-t^2/2}$$

Justification

$$\frac{1}{4-y}\frac{dy}{dt} = t$$

$$\int \frac{1}{y-4}\frac{dy}{dt}\,dt = -\int t\,dt$$

Substitute $\boxed{u = y}$

$du = \frac{dy}{dt}\,dt$

$$\int \frac{du}{u-4} = -\int t\,dt$$

$$\ln|u-4| = -\frac{t^2}{2} + c$$

But $u = y$, so

$$\ln|y-4| = -\frac{t^2}{2} + c$$

$$y-4 = k e^{-t^2/2}$$

define $k = \pm e^{c}$

$y(0) = 0$ so $-4 = k$

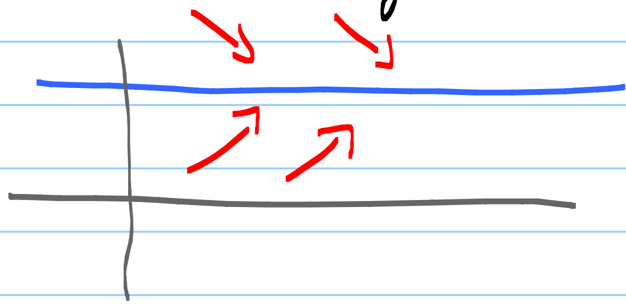
$$\boxed{y = 4\left(1 - e^{-t^2/2}\right)}$$

$$\lim_{t\to\infty} y(t) = 4$$

The solution $y=4$ is a "stable equilibrium".

If $\dot{y} = t(4-y)$, $y(0) = \text{anything}$ then $\lim\limits_{t\to\infty} y(t) = 4$

We can tell that $\lim\limits_{t\to\infty} y(t) = 4$ without finding a formula:



$y=4$

For $t>0$

If $y(t)>4$, $\dot{y}<0$, so $y(t)$ decreases

If $y(t)<4$, $\dot{y}>0$, so $y(t)$ increases

If $y(t)=4$, $\dot{y}=0$, so $y(t)$ doesn't change

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Solving Initial Value Problem with Formulas

Separable Differential Equation

$$\frac{dy}{dt} = F(t)\,G(y) \qquad \text{(DE)}$$

$$y(t_0) = y_0 \qquad \text{(Initial Condition)}$$

Step 1 $\dfrac{dy}{G(y)} = F(t)\,dt$

Step 2 Integrate Both Sides

$$\int \frac{dy}{G(y)} = \int F(t)\,dt + C$$

Step 3 Use (IC) to find C

Step 4 Try to solve for $y(t)$

Sometimes you can

Explicit Solution $y(t) =$

Sometimes you can't

Implicit Solution

Example coming

Problem   Find an "implicit" solution

to

$$\frac{dy}{dt} = \frac{y}{1+y^2}$$

$$y(0) = 1$$

Answer

$$\left(\frac{1+y^2}{y}\right) dy = dt$$

$$\frac{dy}{y} + y\,dy = dt$$

$$\ln|y| + \frac{y^2}{2} = t + c$$

$$\frac{1}{2} = c$$

$$\ln|y| + \frac{y^2}{2} = t + \frac{1}{2}$$

We would like to write an "explicit solution" $y = f(t)$

but sometimes we can't.

Example

$$\frac{dy}{dx} = \frac{-x}{y} \; ; \; y(0) = 1$$

Step 1 $\quad y\,dy = -x\,dx$

Step 2 $\quad \frac{y^2}{2} = -\frac{x^2}{2} + C$

Step 3 Find C using (IC) $y(0) = 1$

$$\frac{1^2}{2} = -\frac{0^2}{2} + C$$

$$C = \frac{1}{2}$$

Implicit Solution

$$\frac{y^2}{2} = -\frac{x^2}{2} + \frac{1}{2}$$

or $\boxed{y^2 + x^2 = 1}$

Explicit Solution

$$y^2 = 1 - x^2$$

$$y = \pm\sqrt{1 - x^2}$$

Step 4 $\quad y(0) = 1$ so $\boxed{y = \sqrt{1 - x^2}}$

(IVP) $\frac{dy}{dx} = \frac{-x}{y}$; $y(0) = 1$

Solution

$\boxed{y = \sqrt{1-x^2}}$

Notice – Formula only makes sense for $-1 \leq x \leq 1$

Recall Theorem – There is exactly one solution to the IVP.

The solution is a function $y(x)$

To be discussed ~~later~~ now

① There are conditions:

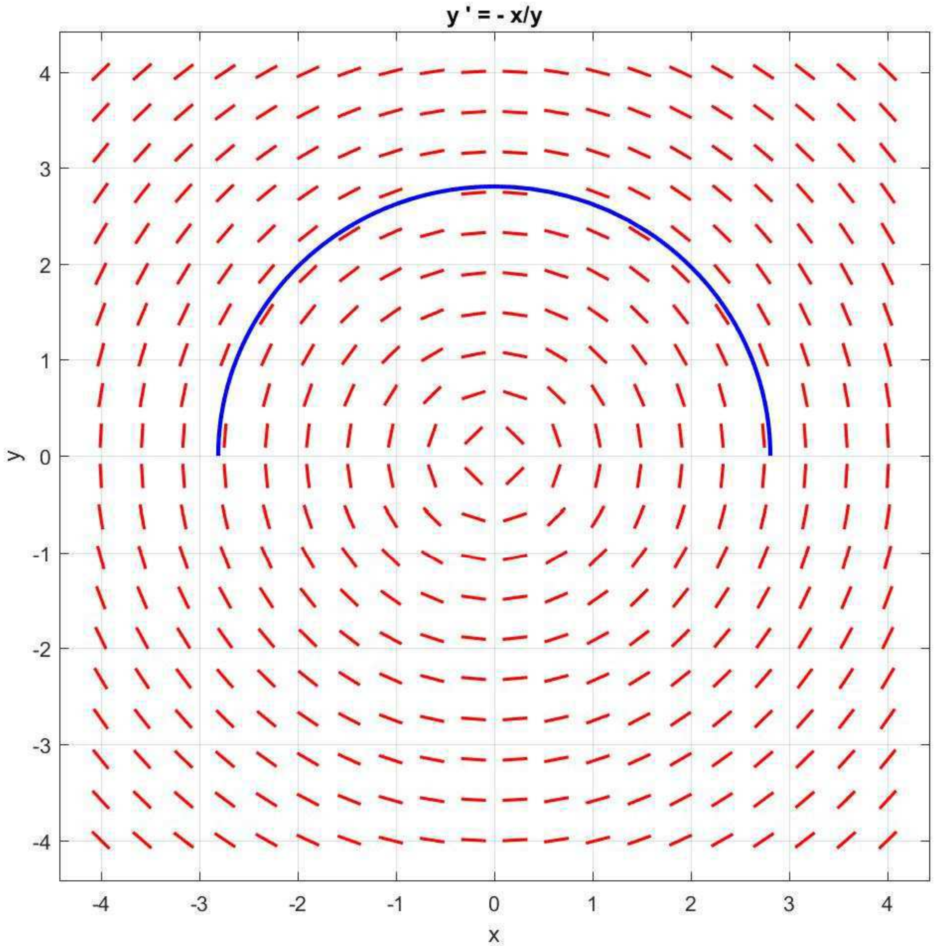
$f(x, y)$ = differentiable function

② Solution may not last forever.

" There is a unique solution defined in some interval about x_0. "

$\frac{-x}{y} = f(x, y)$ is a differentiable function as long as $y \neq 0$, so the solution may "stop" if $y \to 0$, which happens as $x \to \pm 1$

I don't expect you to be able to see from the DE that this will happen



$y = \frac{-x}{y}$

Computer generated solution stops at $x = \pm 1$

* Parametric Equations could find the whole circle

$$\frac{dy}{dx} = \frac{\frac{dy}{ds}}{\frac{dx}{ds}} = \frac{-x}{y}$$

$$\frac{dy}{ds} = -x \qquad \frac{dx}{ds} = y$$

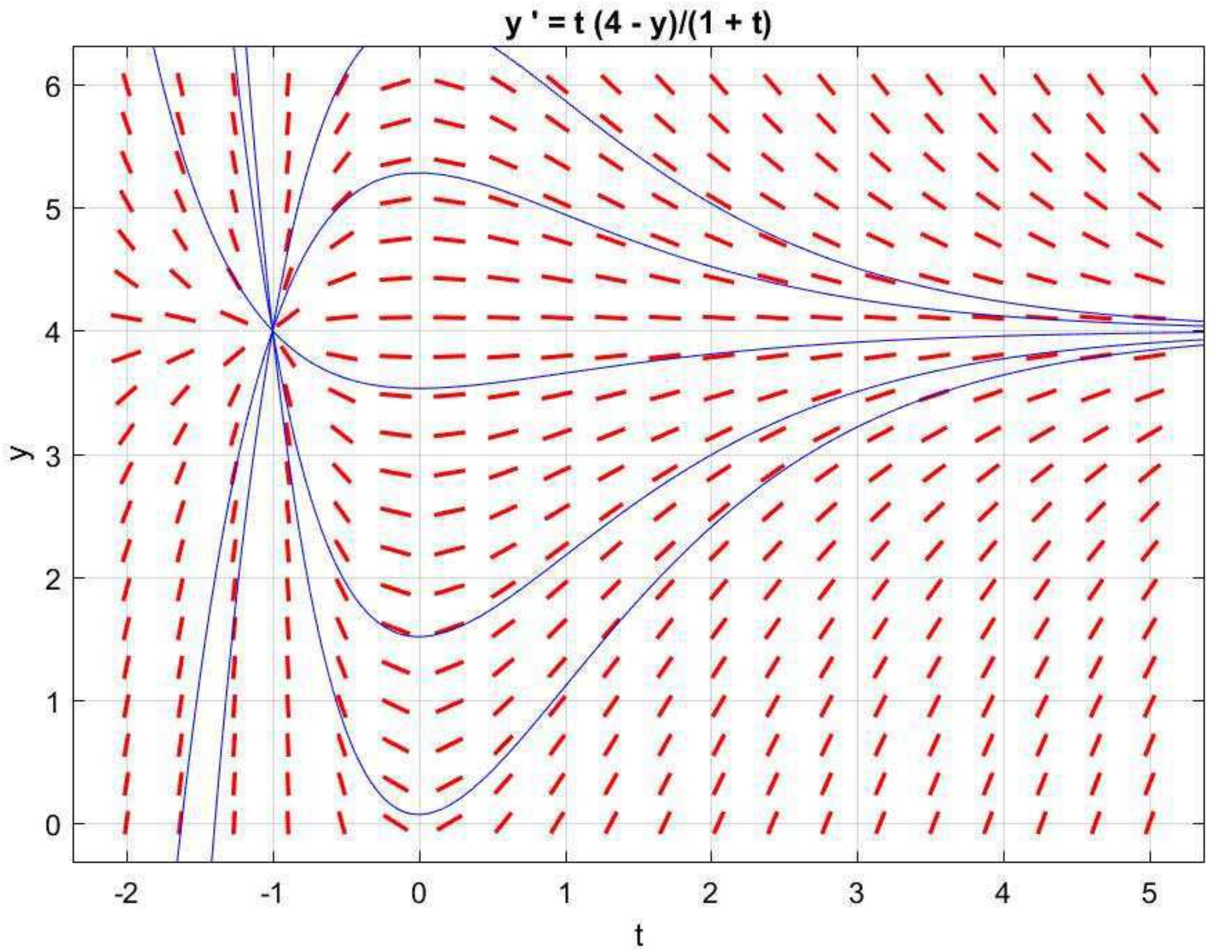
Solution:

$$x(s) = -\cos s$$

$$y(s) = \sin s$$

* I won't test you on this

$$\dot{y} = \frac{t(4-y)}{1+t}$$



What's wrong with this picture?

Does it contradict theorem that says IVP has unique solution? *

* I won't test you on this.

Problem A 100 kg sky diver drops from a great height. The force of air resistance is proportional to his velocity and always opposes the motion. The constant of proportionality is 20 kg/sec.
Take "UP" to be the positive direction.

(a) Formulate the Initial Value Problem

(b) Find the sky diver's "terminal velocity".

Solution:

(a) mass · acceleration = gravitational force + force of air resistance

$$100\dot{v} = -100g \pm 20 \cdot v$$

?

$$\dot{v} = -9.8 - \frac{20}{100} v$$

"drops" means $v(0) = 0$

IVP $\frac{dv}{dt} = -9.8 - \frac{v}{5}$ $\quad v(0) = 0$

$$\frac{dv}{dt} = -9.8 - \frac{v}{5}$$

Equilibrium Solution (Constant Solution)

$$v = -9.8 \cdot 5 = -49$$

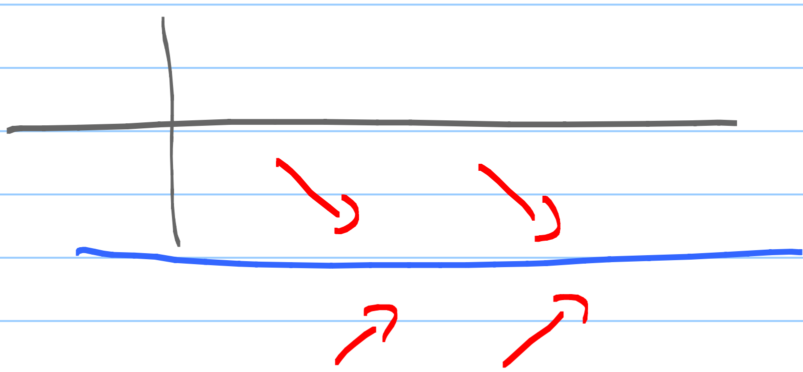
Check $\quad 0 = \frac{dv}{dt} = -9.8 - \frac{(-49)}{5} = 0 \quad \checkmark$

Is $v = -49$ a stable equilibrium?

Answer: Yes because

If $v(t) > -49$, then $\frac{dv}{dt} < 0$, so $v(t) \downarrow$

If $v(t) < -49$, then $\frac{dv}{dt} > 0$, so $v(t) \uparrow$



$v = -49$

From Last Lecture (3)

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Problem A 100 kg sky diver drops from a great height. The force of air resistance is proportional to his velocity and always opposes the motion. The constant of proportionality is 20 kg/sec. Take "UP" to be the positive direction.

(a) Formulate the Initial Value Problem

(b) Find the skydiver's "terminal velocity".

Solution:

(a) mass · acceleration = gravitational force + force of air resistance

$$100\dot{v} = -100g \pm 20 \cdot v$$

?

$$\dot{v} = -9.8 - \frac{20}{100} v$$

"drops" means $v(0) = 0$

IVP $\dfrac{dv}{dt} = -9.8 - \dfrac{v}{5}$ $\quad v(0) = 0$

$$\frac{dv}{dt} = -9.8 - \frac{v}{5}$$

Equilibrium Solution (Constant Solution)

$$0 = \frac{dv}{dt} = -9.8 - \left(\frac{v}{5}\right) = 0$$

$$v = -9.8 \cdot 5 = -49$$

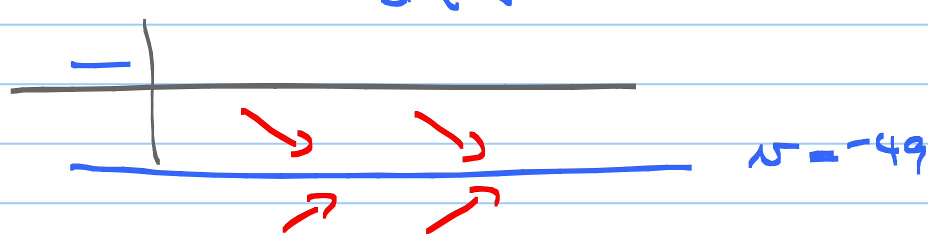
Is $v = -49$ a stable equilibrium?

Answer: Yes because

If $v(t) > -49$, then $\frac{dv}{dt}\left(= \frac{-1}{5}(v+49)\right) < 0$ so $v(t) \downarrow$

If $v(t) < -49$, then $\frac{dv}{dt}\left(= \frac{-1}{5}(v+49)\right) > 0$ so $v(t) \uparrow$

DField



If he starts to fall at any speed < 49 m/s
he speeds up to (almost) 49 m/s
but he never falls faster than 49 m/s.

If he starts falling faster than 49 m/s, he
slows down to 49 m/s

For this reason, we call the stable equilibrium
the terminal velocity.

Find a Formula for the "general solution" to $\frac{dv}{dt} = -9.8 - \frac{v}{5}$

Solution: This is a separable DE, but we will use a different method, called an "integrating factor".

Rewrite as: $\frac{dv}{dt} + \frac{1}{5}v = -9.8$

Multiply by $e^{t/5}$ ← I haven't told you why.

$$e^{t/5}\frac{dv}{dt} + \frac{1}{5}e^{t/5}v = -9.8e^{t/5}$$

Recognize the product rule

$$\frac{d}{dt}\left[e^{t/5}v\right] = -9.8e^{t/5}$$

Integrate both sides

$$e^{t/5}v(t) = -49e^{t/5} + C$$

Solve for $v(t)$

$$v(t) = -49 + Ce^{-t/5}$$

This is the "general solution". It always has an arbitrary constant.

First Order Linear DE's

Examples

y and derivatives of y OK

$$y' = -ty + \cos t$$

$$y' = 3y + e^{-2t}$$

$$y' = y\cos t + \sin t$$

General Example

$$y' = p(t)\,y + g(t)$$

Non-Examples These DE's are NOT linear

other functions of y or derivative NOT OK

$$y' = ty^2 + \sin t$$

$$(y')^2 = ty + \cos t$$

$$y' = \sin y$$

Method of Integrating factors

$$\dot{y} + 4y = 1$$

$$y(0) = 0$$

Multiply both sides by e^{4t} ↖ Not telling you why yet.

$$e^{4t}\dot{y} + 4e^{4t}y = e^{4t}$$

Recognize product rule

$$(e^{4t}y)^{\cdot} = e^{4t}$$

Integrate

$$e^{4t}y = \frac{e^{4t}}{4} + C$$

$$y = \frac{1}{4} + Ce^{-4t} \quad \longleftarrow \text{This is the "general solution"}$$

$$0 = y(0) = \frac{1}{4} + C$$

$$C = -1/4$$

$$y(t) = \frac{1}{4} - \frac{1}{4}e^{-4t} \quad \swarrow$$

This is the solution to the Initial Value Problem

How did we find the integrating Factor?
Call it $m(t)$

$$\dot{y} + 4y = 1$$

Multiply by a function $m(t)$

$$m\dot{y} + 4my = m(t)\cdot\text{something}$$

We want to choose m so that

$$\frac{d}{dt}(my) = m\frac{dy}{dt} + 4my$$

$$\cancel{m\frac{dy}{dt}} + \frac{dm}{dt} = \cancel{m\frac{d}{dt}} + 4ym$$

so we need

$$\frac{dm}{dt} = 4m$$

which is separable

$$\frac{dm}{m} = 4\,dt$$

$$\ln|m| = 4t + C$$

$$m = e^{4t}\cdot k$$

k doesn't matter so we set $k=1$

Example 2

(DE) $\dfrac{dy}{dt} = -\dfrac{1}{t}y + t^2$

(IC) $y(1) = 1$

How to solve?

(DE) $\dfrac{dy}{dt} + \dfrac{1}{t}y = t^2$

$\dfrac{dm}{dt} = \dfrac{m}{t}$

$\dfrac{dm}{m} = \dfrac{dt}{t}$

$\ln|m| = \ln|t|$

$m = \pm t$

$\pm$ doesn't matter

$m = t$

Multiply both sides by t

(DE) $t\dfrac{dy}{dt} + y = t^3$

Recognize the "product rule"

$$t\frac{dy}{dt} + y = \frac{d}{dt}(ty)$$

so

(DE) $\dfrac{d}{dt}(ty) = t^3$

Now integrate both sides using

$$ty = \frac{t^4}{4} + C$$

Divide by t

$$y(t) = \frac{t^3}{4} + \frac{C}{t}$$

This is called the "general solution"

Use the initial condition

$1 = y(1) = \frac{1}{4} + C$ so $C = \frac{3}{4}$

so $\boxed{y(t) = \dfrac{t^3}{4} + \dfrac{3}{4t}}$

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Integrating Factor Summary

① Linear First Order DE

$$\frac{dy}{dt} + p(t)y = q(t)$$

② Find $m(t)$ by solving $\frac{d}{dt} m = p\,m$

we can write a formula $m = e^{\int p(t)}$

③ Multiply First Order DE by m

$m\frac{dy}{dt} + m\,p(t)\,y$ equals $\frac{d}{dt}(m\,y)$

④ Integrate both sides of

$$\frac{d}{dt}(m\,y) = m\,q(t)$$

and divide by m

$$y(t) = \frac{1}{m}\left(\int m\,q(t)\right) + \frac{c}{m}$$

to obtain the general solution.

I don't remember most of these formulas.
I just remember that I want

$m\cancel{\frac{dy}{dt}} + m\,p(t)\,y$ to be $\frac{d}{dt}(m\,y) = \cancel{m\frac{dy}{dt}} + \frac{dm}{dt}y$

so $\frac{dm}{dt} = mp$

Example 2 **Newton's Law of Cooling**

Temperature T of an object changes at a rate proportional to the deviation from ambient temperature. Suppose time is in minutes, ambient temperature is $(20 + 5e^{-.02t})°C$, the constant of proportionality is $0.01\ \text{min}^{-1}$, and the initial temperature of the object is $40°C$.

(a) Write the IVP.

(b) Solve the IVP.

(IVP) $\frac{dT}{dt} = 0.01\left(20+5e^{-0.02t} - T\right)$

$T(0) = 40$

Why not
$(T-(20+5e^{-0.002t}))$?

Solve

$$\frac{dT}{dt} + 0.01\,T = 0.2 + 0.05\,e^{-0.02t}$$

Find integrating factor

Multiply $\frac{dT}{dt} + 0.01T$ by something so that it becomes the derivative of a product.

Answer $e^{0.01t}$

$$e^{0.01t}\frac{dT}{dt} + 0.01\,e^{0.01t}\,T = \frac{d}{dt}\left(e^{0.01t}\,T\right)$$

$$\frac{dT}{dt} + 0.01\,T = 0.2 + 0.05\,e^{-0.02t}$$

Multiply by $e^{0.01t}$

$$e^{0.01t}\frac{dT}{dt} + 0.01\,e^{0.01t}\,T = 0.2\,e^{0.01t} + 0.05\,e^{-0.01t}$$

Recognize product rule

$$\frac{d}{dt}\left(e^{0.01t}\,T\right) = 0.2\,e^{0.01t} + 0.05\,e^{-0.01t}$$

Integrate $\int e^{\alpha t} = \frac{e^{\alpha t}}{\alpha}$

$$e^{0.01t}\,T = \frac{0.2}{0.01}\,e^{0.01t} - \frac{0.05}{0.01}\,e^{-0.01t} + C$$

Multiply by $e^{-0.01t}$

$$\boxed{T(t) = 20 - 5\,e^{-0.02t} + C\,e^{-0.01t}}$$

This is the general solution

Find C $\quad 40 = T(0) = 20 - 5 + C$

$$25 = C$$

$$\boxed{T(t) = 20 - 5\,e^{-0.02t} + 25\,e^{-0.01t}}$$

This is the solution to the IVP

Note Title 1/17/2018

What does Linear Mean?

Linear Equations

$$3x + 4y = b$$

You can add solutions

If $3 \cdot 1 + 4 \cdot 0 = 3$

and $3 \cdot 0 + 4 \cdot 1 = 4$

then $3(1+0) + 4(0+1) = (3+4)$

and multiply by constants

If $3 \cdot 1 + 4 \cdot 0 = 3$

then $3(2 \cdot 1) + 4(2 \cdot 0) = (2 \cdot 3)$

Nonlinear means this doesn't work

$$x^2 + y^2 = b^2$$

$$1^2 + 0^2 = 1^2$$

$$0^2 + 1^2 = 1^2$$

But $(1+0)^2 + (0+1)^2 \neq (1+1)^2$

Superposition Principle

For a linear equation, the sum of solutions is a solution.

Example $\frac{dv}{dt} + \frac{v}{5} = -9.8$

$v_1(t) = -49$ is a solution

$v_2(t) = C e^{-t/5}$ is a solution to

$$\frac{dv_2}{dt} + \frac{v_2}{5} = 0$$

Superposition Principle says that

$v_1(t) + v_2(t)$ solves

$$\frac{d}{dt}(v_1 + v_2) + \frac{1}{5}(v_2 + v_2) = -9.8 + 0$$

so the general solution to

$$\frac{dv}{dt} + \frac{v}{5} = -9.8$$

$$v(t) = v_1 + v_2 = -49 + C e^{-t/5}$$

Question

Find the general solution to:

$$\dot{y} = \frac{-2}{t} y + e^t \qquad \text{(DE)}$$

Solution

write as

$$\frac{dy}{dt} + \frac{2}{t} y = e^t$$

Find Integrating factor

$$\frac{d}{dt} m = \frac{2}{t} m$$

$$\frac{dm}{m} = \frac{2}{t} dt$$

$$\ln m = 2 \ln t = \ln(t^2) + c$$

choose $c = 0$ for convenience

$$m = t^2$$

Now, multiply DE by t^2

$$t^2 \dot{y} + t^2 \frac{2y}{t} = t^2 e^t$$

$$t^2 \frac{dy}{dt} + 2ty = t^2 e^t$$

$$\frac{d}{dt}(t^2 y) = t^2 e^t$$

$$\frac{d}{dt}(t^2 y) = t^2 e^t$$

Integrate $\quad t^2 y = \int t^2 e^t$

$$t^2 y = t^2 e^t - 2te^t + 2e^t + c$$

multiply by $\frac{1}{t^2}$ $\quad \boxed{y = e^t - \frac{2}{t}e^t + \frac{2}{t^2}e^t + \frac{c}{t^2}}$

General Solution

Aside $\quad$ I remember

$$\int t^2 e^t = t^2 e^t + a te^t \; b\, e^t$$

$\downarrow \frac{d}{dt}$

$$t^2 e^t \stackrel{?}{=} (t^2 e^t + 2t e^t) + (a te^t + a e^t) + b e^t$$

$$t^2 e^t \stackrel{?}{=} t^2 e^t + \underbrace{(2+a)}_{=0} te^t + \underbrace{(a+b)}_{=0} e^t$$

-2

$a = -2 \qquad b = -a = 2$

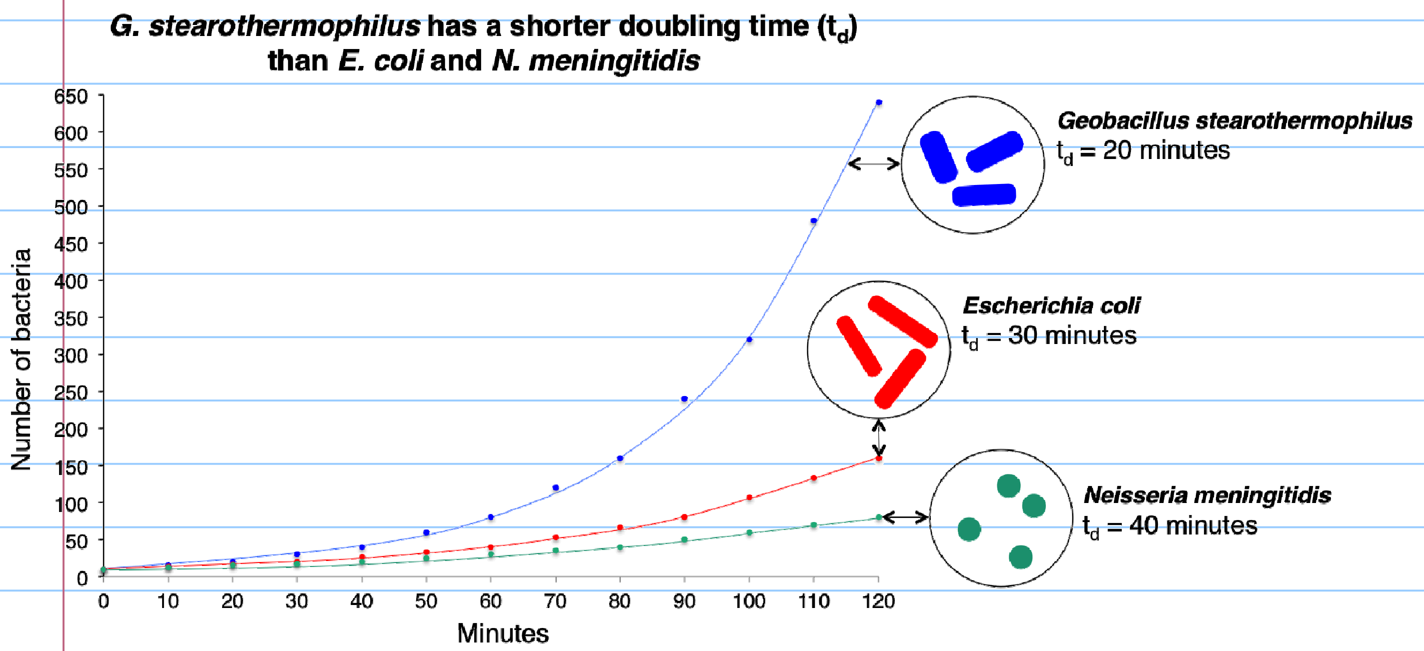
Faster than integration by parts ?

Lecture 06

Note Title 10/6/2017

Population Models

Unlimited Resources (Exponential Growth)



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$P(t)$ = population at time t

r = growth rate = birth rate − death rate

$$\left.\begin{aligned} \frac{dP}{dt} &= rP \\ P(0) &= P_0 \end{aligned}\right\} \text{Initial Value Problem}$$

<u>Question</u> Suppose the doubling time is 30 minutes. Find r.

r is sometimes called the "proportionality constant

$$\left.\begin{array}{l} \dfrac{dP}{dt} = rP \\ P(0) = P_0 \end{array}\right\}$$ Initial Value Problem

<u>Question</u> Suppose the doubling time is 30 minutes. Find r

<u>Answer</u>

$$\frac{dP}{P} = r\,dt$$

$$\ln P = rt + C$$

$$P = k e^{rt}$$

$$P_0 = P(0) = k$$

$$P(t) = P_0 e^{rt}$$

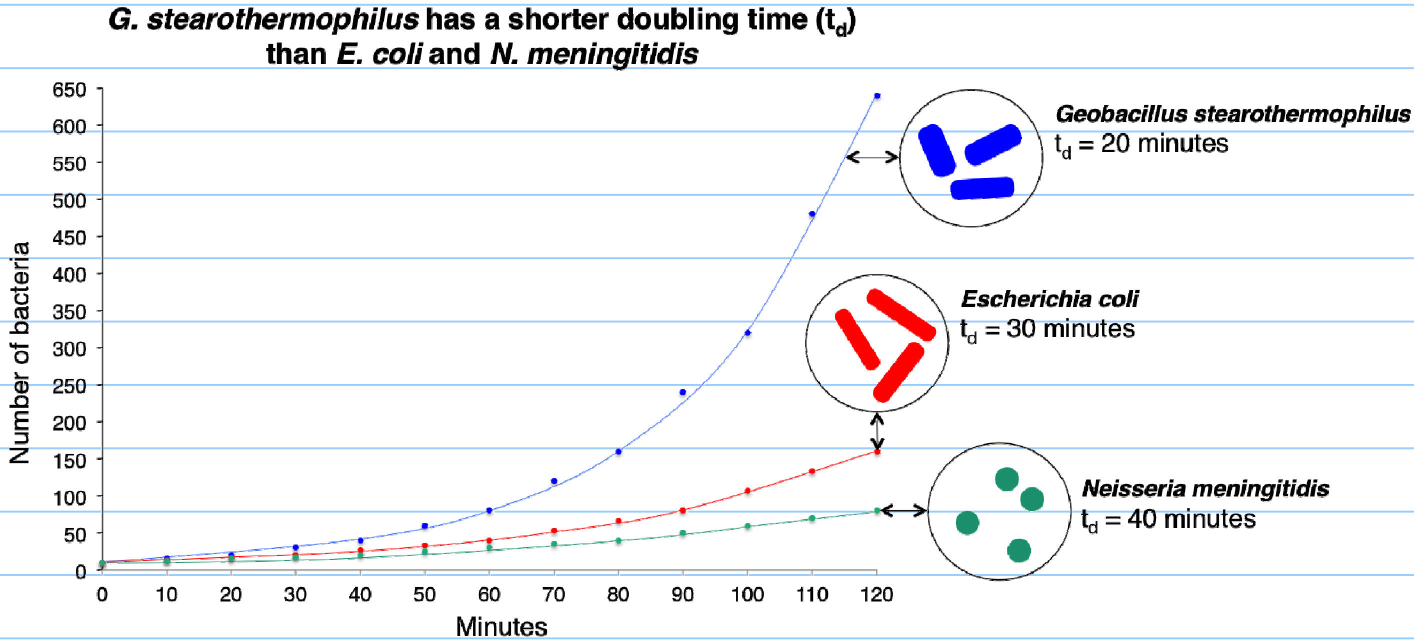
Calculate r

$$2\cancel{P_0} = P(30) = \cancel{P_0} e^{r\cdot 30}$$

$$2 = e^{30r}$$

$$\boxed{\frac{\ln 2}{30} = r}$$

<u>Note:</u> The value of P_0 didn't matter for this problem.



Remark - Its important to work with "letters" (r, P_0) rather than just numbers.

In real applications, we almost never measure parameters like r directly.

Warning: You will see many word problems in textbooks or online where its not clear if they are telling you r or some data from which you should determine r.

Limited Resources (Logistic Growth Model)

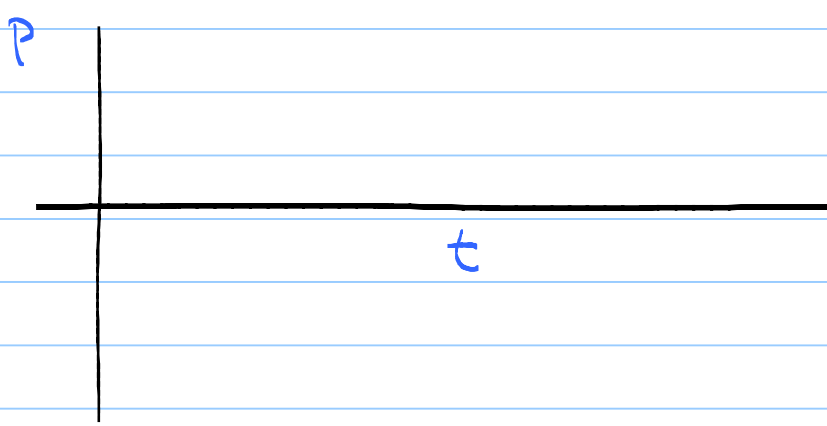
$$\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right) \qquad (\text{DE})$$

r = growth rate

k = carrying capacity

$k = \left\{ \begin{array}{l} \text{maximum population that} \\ \text{available resources (e.g. food supply)} \\ \text{can sustain} \end{array} \right.$

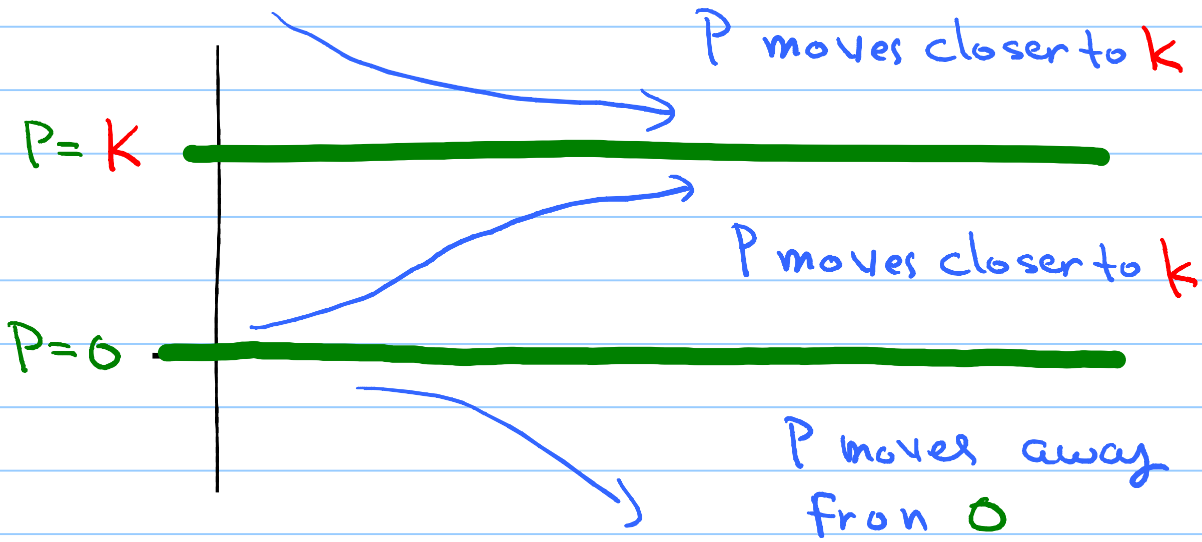
Question Sketch direction field for the DE, label equilibrium solutions and classify as stable or unstable.



What are the two equilibrium solutions

$$\frac{dP}{dt} = rP\left(1-\frac{P}{k}\right)$$

$P<0$	$rP(1-\frac{P}{k}) < 0$
$0<P<k$	$rP(1-\frac{P}{k}) > 0$
$k<P$	$rP(1-\frac{P}{k}) < 0$



Recall - Theorem says IVP has a unique (exactly one) solution, so curves cannot cross.

$P=k$ is a stable equilibrium

$P=0$ is an unstable equilibrium or a threshhold

Question $\dfrac{dP}{dt} = rP\left(1-\dfrac{P}{k}\right)$

Suppose $r=1$ and $k=10$

and $P(0)=1$. Solve the IVP.

(DE) $\dfrac{dP}{dt} = P\left(1-\dfrac{P}{10}\right)$ (IC) $P(0)=1$

Solve $\dfrac{dP}{P\left(1-\frac{P}{10}\right)} = dt$ Separate variables

$$\frac{dP}{P(10-P)} = \frac{dt}{10}$$ Make it look a little neater

$$\frac{1}{10}\frac{dP}{P} + \frac{1}{10}\frac{dP}{10-P} = \frac{dt}{10}$$ partial fractions (details not shown)

Integrate both sides, Note the minus sign

$$\frac{1}{10}\ln|P| - \frac{1}{10}\ln|10-P| = \frac{1}{10}t + c$$

$$\ln\left|\frac{P}{10-P}\right| = t + c$$

$$\left|\frac{P}{10-P}\right| = c_2 e^t$$

$$\left|\frac{P}{10-P}\right| = C_2 e^t$$

Now, worry about absolute values

$$\frac{P}{10-P} = \pm C_2 e^t$$

Fortunately, its easy, set $C_3 = \pm C_2$

The $\pm$ just changes the sign of the constant

$$\frac{P}{10-P} = C_3 e^t$$

Initial Condition:

$$\frac{1}{9} = \frac{1}{10-1} = C_3$$

$$\frac{P}{10-P} = \frac{1}{9} e^t$$

Now solve for P

$$P = \frac{1}{9}(10-P) e^t$$

$$P = \frac{10e^t}{9} - P\frac{e^t}{9}$$

$$9P + e^t P = 10 e^t$$

$$P(9 + e^t) = 10 e^t$$

$$\boxed{P(t) = \frac{10e^t}{9 + e^t}}$$

Reminder

Partial Fractions - cover up method

$$\frac{1}{(s-2)(s-3)(s-4)} = \frac{A}{(s-2)} + \frac{B}{(s-3)} + \frac{C}{s-4}$$

Find A, B, C

To find A

Cover up $(s-2)$ & set $s=2$

$$A = \left.\frac{1}{\cancel{(s-2)}(s-3)(s-4)}\right|_{s=2} = \frac{1}{(2-3)(2-4)}$$

To find B

Cover up $(s-3)$ & set $s=3$

$$B = \left.\frac{1}{(s-2)\cancel{(s-3)}(s-4)}\right|_{s=3} = \frac{1}{(3-2)(3-4)}$$

$(2-3)(2-4)$

Justification - multiply both sides by $(s-2)$

$$\frac{1\,\cancel{(s-2)}}{\cancel{(s-2)}(s-3)(s-4)} = \frac{A\,\cancel{(s-2)}}{\cancel{(s-2)}} + \frac{B(s-2)}{(s-3)} + \frac{C(s-2)}{s-4}$$

Set $s=2$

$$\frac{1}{(2-3)(2-4)} = A + 0 + 0$$

$$\frac{1}{(2-3)(2-4)} = A \qquad \frac{1}{(3-2)(3-4)} = B \qquad \frac{1}{(4-2)(4-3)} = C$$

Note Title 1/4/2020

Problem

DE $\dot{y} = y(y-6)(y-4)$

1. Find the equilibrium solutions

2. Sketch the direction field

3. Sketch the phase line

4. Label equilibrium solutions as stable or unstable

Implicit Solution

$$\left[(y-4)^3 = K e^{-24t} y (y-6)^2\right]$$

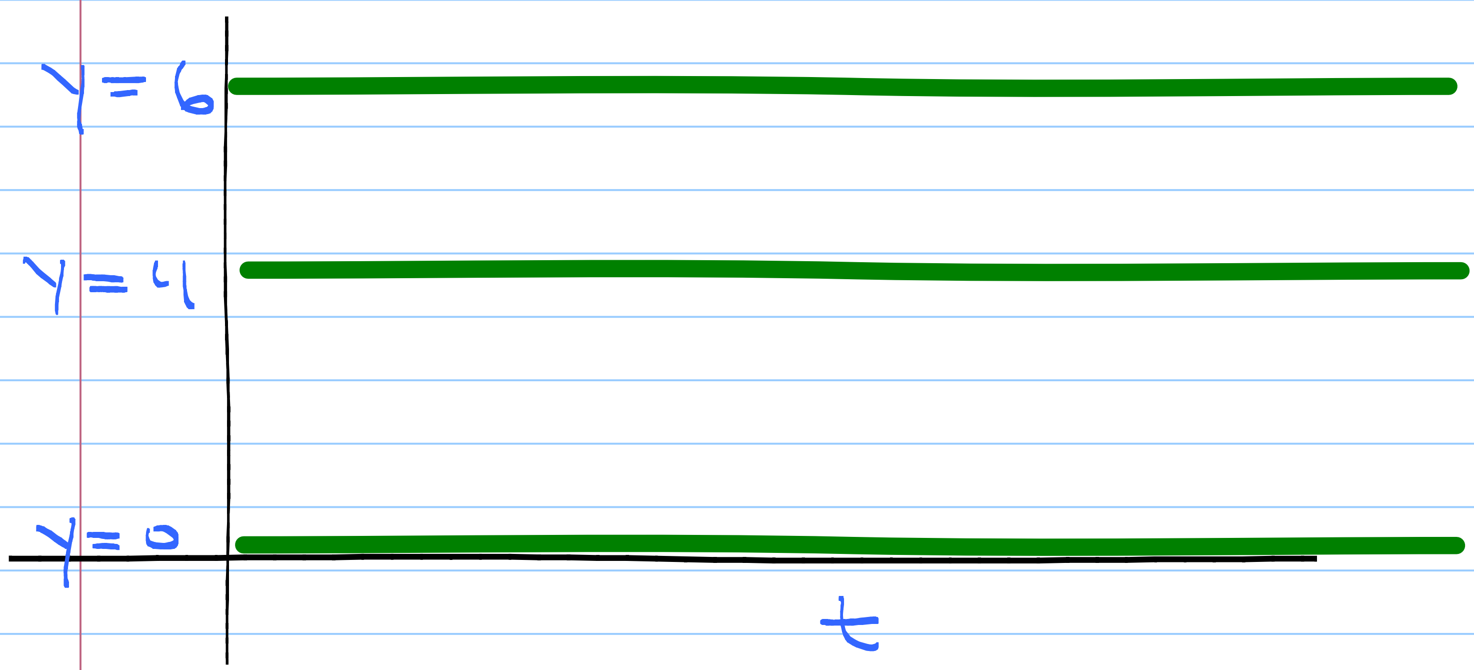
Equilibrium Solutions are $y = \text{constant}$

If $y = \text{constant}$, $\dot{y} = 0$, so we must have $0 = \dot{y} = y(y-4)(y-6)$

So equilibrium solutions are

$$y = 0 \; ; \; y = 4 \; ; \; y = 6$$

Draw them:



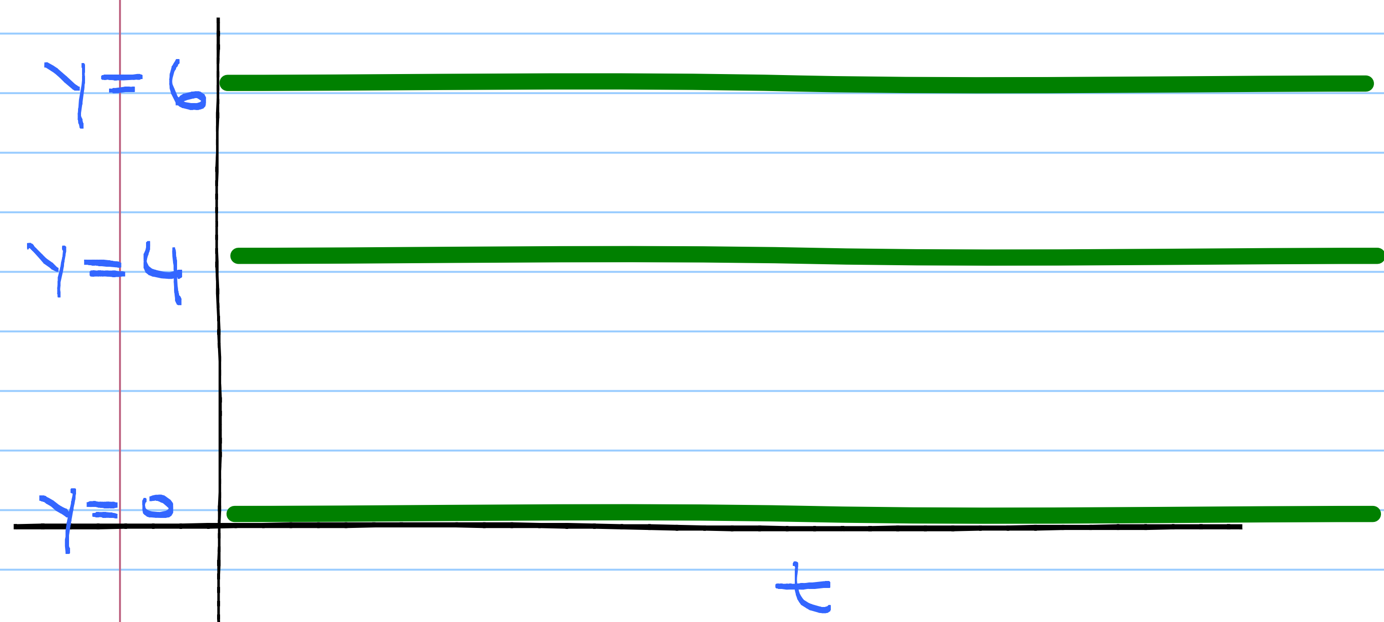
Step ① Sketch Dfield

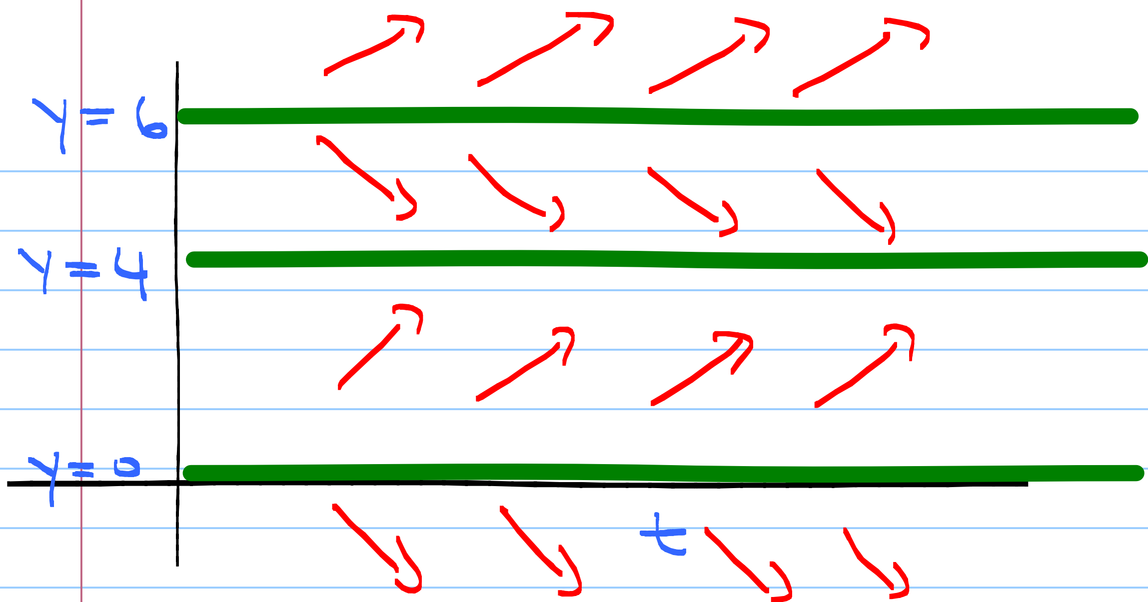
If $y < 0$ $\quad \dot{y} = y(y-6)(y-4) < 0$

If $0 < y < 4$ $\quad \dot{y} = y(y-6)(y-4) > 0$

If $4 < y < 6$ $\quad \dot{y} = y(y-6)(y-4) < 0$

If $6 < y$ $\quad \dot{y} = y(y-6)(y-4) > 0$

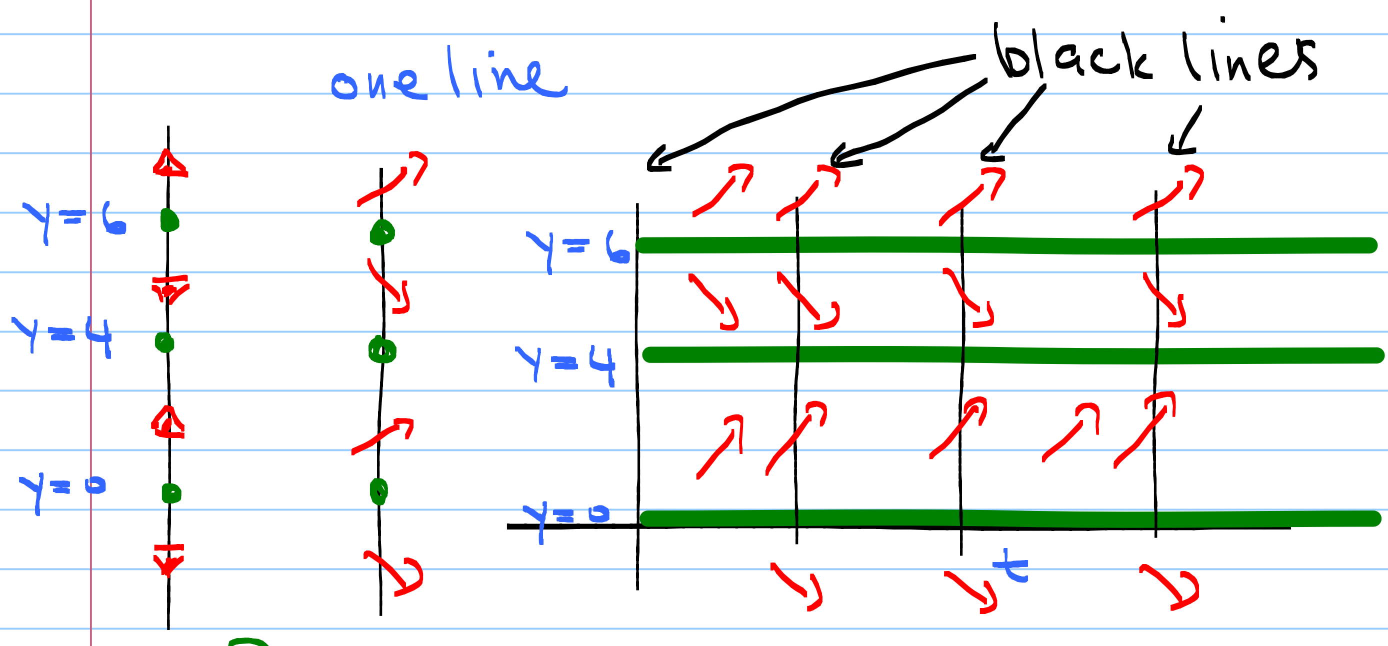




Sketch the Phase Line

Notice that $F(y)$ doesn't involve t.

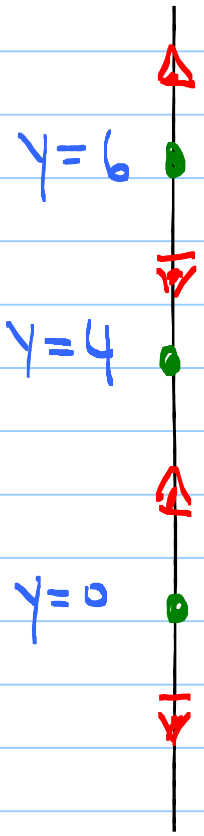
The Direction Field looks the same on each of the black lines below so we can make one line that summarizes all the DField information.



The Phase line

But we usually draw it differently.

Label equilibrium solutions as stable or unstable

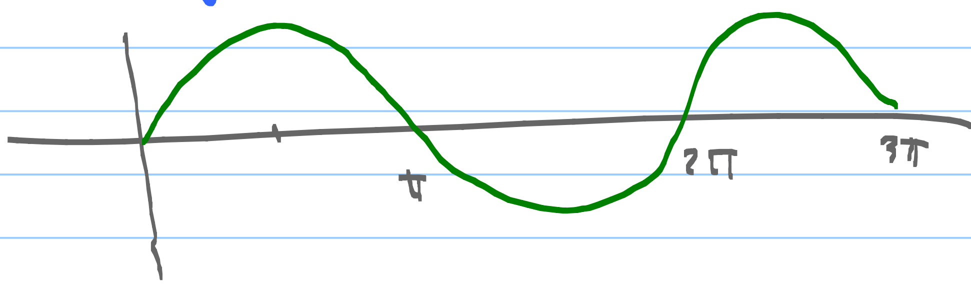


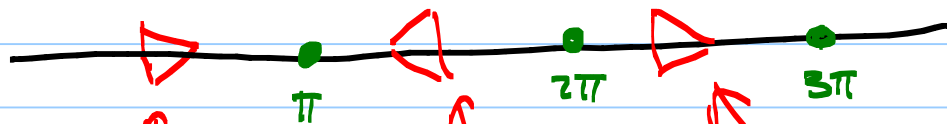
← solutions near $y=6$ move away from 6
Unstabe

← solutions near $y=4$ move towards 4
Stabe

← solutions near $y=0$ move away from 0
Unstabe

Problem $\dot{y} = \sin y$ Draw the Phase Line



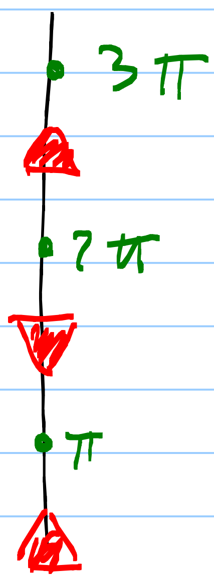


because sine is positive

because sine is negative

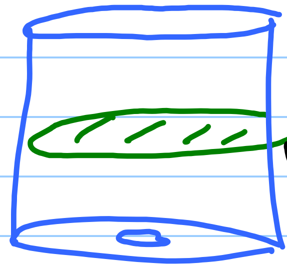
because sine is positive

Now turn it sideways



Toricelli's Law (problem 7 HW3)

cross section



ylindrical Tank

Water leaks out hole

Determine $h(t)$ = height of water at time t

Motivation – Build a water clock by marking $h(t)$ on the sides of the cylinder.

Two Equations

h = height of water
v = speed of water exiting

Water Out = decrease in Volume

$$A_{hole} \cdot v = A_{\substack{cylinder \\ cross\ section}} \cdot \frac{dh}{dt} \qquad (T1)$$

Increase in kinetic energy = { Decrease in potential energy

$$\frac{1}{2} \Delta m\, v^2 = \Delta m\, g\, h \qquad (T2)$$

$\Delta m = A_{hole}\, \rho\, v\, dt$ = mass that exits in time dt

Find DE for h(t)

$$\frac{1}{2} \cancel{\Delta} m v^2 = \cancel{\Delta} m g h \qquad (I2)$$

$$v = \pm \sqrt{2gh}$$

$$A_{cyl} \cdot \frac{dh}{dt} = \pm A_{hole} v \qquad (I1)$$

$$A_{cyl} \frac{dh}{dt} = \pm A_{hole} \sqrt{2gh}$$

$$\frac{dh}{dt} = \pm \underbrace{\frac{\pi r_{hole}^2}{\pi r_{cyl}^2}}_{\text{constant}} \cdot \sqrt{2000}\, h^{1/2}$$

Now I know what sign to choose.

h(t) must decrease so choose minus sign.

Question Find DE for $v(t)$

Answer

minus, because up is positive

$$v^2 = 2gh \Rightarrow v = -\sqrt{2g}\sqrt{h}$$

$$\cancel{2}v\frac{dv}{dt} = \cancel{2}g\frac{dh}{dt}$$

$$= g\frac{r_{hole}^2}{r_{cyl}^2}\cdot\sqrt{2g}\quad h^{1/2}$$

$$\cancel{v}\frac{dv}{dt} = g\frac{r_{hole}^2}{r_{cyl}^2}\cancel{\sqrt{2g}}\left(\frac{-\cancel{v}}{\cancel{\sqrt{2g}}}\right)$$

$$\frac{dv}{dt} = -g\frac{r_{Hole}^2}{r_{cylinder}^2}$$

A pond has an initial volume of $10{,}000\ m^3$. Two streams flow in and one stream flows out.

<u>Stream A</u>
$\left\{ \begin{array}{l} \dfrac{500\ m^3}{day} \text{ influx} \\ \text{water contains } \dfrac{5\,kg}{1000\ m^3} \text{ salt} \end{array} \right.$

<u>Stream B</u>
$\left\{ \begin{array}{l} \dfrac{750\ m^3}{day} \text{ influx} \\ \text{No salt} \end{array} \right.$

<u>Stream C</u> $\dfrac{1300\ m^3}{day}$ outflux

Find the differential Equation for S = total amount of salt in the pond

$$\frac{dS}{dt} = \begin{array}{c}\text{rate}\\\text{of}\\\text{salt}\\\text{influx}\end{array} - \begin{array}{c}\text{rate}\\\text{of}\\\text{salt}\\\text{outflux}\end{array}$$

$$= \frac{500\ m^3}{\text{day}} \cdot \frac{5\ \text{kg}}{1000\ m^3} - \frac{1300\ m^3}{\text{day}} \cdot \begin{array}{c}\text{concentration}\\\text{of}\\\text{salt}\end{array}$$

$$\begin{array}{c}\text{concentration}\\\text{of}\\\text{Salt}\end{array} = \frac{S}{\begin{array}{c}\text{Volume}\\\text{of}\\\text{lake}\end{array}} = \frac{S}{10{,}000 - 50t}$$

$$\text{Volume} = \underbrace{10{,}000}_{\text{initial volume}} + \underbrace{500t}_{\text{stream A in}} \underbrace{+750t}_{\text{stream B in}} \underbrace{-1300t}_{\text{stream C out}}$$

$$\frac{dS}{dt} = \frac{5}{2}\,\frac{\text{kg}}{\text{day}} - \frac{1300}{10{,}000 - 50t}\,S$$

small times

$$\frac{dS}{dt} = \frac{5}{2} - \frac{13}{100}S$$

$$S = 3\left(1 - e^{-\frac{13}{100}t}\right)$$

$$\frac{dS}{dt} - \frac{1300}{50t - 10000}\,S = \frac{5}{2}$$

$$\boxed{\frac{dS}{dt} - \frac{26}{(t-200)}\,S = \frac{5}{2} \qquad S(0) = 10{,}000}$$

$$\frac{dS}{dt} - \frac{26}{(t-200)} S = \frac{5}{2} \qquad S(0) = 0$$

No salt at time = 0

Find Integrating Factor

$$\frac{dm}{dt} = \frac{-26}{(t-200)} m$$

$$\frac{dm}{m} = \frac{-26}{t-200} dt$$

$$\ln|m| = -26 \ln|t-200| + c$$

$$= \ln|(t-200)^{-26}| + c$$

$$m = (t-200)^{-26}$$

Multiply Both sides by m

$$(t-200)^{-26} \frac{dS}{dt} - 26(t-200)^{-27} = \frac{5}{2}(t-200)^{-26}$$

$$\frac{d}{dt}\left[(t-200)^{-26} S\right] = \frac{5}{2}(t-200)^{-26}$$

Integrate

$$(t-200)^{-26} S = \frac{5}{2} \cdot \frac{(t-200)^{-25}}{-25} + c$$

$$S = -\frac{1}{10}(t-200) + c(t-200)^{26}$$

$$S = 20 - \frac{t}{10} + c(t-200)^{26}$$

Find C $\qquad 0 = 20 + c(200)^{26}$

$$\boxed{S = 20 - \frac{t}{10} - \frac{20}{(200)^{26}}(t-200)^{26}}$$

Math 307AB **Practice Midterm 1** **Winter 2020**

Your Name

Your Signature

Section (circle one) AA AB AC BA bB BC

Problem	Total Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

- This exam is closed book. You may use one side of one $8\frac{1}{2} \times 11$ sheet of handwritten notes. You may not share notes.
- No graphing or symbolic calculators are allowed. You may not use cell phones during the exam.
- Show your work. Do not do computations in your head. Instead, write them out on the exam paper.
- **Place a box around** $\boxed{\textbf{YOUR FINAL ANSWER}}$ **to each question.**
- If you need more room, use the backs of the pages and indicate that you have done so.
- If you are not sure what a question means, raise your hand and ask us.
- The hints are suggestions only.

1 (20 points) Solve the Initial Value Problem:

$$\begin{aligned} y' - 2y &= t \\ y(0) &= -1 \end{aligned}$$

Integrating Factor

$m = e^{\int -2\,dt} = e^{-2t}$

$(e^{-2t}y)' = t e^{-2t}$

$e^{-2t}y = -\frac{t e^{-2t}}{2} - \frac{e^{-2t}}{4} + C$

$y = -\left(\frac{t}{2} + \frac{1}{4}\right) + C e^{2t}$

Initial Condition

$-1 = -\frac{1}{4} + C$

$C = -\frac{3}{4}$

$$\boxed{y(t) = -\left(\frac{t}{2} + \frac{1}{4}\right) - \frac{3}{4} e^{2t}}$$

$\boxed{2}$ (20 points) A 1 kg model car is moving so fast that the force of air resistance is proportional to the square of the speed (and opposes the direction of motion). Let k represent the positive proportionality constant. No other forces act on the car. *Assume that the car is always moving forward, so that speed and velocity are always the same.*

(a) Formulate the first order differential equation for the velocity.

$$1\frac{dv}{dt} = -kv^2$$

(b) Suppose the initial velocity is 100 meters/sec and the velocity after 10 seconds is 90 meters/sec. Find k.

$$\frac{dv}{v^2} = -k\,dt$$

$$-\frac{1}{v} = -kt + C$$

$$v = \frac{1}{kt - C}$$

$$100 = v(0) = \frac{1}{-C}$$

$$v(t) = \frac{1}{kt + \frac{1}{100}}$$

$$90 = v(10) = \frac{1}{10k + \frac{1}{100}}$$

$$\frac{1}{90} = 10k + \frac{1}{100}$$

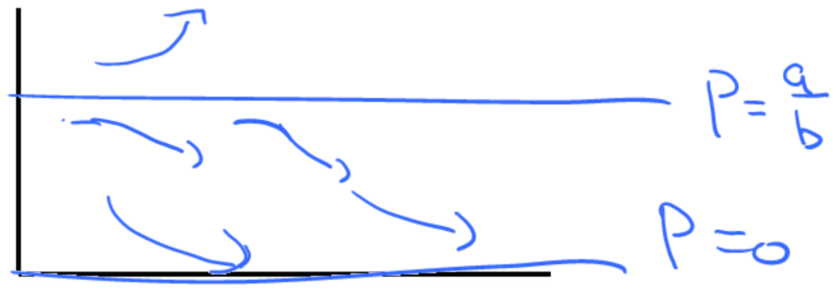
$$\frac{\frac{1}{90} - \frac{1}{100}}{10} = k$$

$$\frac{1}{9000} = k$$

k = $\boxed{1/9000}$ kg/

$\boxed{3}$ (20 points) Nurgaliev's Law models the evolution of a fish population as the solution $P(t)$ to the initial value problem below. Suppose that a and b are positive constants. Sketch a direction field. Label all equilibrium solutions and classify them as stable or unstable.

$$\begin{aligned} \frac{dP}{dt} &= bP^2 - aP \\ P(0) &= P_0 \end{aligned}$$



In this model, the population will either grow or die out as time progresses. State conditions under which the population will die out, and under which the population will grow.

If $P_0 > \frac{a}{b}$ population grows $P(t) \uparrow \infty$

If $P_0 < \frac{a}{b}$ population dies out $P(t) \to 0$

4 (20 points) Solve the initial value problem:

$$\frac{dy}{dx} = 4x(y^2 - 9) \qquad y(0) = 0$$

$$\frac{dy}{y^2-9} = 4x\,dx$$

$$\frac{1}{(y-3)(y+3)} = \frac{1/6}{y-3} - \frac{1/6}{y+3}$$

$$\int \frac{dy}{y-3} - \int \frac{dy}{y+3} = 24 \int x\,dx$$

$$\ln|y-3| - \ln|y+3| = 12x^2 + C_1$$

$$\ln\left|\frac{y-3}{y+3}\right| = 12x^2 + C_1$$

$$\frac{y-3}{y+3} = C_2 e^{12x^2} \quad \text{where } C_2 = \pm e^{C_1}$$

Initial Condition

$$-1 = \frac{0-3}{0+3} = C_2$$

$$y - 3 = -e^{12x^2}(y+3)$$

$$y(1 + e^{12x^2}) = 3(1 - e^{12x^2})$$

$$y(x) = 3\left(\frac{1 - e^{12x^2}}{1 + e^{12x^2}}\right) = 3\left(\frac{e^{-12x^2} - 1}{e^{-12x^2} + 1}\right)$$

What are $\lim_{x\to\infty} y(x)$ and $\lim_{x\to-\infty} y(x)$?

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty} 3\left(\frac{e^{-17x} - 1}{e^{-12x^2} + 1}\right) = \frac{0-1}{0+1} = \boxed{-3}$$

$$\lim_{x\to-\infty} y(x) = \lim_{x\to-\infty} 3\left(\frac{e^{-17x} - 1}{e^{-12x^2} + 1}\right) = \frac{0-1}{0+1} = \boxed{-3}$$

$\boxed{5}$ (20 points) A rocket engine generates a constant thrust (upward force) of 100 newtons. It has a base mass of 8kg, and, initially, it carries 2kg of fuel. The fuel burns at a rate of 2 kg per second. The rocket has a coefficient of air resistance of 2 Newton seconds per meter. Assume the rocket starts from rest, and that up is the positive direction. Take the gravitational acceleration to be -10m/sec^2 for simplicity. **Write an initial value problem for the velocity of the rocket during the time the fuel is burning. Then solve the IVP.**

$m \quad \dot{v} = AR + THRUST + Gravity$

$(10-2t)\dot{v} = -2v + 100 - (10-2t)10$

$(10-2t)\dot{v} + 2v = 20t$

$(t-5)\dot{v} - v = -10t$

$\dot{v} - \frac{1}{t-5}v = -10\left(\frac{t}{t-5}\right)$

$\left((t-5)^{-1}v\right)^{\bullet} = -10\left(\frac{t}{(t-5)^2}\right)$

$(t-5)^{-1}v = -10\left(\ln|t-5| - \frac{5}{t-5}\right) + c$

Integrating Factor

$-\int \frac{dt}{t-5} = -\ln(t-5)$

$e^{-\ln(t-5)} = (t-5)^{-1}$

Integral

$\int \frac{t}{(t-5)^2} = \int \frac{u+5}{u^2}$

$= \int \frac{1}{u} + 5\int \frac{1}{u^2}$

$= \ln|u| - \frac{5}{u}$

$= \ln|t-5| - 5/(t-5)$

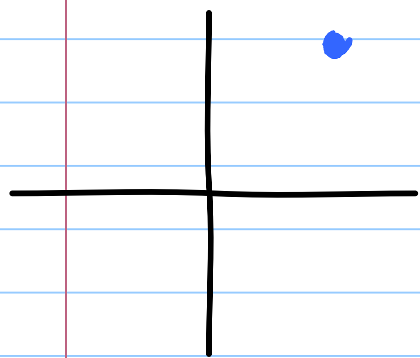
$v = -10(t-5)\ln|t-5| + 50 + c(t-5)$

$0 = 50\ln|5| + 50 - 5c$

$c = 10\ln|5| + 10$

$$\boxed{v(t) = -10(t-5)\ln|t-5| + 50 + (10\ln|5|+10)(t-5)}$$

Complex Numbers



$z = x + iy$

These are just the x, y coordinates

Multiplication There is no natural way to multiply $x\,y$ coordinates

but there is a way to multiply complex numbers.

$$(a+ib)\cdot(c+id) = ac + iad + ibc + i^2 bd$$

$$\boxed{i^2 = -1} \qquad = (ac - bd) + i(ad + bc)$$

We write $z = a + ib$ and say

a is the **real part** of z and

b is the **imaginary part** of z

We also write $\bar{z} = a - ib$ and call

$\bar{z}$ the **complex conjugate** of z

spoken as "zee-bar"

Let $z = a + ib$

$|z| = |a+ib|$ is the length (or modulus) of z.

We calculate $|z|$ using the formula

$$|z|^2 = z \cdot \bar{z} = (a+ib)(a-ib)$$
$$= a^2 + b^2$$

Reciprocals and Division

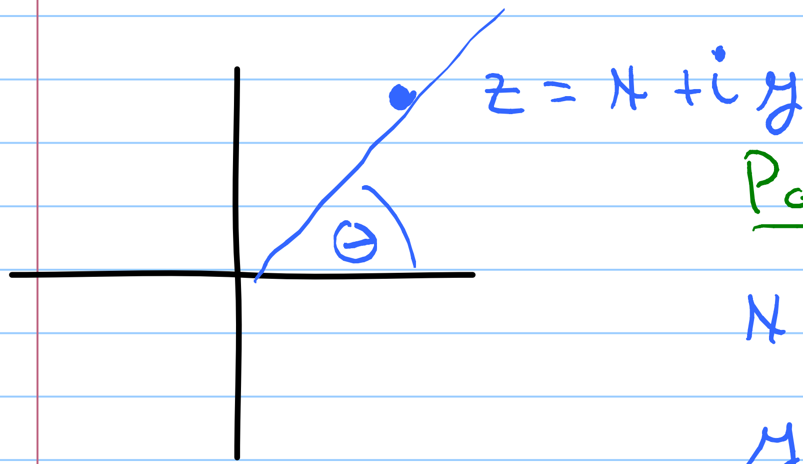
$$\frac{1}{a+ib} = \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2}$$

$$= \frac{a}{a^2+b^2} + i \, \frac{-b}{a^2+b^2}$$

This can also be written as

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} \quad \left[= \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} \right]$$

Polar Representation



Polar Coordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

so $\quad z = r\cos\theta + i r\sin\theta$

$$= r(\cos\theta + i\sin\theta)$$

$$= r(e^{i\theta})$$

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Polar Representation of a Complex Number
(also called phasor representation)
(the same as polar coordinates)

What is the polar representation of $1+\sqrt{3}i$?

This means, find r and θ so that

$$1+\sqrt{3}i = r e^{i\theta}$$

$$= r\cos\theta + i r\sin\theta$$

so we require

and solve

$$1 = r\cos\theta$$
$$\sqrt{3} = r\sin\theta$$

Solve for r

$$1^2+(\sqrt{3})^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta+\sin^2\theta) = r^2$$
$$4 = r^2$$

Solve for θ

$$\frac{\sqrt{3}}{1} = \tan\theta$$

$$\theta = \operatorname{atan}(\sqrt{3}) = \frac{\pi}{3}$$

so

$$\boxed{1+\sqrt{3}i = 2e^{i\frac{\pi}{3}}}$$

Question

Write the real and imaginary parts of

$$(1+\sqrt{3}\,i)\cdot e^{i(2t+\frac{\pi}{3})}.$$

$$(1+\sqrt{3}\,i)\cdot e^{i(2t+\frac{\pi}{3})} =$$

$$2e^{i\frac{\pi}{3}}\cdot e^{i(2t+\frac{\pi}{3})} = 2e^{i(2t+\frac{2\pi}{3})}$$

$$= \underbrace{2\cos\left(2t+\frac{2\pi}{3}\right)}_{\text{Real part}} + i\,\underbrace{2\sin\left(2t+\frac{2\pi}{3}\right)}_{\text{Imaginary part}}$$

Justifying Euler's Identity

$$e^{it} = \cos t + i \sin t$$

Characterization of e^t

Theorem If $f(t+s) = f(t)f(s)$,

then ① $f'(t) = f(t) \cdot f'(0)$ and $f(0) = 1$

② $f(t) = e^{f'(0)t}$

Proof
①
$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

$$= \lim_{h \to 0} \frac{f(t)f(h) - f(t)}{h}$$

$$= f(t) \lim_{h \to 0} \frac{f(h) - 1}{h}$$

$$= f(t) f'(0)$$

② $f'(t) = f(t) \cdot b \qquad b := f'(0)$

$$\frac{df}{f} = b\,dt$$

$$\ln |f| = bt + c_1$$

$$f = c_2 e^{bt}$$

$$1 = f(0) = c_2$$

$$f(t) = e^{bt} \qquad \blacksquare$$

Let $f(t) = \cos t + i \sin t$

$$f'(t) = -\sin t + i \cos t$$

$$= i(\cos t + i \sin t)$$

$$f'(t) = i\, f(t)$$

$$f(0) = 1$$

So $f(t) = e^{it}$ $\blacksquare$

Sum of angle Formulas

$$\cos(t+s) + i\sin(t+s) = e^{i(t+s)}$$

$$= e^{it} \cdot e^{is}$$

$$= (\cos t + i\sin t)\cdot(\cos s + i\sin s)$$

$$= \cos t\cos s + i\cos t\sin s$$

$$+ i\sin t\cos s + i^2 \sin t\sin s$$

$$= (\cos t\cos s - \sin t\sin s)$$

$$+ i(\cos t\sin s + \sin t\cos s)$$

Note Title 2/1/2020

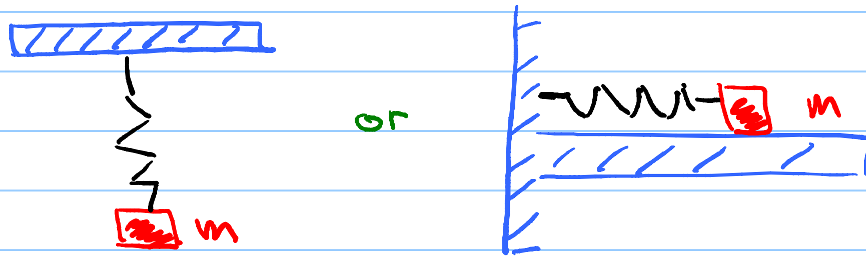
Second Order Differential Equations

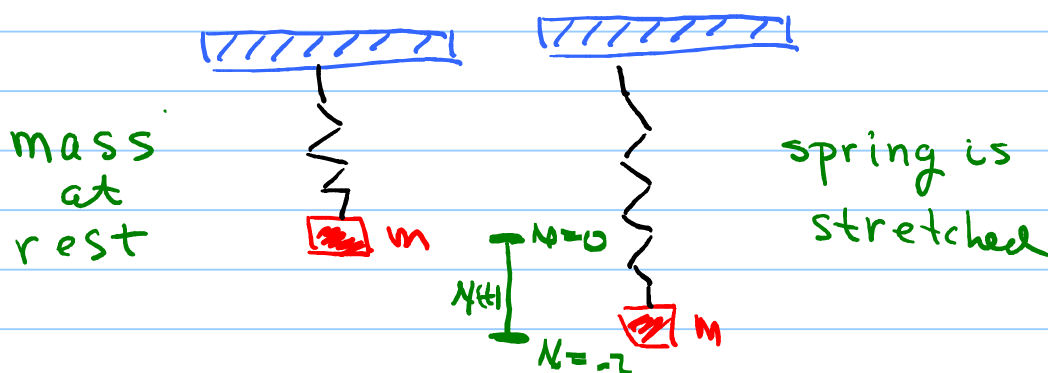
Force = Mass · Acceleration

$= m \cdot \ddot{x}$

Primary Example Mass and Spring

a.k.a Simple Harmonic Oscillator





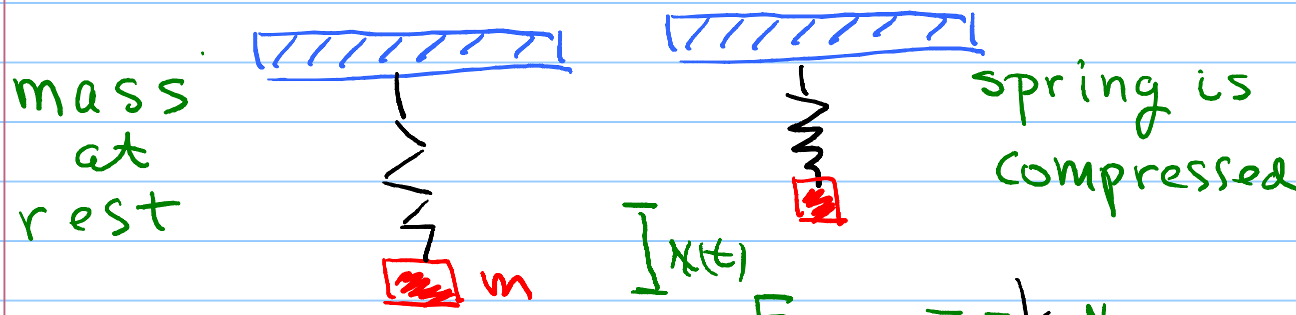
Hook's Law

$$F_{spring} = -kx$$

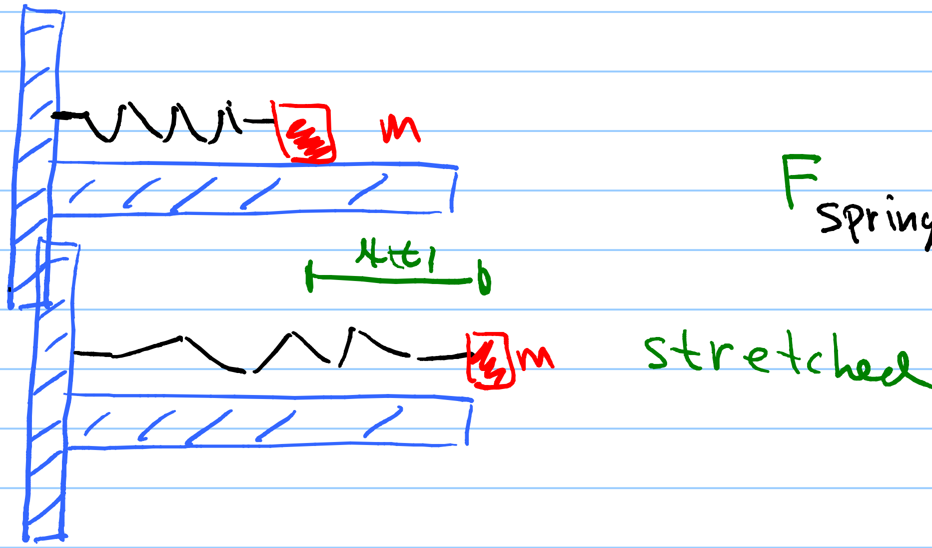
Spring force opposes displacement.

You may choose up [or ~~down~~] to be the positive direction.

Hook's Law



$$F_{spring} = -k\,x$$

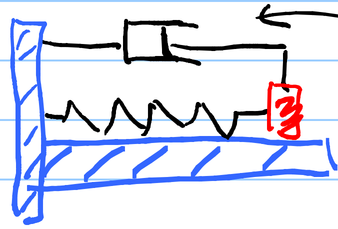


$$F_{spring} = -k\,x$$

$$\boxed{\begin{array}{c}\underline{\text{Equation of Motion}} \\ m\ddot{x} = -k\,x\end{array}}$$

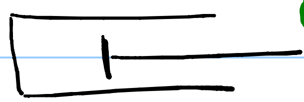
k is called the spring constant

Damped Harmonic Oscillator



Damper

Resists motion with a force proportional to velocity and in the opposite direction to the velocity



γ = damping coefficient

piston moving through a fluid

$\boxed{\text{damping isn't dry friction}}$

examples of damping

air resistance

shock absorber

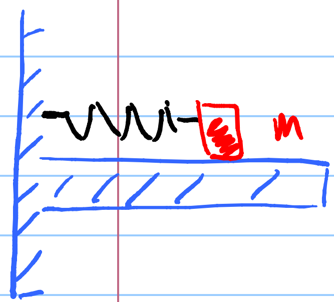
$$F_{damping} = -\gamma \dot{x}$$

Equations of Motion for Damped Harmonic Oscillator

$$m\ddot{x} = -\gamma \dot{x} - kx$$

Usually written

$$m\ddot{x} + \gamma \dot{x} + kx = 0$$



Suppose a 2 kg mass is attached to a spring with spring constant 24 kg/sec^2 and a damping coefficient of 16 kg/sec. The mass is set in motion from its equilibrium position with an initial velocity of 1 m/sec. Formulate and solve the Initial Value Problem.

Solution

y = displacement from equilibrium

$$m\ddot{y} = -\gamma\dot{y} - ky$$

$$2\ddot{y} = -16\dot{y} - 24y$$

starts from equilibrium $y(0) = 0$

initial velocity $\dot{y}(0) = 1$

$$2\ddot{y} + 16\dot{y} + 24y = 0$$

$$y(0) = 0 \quad \dot{y}(0) = 1$$

(IVP)

IVP

$$\ddot{y} + 8\dot{y} + 12y = 0 \quad \text{(DE)}$$

$$y(0) = 0 \quad \dot{y}(0) = 1 \quad \text{(IC)}$$

Seek $y(t) = e^{rt}$

then $\dot{y}(t) = r e^{rt}$ and $\ddot{y}(t) = r^2 e^{rt}$

Insert into (DE)

$$r^2 e^{rt} + 8 r e^{rt} + 12 e^{rt} = 0$$

$$r^2 + 8r + 12 = 0$$

$$(r+6)(r+2) = 0$$

$$r = -6 \quad \text{or} \quad r = -2$$

General Solution $y(t) = c_1 e^{-6t} + c_2 e^{-2t}$

Recall – This is a linear equation, so we may add solutions and multiply them by constants

Impose Initial Conditions

$$0 = y(0) = c_1 + c_2$$

$$1 = \dot{y}(0) = -6c_1 - 2c_2$$

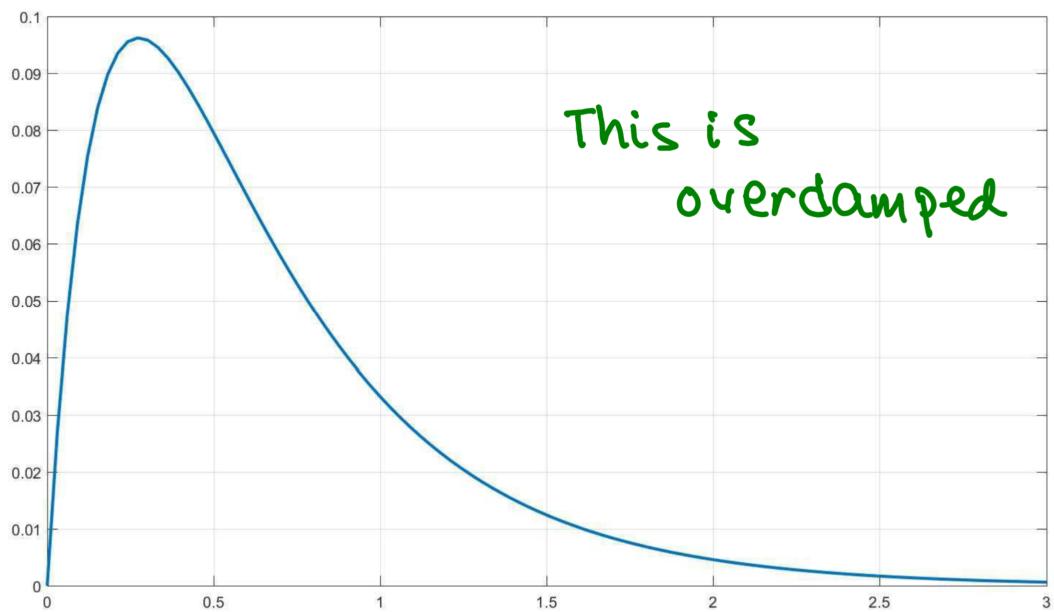
$$0 = y(0) = c_1 + c_2$$

$$1 = \dot{y}(0) = -6c_1 - 2c_2$$

$$
\begin{aligned}
0 &= 2c_1 + 2c_2 \\
+\quad 1 &= -6c_1 - 2c_2 \\
\hline
1 &= -4c_1
\end{aligned}
$$

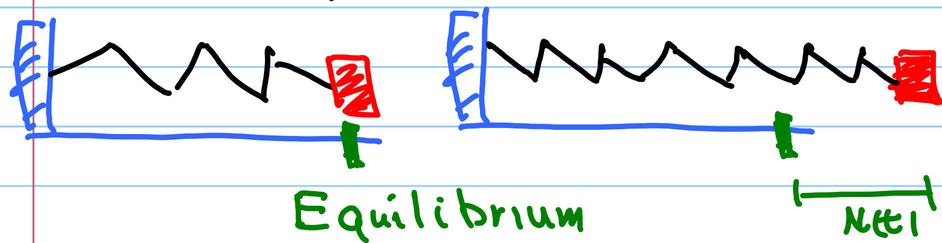
so $c_1 = -\frac{1}{4}$ and $c_2 = \frac{1}{4}$

so $y(t) = -\frac{1}{4}e^{-6t} + \frac{1}{4}e^{-2t}$



This is overdamped

Undamped Harmonic Oscillator



Suppose the spring constant is 50 $kg\,sec^{-2}$

<u>No damping</u> $\quad m = 2$ kg

Initial displacement = 5 meters

Initial velocity = 20 meters/sec

① Write the Initial Value Problem

② Find a formula for $x(t)$

<u>Answer</u> mass · acceleration = Σ Forces

$$m\ddot{x} = -kx$$

$$2\ddot{x} = -50x$$

or
$$\ddot{x} + 25x = 0$$

$$x(0) = 5 \qquad \dot{x}(0) = 20$$

IVP $\ddot{x} + 25x = 0$

$x(0) = 5 \qquad \dot{x}(0) = 20$

Seek $x(t) = e^{rt}$

$$\begin{aligned} \ddot{x} &= r^2 e^{rt} \\ + \quad 25x &= 25 e^{rt} \end{aligned}$$

$$\ddot{x} + 25x = (r^2+25) e^{rt} = 0$$

so $r^2 + 25 = 0$

or $r = \pm 5i$

Two solutions:

$x_1(t) = e^{5ti}$ and $x_2(t) = e^{-5ti}$

Because this is a Linear DE

$$x(t) = c_1 e^{5ti} + c_2 e^{-5ti}$$

is a solution for any constants c_1 and c_2

But we seek real solutions, so we use Euler's formula, and write our general solution as:

$$x(t) = D_1 \cos 5t + D_2 \sin 5t$$

and solve for D_1 and D_2.

General Solution

$$y(t) = C_1 e^{5it} + C_2 e^{-5it}$$

This is a real physical problem. It should always have a real solution. The displacement shouldn't be a complex number.

$$y(t) = C_1 e^{5it} + C_2 e^{-5it}$$

Euler's Formula

$$e^{5it} = \cos(5t) + i\sin(5t)$$

$$e^{-5it} = \cos(5t) - i\sin(5t)$$

why?

$\cos(-4t) = \cos(4t)$

$\sin(-4t) = -\sin(4t)$

cosine is even and sine is odd

Add two solutions and divide by 2

$$\frac{e^{5it} + e^{-5it}}{2} = \cos(5t)$$

Subtract and divide by $2i$

$$\frac{e^{5it} - e^{-5it}}{2i} = \sin(5t)$$

So the general solution may be written as

$$y(t) = D_1\cos(5t) + D_2\sin(5t)$$

$$x(t) = D_1 \cos 5t + D_2 \sin 5t$$

$$5 = x(0) = D_1 \cos(0) + D_2 \sin(0) = D_1$$

$$20 = \dot{x}(0) = -5D_1 \sin(0) + 5D_2 \cos(0) = 5D_2$$

so $D_1 = 5$ and $D_2 = 4$

$$\boxed{x(t) = 5\cos 5t + 4\sin 5t}$$

Summary - Linear Constant Coefficient Differential Equations, Second Order Homogeneous

$$\ddot{x} + r\dot{x} + kx = 0$$

① If you find 2 different solutions y_1 and y_2, then every solution is a linear combination of y_1 and y_2

i.e. $y(t) = C_1 y_1(t) + C_2 y_2(t)$

② All solutions are sums of exponentials.*

* This is a little bit of a lie. I'll explain more as we go on.

Examples of Damped Harmonic Oscillators

I won't test you on this.

A single story shear building consists of a rigid girder with mass m, which is supported by columns with combined stiffness k. The columns are assumed to be weightless, inextensible in the axial (vertical) direction, and they can only take shear forces but not bending moments. In the horizontal direction, the columns act as a spring of stiffness k. As a result, the girder can only move in the horizontal direction, and its motion can be described by a single variable $x(t)$; hence the system is called a single degree-of-freedom (DOF) system. The number of degrees-of-freedom is the total number of variables required to describe the motion of a system.

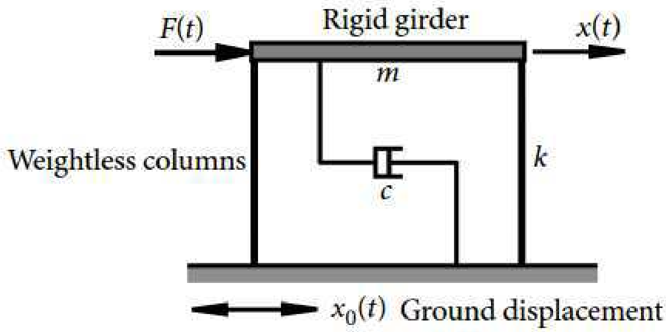


Figure 5.1 A single-story shear building.

The combined stiffness k of the columns can be determined as follows. Apply a horizontal static force P on the girder. If the displacement of the girder is Δ as shown in Figure 5.2, then the combined stiffness of the columns is $k = P/\Delta$.

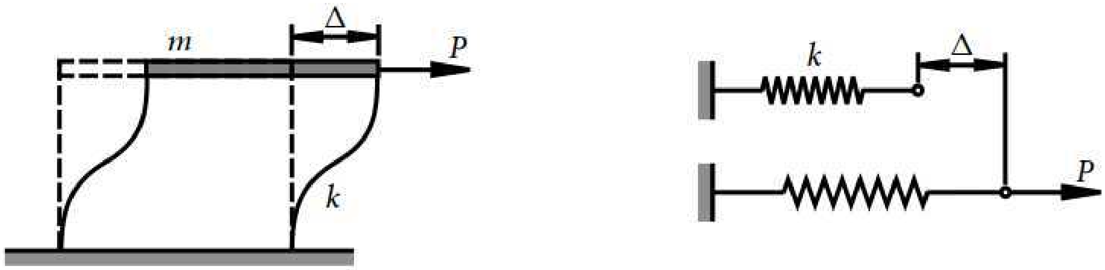
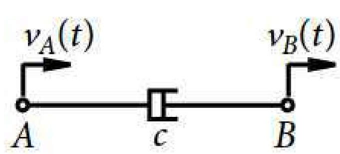


Figure 5.2 Determination of column stiffness.

The internal friction between the girder and the columns is described by a viscous dashpot damper with damping coefficient c. A dashpot damper is shown schematically in Figure 5.3 and provides a damping force $-c\,(v_B - v_A)$, where v_A and v_B are the velocities of points A and B, respectively, and $(v_B - v_A)$ is the relative velocity between points B and A. The damping force is opposite to the direction of the relative velocity.



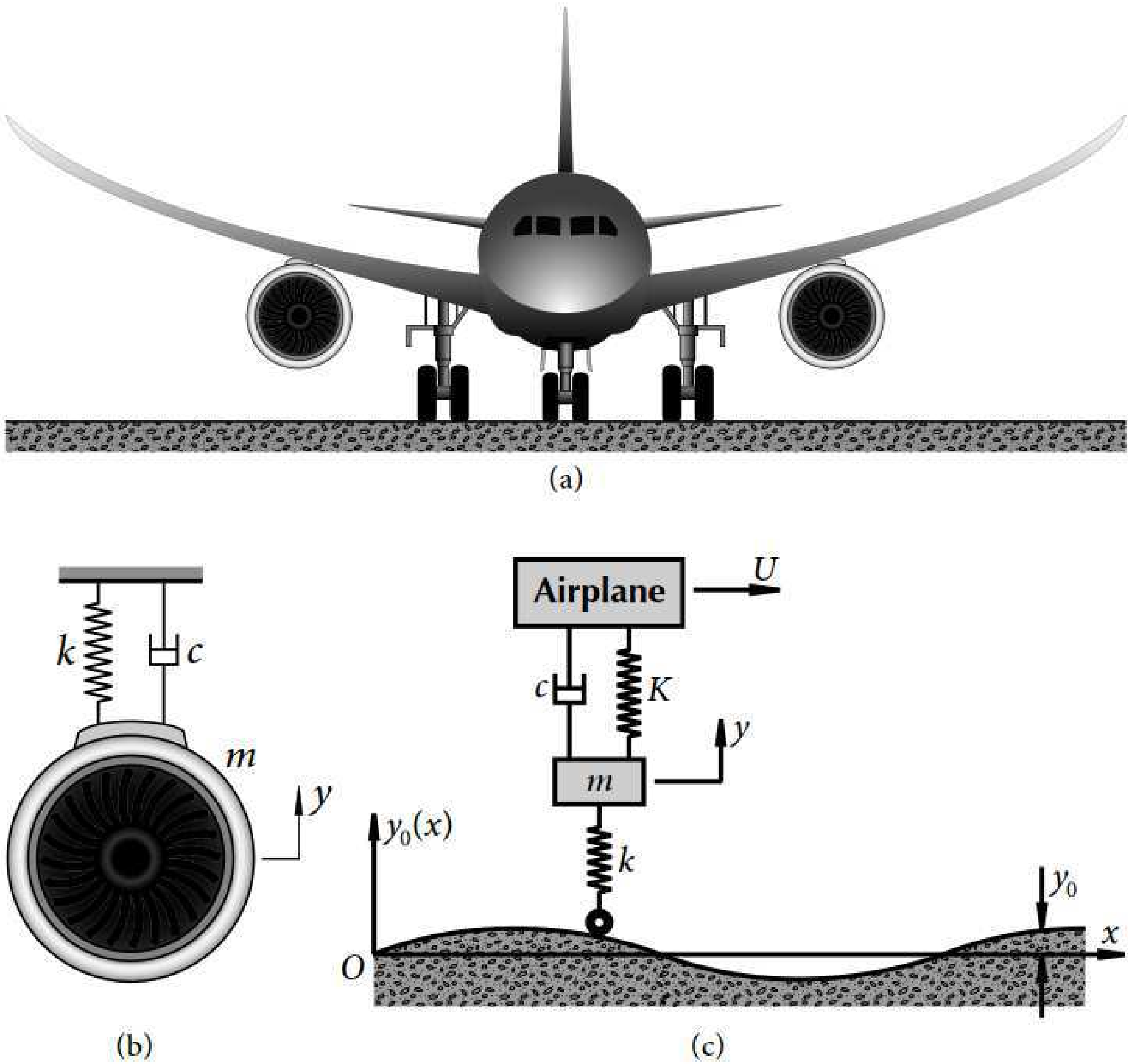
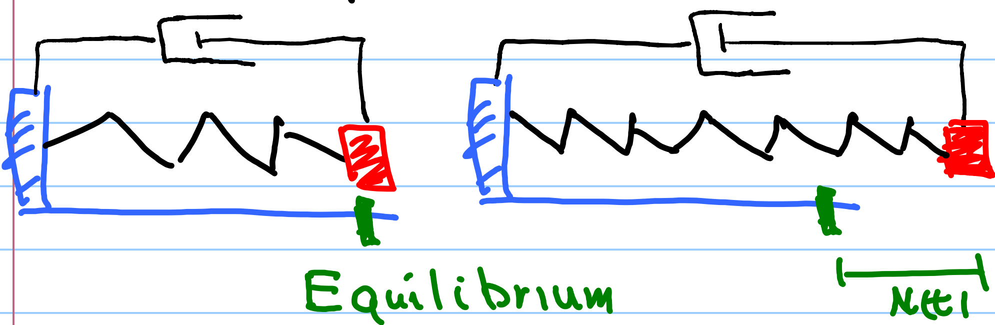


Figure 5.8 Mathematical modeling of jet engine and landing gear.

Note Title 2/1/2020

Damped Harmonic Oscillator



Suppose the spring constant is 145 kg sec^{-2}

Damping coeff. = 2 kg sec^{-1}. m = 1 kg

Initial displacement = 1 meters

Initial velocity = 2 meters/sec

① Write the Initial Value Problem

② Find a formula for $x(t)$

Answer $m\ddot{x} = -r\dot{x} - kx$

$$1\ddot{x} = -2\dot{x} - 145x$$

$$\ddot{x} + 2\dot{x} + 145x = 0 \qquad \text{(IVP)}$$

$$x(0) = 1 \qquad \dot{x}(0) = 2$$

$$\ddot{x} + 2\dot{x} + 145x = 0 \quad \text{(IVP)}$$

$$x(0) = 1 \qquad \dot{x}(0) = 2$$

Seek $x(t) = e^{rt}$

$$r^2 e^{rt} + 2r e^{rt} + 145 e^{rt} = 0$$

$$r^2 + 2r + 145 = 0$$

$$r^2 + 2r + 1 = -144$$

$$(r+1)^2 = -144$$

$$r + 1 = \pm 12 i$$

$$r = -1 \pm 12 i$$

We could write

$$x(t) = c_1 e^{(-1+12i)t} + c_2 e^{(-1-12i)t}$$

But, instead we write

$$x(t) = D_1 e^{-t} \cos 12t + D_2 e^{-t} \sin 12t$$

$$x(t) = D_1 e^{-t}\cos 12t + D_2 e^{-t}\sin 12t$$

and require $\quad x(0) = 1 \quad \dot{x}(0) = 2$

$$1 = x(0) = D_1 + \cancel{D_2 \cdot 0}$$

$$2 = \dot{x}(0) = D_1\left(-e^{-t}\cos 12t - 12e^{-t}\sin 12t\right)\Big|_{t=0} + D_2\left(-e^{-t}\sin 12t + 12e^{-t}\cos 12t\right)\Big|_{t=0}$$

$$2 = D_1(-1 - 0) + D_2(0 + 12)$$

so $\quad 1 = D_1$

and $\quad 2 = -D_1 + 12D_2$

so $\quad D_2 = \frac{3}{12} = \frac{1}{4}$

$$\boxed{x(t) = 1e^{-t}\cos 12t + \frac{1}{4}e^{-t}\sin 12t}$$

Critically Damped

$$\ddot{x} + 2\dot{x} + x = 0 \quad \text{(DE)}$$

$$x(0) = 0 \quad \dot{x}(0) = 10 \quad \text{(IC)}$$

① Solve the IVP ② Is the displacement ever $= 0$?

③ What is the maximum displacement?

Seek $x(t) = e^{rt}$ $\quad r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$ so $r = -1$

When there is only one value of r that solves the indicial equation, the general solution is $c_1 e^{rt} + c_2 t e^{rt}$.

explanation after we finish the problem.

General Solution $\quad c_1 e^{-t} + c_2 t e^{-t}$

Impose IC's

$0 = x(0) = c_1 + 0c_2$ so $c_1 = 0$

$10 = \dot{x}(0) = -c_1 + c_2$ so $c_2 = 10$

① $x(t) = 0e^{-t} + 10te^{-t} = 10te^{-t}$

$$x(t) = 10\,t\,e^{-t}$$

② where is displacement $= 0$

$$0 \stackrel{?}{=} 10t\,e^{-t}$$

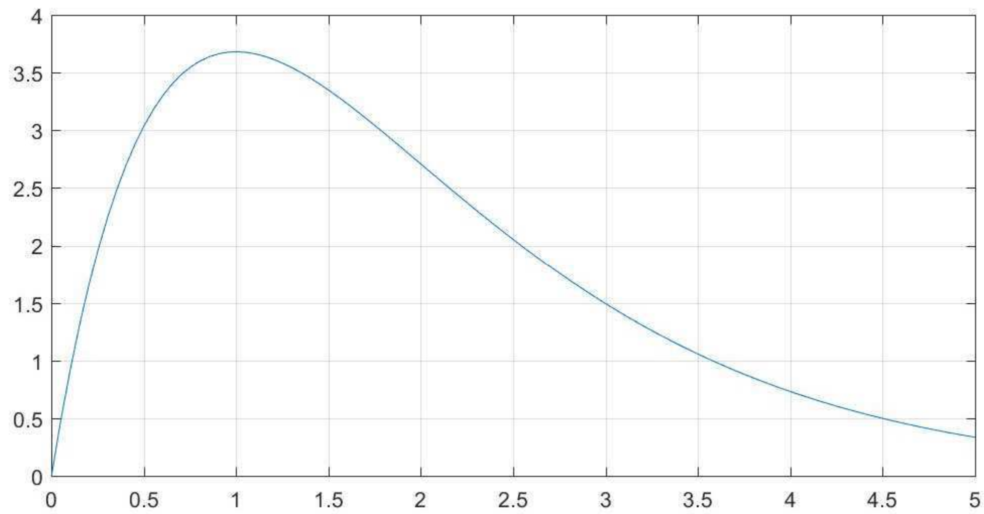
Only zero at $t = 0$

③ What is the max displacement?

Max displacement occurs when $\dot{x} = 0$

$$0 = \dot{x}(t) = 10(e^{-t} - t\,e^{-t}) = 10\,e^{-t}(1-t)$$

$\dot{x}$ vanishes at $t=1$. Max displacement $= 10\,e^{-1}$



Looks just like overdamped

Find the ge eral Solution to

$$\ddot{x} + 2\dot{x} + (1-\alpha)x = 0 \qquad \text{(DE)}$$

The form of the solution will look different for different values of α.

Seek $x = e^{rt}$ $\quad \dot{x} = re^{rt}$ $\quad \ddot{x} = r^2 e^{rt}$

(DE) $\quad r^2 e^{rt} + 2re^{rt} + (1-\alpha)e^{rt} = 0$

$$r^2 + 2r + 1 - \alpha = 0$$

$$r^2 + 2r + 1 = \alpha$$

$$(r+1)^2 = \alpha$$

$$r = -1 \pm \sqrt{\alpha}$$

$$x(t) = C_1 e^{(-1+\sqrt{\alpha})t} + C_2 e^{(-1-\sqrt{\alpha})t}$$

__Case 1__ $\underline{\alpha > 0}$ $\sqrt{\alpha}$ is a real number

$$x(t) = c_1 e^{(-1+\sqrt{\alpha})t} + c_2 e^{(-1-\sqrt{\alpha})t}$$

General Solution is a linear combination of decaying exponentials. If $\alpha = \frac{1}{2}$

$$x(t) = c_1 e^{\left(-1+\frac{1}{\sqrt{2}}\right)t} + c_2 e^{\left(-1-\frac{1}{\sqrt{2}}\right)t}$$

Overdamped – No oscillations

__Case 2__ $\alpha < 0$ $\sqrt{\alpha} = i\sqrt{-\alpha}$

$$x(t) = c_1 e^{(-1+i\sqrt{-\alpha})t} + c_2 e^{(-1+i\sqrt{-\alpha})t}$$

e.g. $\underline{\alpha = -1}$

$$x(t) = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t}$$

$$x(t) = D_1 e^{-t} \cos t + D_2 e^{-t} \sin t$$

Underdamped – Decaying oscillations

Case 3 $\alpha = 0$ Critically Damped

$$x(t) = c_1 e^{(-1+0)t} + c_2 e^{(-1-0)t}$$

$$x(t) = c_1 e^{-t} + c_2 e^{-t} \leftarrow$$ Both terms are the same

what is the other solution?

Fact

when the indicial equation only has one root r, both e^{rt} and te^{rt} are solutions to the DE1.

How do you see this?

Boyce-DiPrima problems 20-21-22 §3.5

I'll follow 21

Philosophy

General Solutions - without initial conditions aren't necessarily physical, so they can behave strangely. Solutions to the (IVP) are physical, so they can't do strange things.

Example

$$\ddot{x} + 2\dot{x} + (1-\alpha)x = 0$$

$$x(0) = 0$$
$$\dot{x}(0) = 1$$

We will solve this IVP explicitly for $\alpha > 0$ and then take the limit as $\alpha \to 0$

$$x(t) = c_1 e^{(-1+\sqrt{\alpha})t} + c_2 e^{(-1-\sqrt{\alpha})t}$$

Impose initial conditions

$$0 = x(0) = c_1 + c_2$$

$$1 = \dot{x}(0) = (-1+\sqrt{\alpha})c_1 + (-1-\sqrt{\alpha})c_2$$

$$0 = x(0) = C_1 + C_2$$

$$1 = \dot{x}(0) = (-1+\sqrt{\alpha})C_1 + (-1-\sqrt{\alpha})C_2$$

so $\quad C_1 = -C_2$ and $\quad 1 = 2\sqrt{\alpha}\, C_1$

$$x(t) = \frac{1}{2\sqrt{\alpha}} e^{(-1+\sqrt{\alpha})t} - \frac{1}{2\sqrt{\alpha}} e^{(-1-\sqrt{\alpha})t}$$

$$x(t) = e^{-t}\left(\frac{e^{\sqrt{\alpha}\,t} - e^{-\sqrt{\alpha}\,t}}{2\sqrt{\alpha}} \right)$$

Now, let $\alpha \to 0 \quad x(t) \to e^{-t}\left(\frac{0}{0}\right)$

The substitution $\alpha = b^2$ makes it easier to apply L'Hopital's rule

$$\lim_{\alpha \to 0} \left(\frac{e^{\sqrt{\alpha}\,t} - e^{-\sqrt{\alpha}\,t}}{2\sqrt{\alpha}} \right) = \lim_{b \to 0} \left(\frac{e^{bt} - e^{-bt}}{2b} \right)$$

$$\lim_{\to 0} \left(\frac{e^{bt} - e^{-bt}}{2b} \right) = \lim_{\to 0} \frac{\frac{d}{db}\left(e^{bt} - e^{-bt}\right)}{\frac{d}{d}(2b)} = \frac{t}{2}$$

$$\boxed{x(t) = t e^{-t}}$$ This is the second solution.

I won't ask you to reproduce this, but I will ask you to solve an IVP with a parameter.

e.g.

Find the solution to

$$\ddot{x} + \omega^2 x = 0$$

$$x(0) = 0$$

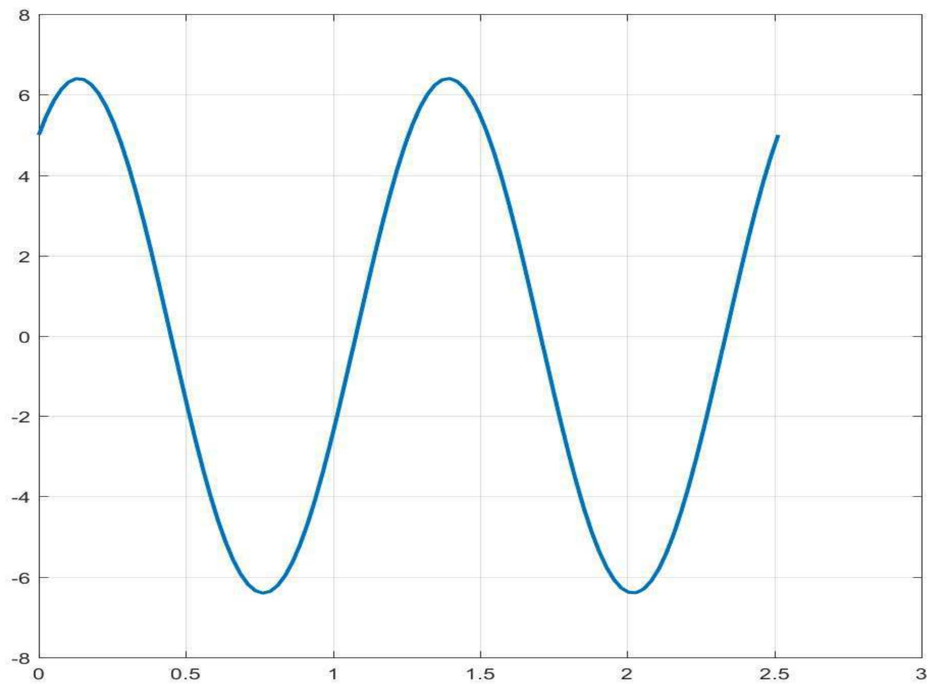
$$\dot{x}(0) = 1$$

Solution

$$x(t) = \frac{\sin \omega t}{\omega}$$

Find Frequency Amplitude Phase
from the graph

$$y(t) = A\cos(\omega t - \phi)$$



A = amplitude ω = angular frequency (omega)

ϕ = phase (lag)

slightly different form

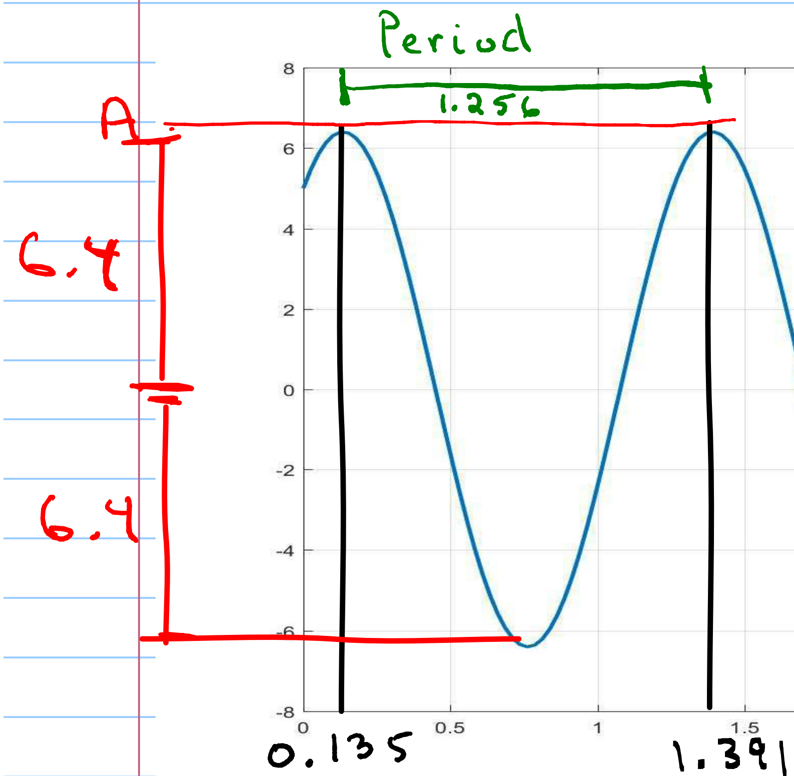
$$y(t) = A\cos(\omega(t - t_0))$$

Because its easier to find t_0 first

t_0 = time lag

$\phi = \omega t_0$

$$y(t) = A\cos(\omega(t - t_0))$$



$A = 6.4$ $\quad \omega = 2\pi/\text{Period} = 2\pi/1.256 = 5$

$t_0 = 0.135$ Because $A\cos(\omega(t - t_0)) = A$

when $\omega(t - t_0) = 0$

$$y(t) = 6.4\cos(5(t - 0.135))$$

$$= 6.4\cos(5t - 0.675)$$

$$\varphi = 0.675$$

Linear DE's $\ddot{y} + a\dot{y} + by = f(t)$

Two Principles

① **If** $y_1(t)$ solves and $y_2(t)$ solves **then** $c_1 y_1(t) + c_2 y_2(t)$ solves

② **If** $y_1(t)$ and $y_2(t)$ are independent $(y_1(t) \neq C\, y_2(t))$ solutions **then** every solution $y(t)$ can be written as $y(t) = C_1\, y_1(t) + C_2\, y_2(t)$

① is easy to understand

② is harder

I will expect you to use these two principles. You don't need to remember the proofs.

Example of ①

If
$$\ddot{y}_1 + 4y_1 = f_1$$
$$\ddot{y}_2 + 4y_2 = f_2$$

then
$$w = 3y_1 + 8y_2$$

Solves
$$\ddot{w} + 4w = 3f_1 + 8f_2$$

Proof
$$3\ddot{y}_1 + 3\cdot 4\, y_1 = 3f_1$$
$$+\quad 8\ddot{y}_2 + 8\cdot 4\, y_2 = 8f_2$$

$$3\ddot{y}_1 + 8\ddot{y}_2 + 4\cdot(3y_1 + 8y_2) = 3f_1 + 8f_2$$

derivative is linear

$$(3y_1 + 8y_2)^{\cdot\cdot} + 4\cdot(3y_1 + 8y_2) = 3f_1 + 8f_2$$

$$\ddot{w} + 4w = 3f_1 + 8f_2$$

So far, we have only used this in the case that $f_1 = 0$ and $f_2 = 0$

Thm There is exactly one solution to a second order initial value problem.

Proof by example

$\ddot{y} = 5y \qquad y(0) = a \qquad \dot{y}(0) = b$

$\ddot{y}(0) = 5y(0) = 5a$

$\dddot{y}(0) = 5\dot{y}(0) = 5b$

$y^{(iv)}(0) = 5\ddot{y}(0) = 25a$

$y^{(v)}(0) = 5\dddot{y}(0) = 25b$

etc.

$$y(t) = \sum_{t=0}^{\infty} \frac{y^{(n)}(0)}{n!} t^n$$

We have produced $y(t)$ as a formula involving only a and b

Corollary

$$y_1(t) = y_2(t) \iff y_1(0) = y_2(0) \text{ and } \dot{y}_1(0) = \dot{y}_2(0)$$

Corollary

$$y_1(t) = c\,y_2(t) \iff y_1(0) = c\,y_2(0) \text{ and } \dot{y}_1(0) = c\,\dot{y}_2(0) \iff y_1(0)\dot{y}_2(0) - \dot{y}_1(0)\,y_2(0) = 0$$

Corollary

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \iff y(0) = c_1 y_1(0) + c_2 y_2(0) \text{ and } \dot{y}(0) = c_1 \dot{y}_1(0) + c_2 \dot{y}_2(0)$$

Every solution $y(t) = c_1 y_1(t) + c_2 y_2(t)$

$\iff$ I can solve for c_1 and c_2 from

$$e = c_1 y_1(0) + c_2 y_2(0)$$
and
$$f = c_1 \dot{y}_1(0) + c_2 \dot{y}_2(0)$$

for every e and f

Two equations in two unknowns

$$a c_1 + b c_2 = e$$

$$c c_1 + d c_2 = f$$

Formula for solution

$$c_1 = \frac{be - cf}{ad - bc}$$

$$c_2 = \frac{ae - df}{ad - bc}$$

as long as $ad - bc \neq 0$

Corollary

I can solve for c_1 and c_2 from

$e = c_1 y_1(0) + c_2 y_2(0)$

and

$f = c_1 \dot{y}_1(0) + c_2 \dot{y}_2(0)$

for every e and f

$$\Longleftrightarrow$$

$y_1(0)\dot{y}_2(0) - \dot{y}_1(0)\, y_2(0) \neq 0 \iff y_1(t) \neq C\, y_2(t)$

$ad - bc \neq 0$

Back to stuff you're responsible for

Example $\ddot{y} + 4y = 0$

Some solutions

$y_1 = \cos 2t$

$y_2 = \sin 2t$

$y_3 = \cos(2t-3)$

$y_4 = 6\cos(2t-3) + P\sin 2t$

I can always find a solution to the

(IVP) $\ddot{y} + 4y = 0$

$y(0) = a$

$\dot{y}(0) = b$

of the form

$$c_1 \cos 2t + c_2 \cos(2t-3)$$

or

$$c_1 \sin 2t + c_2 \cos(2t-3)$$

or

$\vdots$

choose any 2

$c_1 \sin(2t) + c_2\, 2\sin(2t)$ doesn't work

$c_1 \sin(2t) + c_2 \cos(2t - \frac{\pi}{2})$ doesn't work

Forced Harmonic Oscillator

$m\ddot{y} + \gamma\dot{y} + ky = F(t)$ = external force

Problem Solve the (IVP)

$$\ddot{y} + 3\dot{y} + 2y = e^{-4t}$$

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

Technique

Seek $y = y_p + y_H$

where

(1) Find general solution to

$\ddot{y}_H + 3\dot{y}_H + 2y_H = 0$ called "homogeneous solution"

(2) Find any solution to

$\ddot{y}_p + 3\dot{y}_p + 2y_p = e^{-4t}$ called "particular solution" or "steady state solution"

(3) Choose the constants from (1) to satisfy IC's.

① Find general solution to homogeneous equation.

$$\ddot{y}_H + 3\dot{y}_H + 2y_H = 0$$

Seek $y_H = e^{rt}$ $\qquad r^2 + 3r + 2 = 0$

$\qquad (r+2)(r+1) = 0$

$$y_H = C_1 e^{-2t} + C_2 e^{-t}$$

② Find any solution to

$$\ddot{y}_p + 3\dot{y}_p + 2y_p = e^{-4t}$$

Seek $y_p = Ae^{-4t}$

Substitute y_p into (DE) and solve for A

$$(-4)^2 A e^{-4t} + 3(-4) A e^{-4t} + 2Ae^{-4t} = e^{-4t}$$

$$(16 - 12 + 2) A = 1$$

$6A = 1$

$A = \frac{1}{6}$ $\qquad \boxed{y_p = \frac{1}{6} e^{-4t}}$

Recall $y(t) = y_H + y_P$

so $y = \frac{1}{6} e^{-4t} + C_1 e^{-2t} + C_2 e^{-t}$

Now use IC's to find C_1 and C_2.

(IC)'s $\quad y(0) = 0 \qquad \dot{y}(0) = 0$

$$\left.\begin{aligned} 0 &= y(0) = \tfrac{1}{6} + C_1 + C_2 \\ 0 &= \dot{y}(0) = -\tfrac{4}{6} - 2C_1 - C_2 \end{aligned}\right\} \quad \begin{aligned} C_1 &= -1/2 \\ C_2 &= 1/3 \end{aligned}$$

$$\boxed{y = \tfrac{1}{6} e^{-4t} - \tfrac{1}{2} e^{-2t} + \tfrac{1}{3} e^{-t}}$$

lecture 14

Note Title 10/26/2017

Forced Harmonic Oscillator

$m\ddot{y} + \gamma\dot{y} + ky = F(t)$ = external force

Problem Solve the (IVP)

$$\ddot{y} + 3\dot{y} + 2y = \cos 2t$$

$$y(0) = 0 \qquad \dot{y}(0) = 1$$

Technique

Seek $y = y_p + y_H$

where

(1) Find general solution to

$\ddot{y}_H + 3\dot{y}_H + 2y_H = 0$ called "homogeneous solution"

(2) Find any solution to

$\ddot{y}_p + 3\dot{y}_p + 2y_p = \cos 2t$

called "particular solution", or "steady state solution"

(3) Choose the constants from (1) to satisfy IC's.

$$\ddot{y} + 3\dot{y} + 2y = \cos 2t$$

$$y(0) = 0 \qquad \dot{y}(0) = 1$$

---

① $\ddot{y}_H + 3\dot{y}_H + 2y_H = 0$

Seek $y = e^{rt}$ $\quad r^2 + 3r + 2 = 0$

$$\boxed{y_H(t) = C_1 e^{-t} + C_2 e^{-2t}}$$

---

② $\ddot{y}_p + 3\dot{y}_p + 2y_p = \cos 2t$

Seek $y_p = A\cos 2t$ ← Not good enough

Seek $y_p = A\cos 2t + B\sin 2t$ ← This one works

Rule: The particular solution must contain constants times

① The forcing term

② All terms that are derivatives of forcing terms

$$\ddot{y}_p + 3\dot{y}_p + 2y_p = \cos 2t$$

This doesn't work

Try $y_p = A\cos 2t$

$$2y_p = 2A\cos 2t$$

$$3\dot{y}_p = -2A\sin 2t \cdot 3$$

$$+\ \ddot{y}_p = -4A\cos 2t$$

$$\cos 2t = \underbrace{-2A}_{=1}\cos 2t \underbrace{-6A}_{=0}\sin 2t$$

Can't solve this

$$2\, y_p = 2A\cos 2t + 2B\sin 2t$$

$$3 \cdot \dot{y}_p = 3\cdot(-2)A\sin 2t + 3\cdot 2B\cos 2t$$

$$+ \quad \ddot{y}_p = -4A\cos 2t - 4B\sin 2t$$

$$\cos 2t = (2A + 6B - 4A)\cos 2t$$

$$+ (2B - 6A - 4B)\sin 2t$$

$$0\sin 2t + 1\cos 2t$$

$$= (-2B - 6A)\sin 2t + (6B - 2A)\cos 2t$$

$$0 = (-2B - 6A) \qquad B = -3A$$

$$1 = (6B - 2A) \qquad 1 = -20A$$

$$A = -\frac{1}{20} \qquad B = \frac{3}{20}$$

$$y_p = -\frac{1}{20}\cos 2t + \frac{3}{20}\sin 2t$$

Now go to step 3

$$y(t) = y_p + y_H$$

$$= \frac{-1}{20}\cos 2t + \frac{3}{20}\sin 2t + c_1 e^{-t} + c_2 e^{-2t}$$

$$0 = y(0) = \frac{-1}{20}\cdot 1 + \frac{3}{20}\cdot 0 + c_1\cdot 1 + c_2\cdot 1$$

$$1 = \dot{y}(0) = \frac{2}{20}\cdot 0 + \frac{6}{20}\cdot 1 - c_1 - 2c_2$$

$$\left.\begin{aligned} c_1 + c_2 &= \frac{1}{20} \\ -c_1 - 2c_2 &= \frac{14}{20} \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= \frac{16}{20} \\ c_2 &= \frac{-15}{20} \end{aligned}$$

$$y(t) = \frac{-1}{20}\cos 2t + \frac{3}{20}\sin 2t + \frac{16}{20}e^{-t} - \frac{15}{20}e^{-2t}$$

Notice: As t gets larger, $y_H \to 0$

so, after a long time

$$y(t) \approx \frac{-1}{20}\cos 2t + \frac{3}{20}\sin 2t$$

so we call $y_p(t)$ the steady state solution.

Back to discussion of particular solution

(2) $\ddot{y}_p + 3\dot{y}_p + 2y_p = \cos 2t$

why make this choice?

Seek $y_p = A\cos 2t + B\sin 2t$

I want sums of functions $y_p, \dot{y}_p, \ddot{y}_p$

that will sum up to $\cos 2t$.

Start with $y_p = A\cos 2t$

but y_p will get differentiated, so

I will see terms like $B\sin 2t$

and $B\sin 2t$ will get differentiated,

so I'll see terms like $C\cos 2t$.

But I already have this term, so

I stop here.

Same as previous page with fewer words

$\cos 2t$ ①

$\Big\downarrow \frac{d}{dt}$

$2\sin 2t$ ②

$\Big\downarrow \frac{d}{dt}$

$-4\cos 2t^{\text{old}}$ STOP

$y_p(t) =$

$A\cos 2t + B\sin 2t$

Another example

$\ddot{y}_p + 2y_p = t\cos t$

$t\cos t$ ①

$\Big\downarrow \frac{d}{dt}$

② $\cos t$ $-$ ③ $t\sin t$

$\Big\downarrow \frac{d}{dt}$ $\qquad$ $\Big\downarrow \frac{d}{dt}$

$-\sin t$ $\qquad$ $t\cos t^{\text{old}}$

④ $\Big\downarrow$ $\qquad$ $+ \underset{\text{old}}{\sin t}$

$-\underset{\text{old}}{\cos t}$ $\qquad$ stop

$y_p =$
$(At+B)\cos t$
$+ (Ct+D)\sin t$

Slightly More Complicated Example

$$\ddot{y} + 3\dot{y} + 2y = t^2 e^{-4t}$$

Homogeneous Solution is the same

What is the form of y_p?

Start with $y_p = At^2 e^{-4t} + ??$

<u>Differentiate</u>

$$\dot{y}_p = 2t\,e^{-4t} - 8t^2 e^{-4t}$$

I could add the whole derivative

$$\left[\begin{aligned} y_p &= At^2 e^{-4t} + B\left(2t\,e^{-4t} - 8t^2 e^{-4t}\right) \\ &= (A - 8B)\,t^2 e^{-4t} + 2Bte^{-4t} \\ &= A_1 t^2 e^{-4t} + B_1 te^{-4t} \end{aligned}\right]$$

but its simpler to just add the new term

$$\dot{y}_p = \underbrace{2t\,e^{-4t}}_{\text{new term}} \underbrace{- 8t^2 e^{-4t}}_{\text{old term}}$$

$$y_p = At^2 e^{-4t} + Bte^{-4t} + ??$$

$$y_p = At^2e^{-4t} + Bte^{-4t} + ??$$

Now differentiate again. Its good enough to just differentiate the new term.

$$(te^{-4t})' = \underbrace{e^{-4t}}_{\text{new term}} - 4\underbrace{te^{-4t}}_{\text{old term}}$$

Add the new term

$$y_p = At^2e^{-4t} + Bte^{-4t} + Ce^{-4t} + ??$$

Now differentiate again. Its good enough to just differentiate the new term.

$$(e^{-4t})' = -4\underbrace{e^{-4t}}_{\text{old term}}$$

No new term - I can stop!!

$$y_p = At^2e^{-4t} + Bte^{-4t} + Ce^{-4t}$$

This is exactly the same calculation as the previous pages, with fewer words

Purple means "not important"

$$t^2 e^{-4t} \quad ①$$

$$\Big\downarrow \frac{d}{dt}$$

$$2t e^{-4t} \; ② \quad -4 t^2 e^{-4t} \; \text{old}$$

STOP

$$\Big\downarrow \frac{d}{dt}$$

$$2 e^{-4t} \; ③ \quad -4 t e^{-4t} \; \text{old}$$

STOP

$$\Big\downarrow$$

$$-8 e^{-4t} \; \text{old}$$

STOP

$$y_p(t) = At^2 e^{-4t} + Bte^{-4t} + Ce^{-4t}$$

or

$$(At^2 + Bt + c)\, e^{-4t}$$

$$y_p = At^2e^{-4t} + Bte^{-4t} + Ce^{-4t}$$

Now substitute y_p into

$$\ddot{y}_p + 3\dot{y}_p + 2y_p = t^2 e^{-4t}$$

and solve for A, B, C

$$y_p = At^2e^{-4t} + Bte^{-4t} + Ce^{-4t}$$

$$\dot{y}_p = -4At^2e^{-4t} + (2A-4B)te^{-4t} + (B-4C)e^{-4t}$$

$$\ddot{y}_p = 16At^2e^{-4t} + (-16A+16B)te^{-4t} + (2A-8B+16C)e^{-4t}$$

$$2y_p = 2At^2e^{-4t} + 2Bte^{-4t} + 2Ce^{-4t}$$

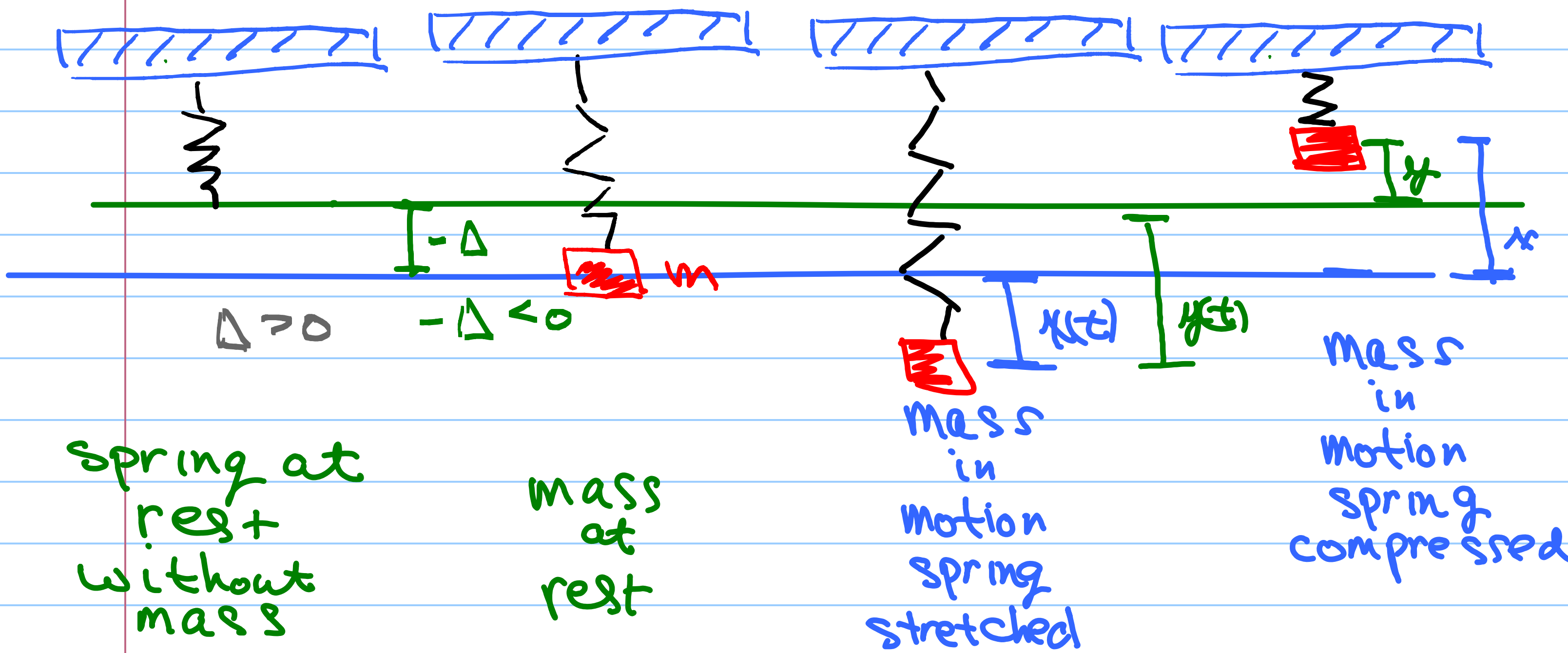
$$+\ 3\dot{y}_p = -12At^2e^{-4t} + (6A-12B)te^{-4t} + (3B-12C)e^{-4t}$$

$$t^2e^{-4t} = 6At^2e^{-4t} + (-10A-6B)te^{-4t} + (2A-5B+6C)e^{-4t}$$

$$\left.\begin{aligned} 6A &= 1 \\ -10A-6B &= 0 \\ 2A-5B+6C &= 0 \end{aligned}\right\} \qquad \begin{aligned} A &= \frac{1}{6} \\ B &= \frac{-10}{36} = \frac{-5}{18} \\ C &= \frac{-31}{18\cdot 6} \end{aligned}$$

I didn't check this carefully.

where did gravity go?



Mass at Rest - Net Force = 0

$0 = -k(\Delta) - mg$ = Spring Force + Gravitational Force

DE for $y(t)$ $\qquad m\ddot{y} = -ky - mg$

But Everyone writes DE for $x(t) = y + \Delta$

$$m\ddot{x} = m\ddot{y} = -ky - mg = -k(x-\Delta) - mg$$

$$= -kx + \underbrace{k\Delta - mg}_{=0}$$

$$\boxed{m\ddot{x} = -kx}$$

Experimental Method to determine k

① Attach m

② measure Δ $\qquad$ ③ $k = \dfrac{mg}{\Delta}$

You are not responsible for this

How do you force a mass spring system?

The forcing comes from driving over an uneven surface, or from a moving surface, as in an earthquake. The force is transmitted through a spring.

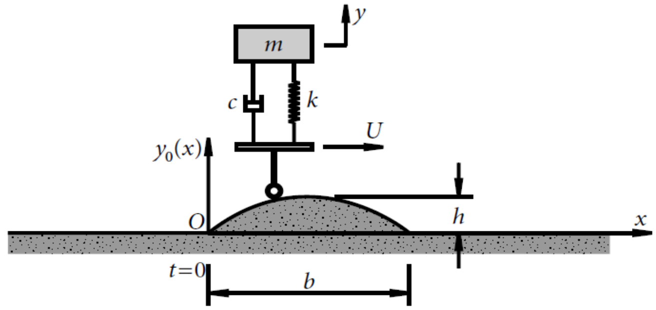
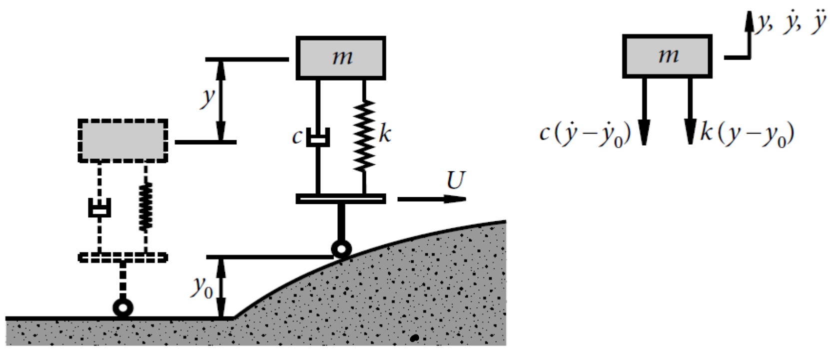


Figure 5.20 A vehicle passing a speed bump.



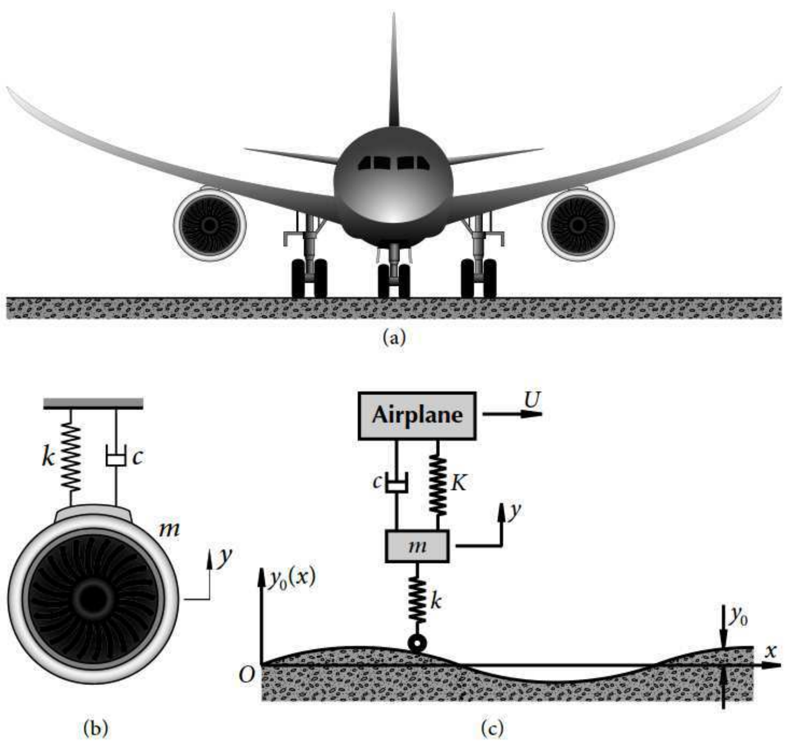
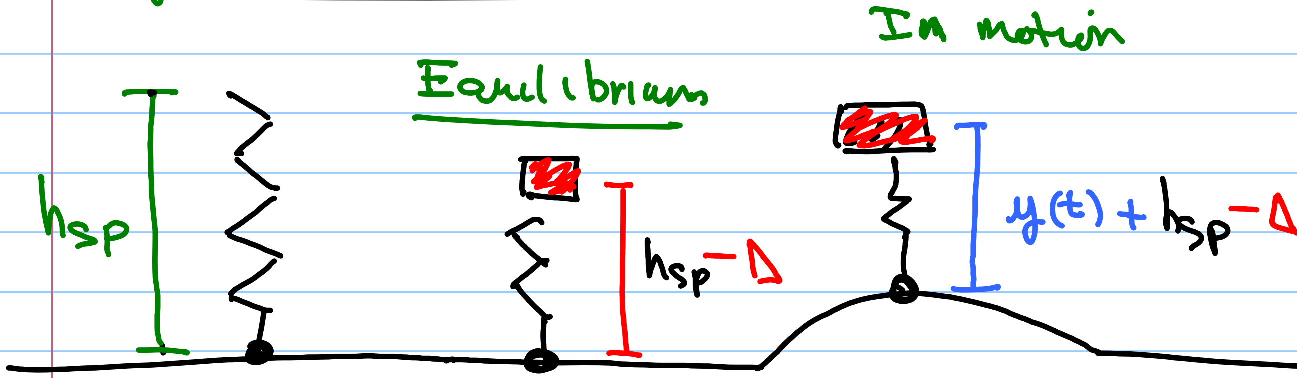


Figure 5.8 Mathematical modeling of jet engine and landing gear.

Height of the ground

e.g. $H_g(x) = A\cos(\alpha x)$

Mass Spring moving along bumpy ground at speed c



$h_g(t)$ = height of the ground

$\left[= H_g(ct) \quad \text{where} \right.$

$\left. H_g(x) = \text{height at position } x \right\}$

$$\boxed{\begin{array}{l} \underline{\text{At equilibrium}} \\ 0 = -mg - k(-\Delta) \\ \text{or} \quad k\Delta = mg \end{array}}$$

Newton

$$m \begin{pmatrix}\text{block} \\ \text{acceleration}\end{pmatrix} = \sum \text{Forces}$$

$$m \ddot{(\text{block height})} = -k \begin{pmatrix}\text{spring} \\ \text{compressed}\end{pmatrix} - mg$$

$$m \ddot{\left(y(t) + h_{sp} - \Delta + h_g(t)\right)} = -k(y - \Delta) - mg$$

$$m\ddot{y} + m\ddot{h}_g = -ky + \underbrace{k\Delta - mg}_{= 0}$$

$$\boxed{m\ddot{y} + ky = -m\ddot{h}_g(t)}$$

$$m\ddot{y} + ky = -m\ddot{h}_g(t)$$

$$h_g(t) = H_g(ct) =$$

$$\ddot{h}_g = c^2 \ddot{H}_g(ct)$$

e.g. $\quad H_g(x) = A\cos(\alpha x)$

$$\ddot{h}_g = \alpha^2 c^2 A\cos(\alpha c t)$$

$$m\ddot{y} + ky = m\alpha^2 c^2 A\cos(\alpha c t)$$

Notice: Forcing frequency depends on speed c and "wavenumber" α.

lecture 16

Note Title 10/28/2017

Method of Undetermined Coefficients

Problem

Find a particular solution to

$$\ddot{y}_p + y_p = \cos t$$

General Proceedure

① Find general homogeneous solution to $\ddot{y}_H + y_H = 0$

Answer $y_H = c_1 \cos t + c_2 \sin t$

② Seek particular solution as a linear combination of the forcing term and its derivatives

Answer $y_p = A \underbrace{\cos t}_{\text{Forcing term}} + B \underbrace{\sin t}_{\text{derivative of forcing term}}$

③ Ask homogeneous question: Are any terms of y_p solutions to the homogeneous equation?

No: Insert y_p in (1)(2) and solve for coefficients

Yes: Multiply those terms by t and repeat step ③

$$\ddot{y}_p + y_p = \cos t$$

$$y_H = C_1 \cos t + C_2 \sin t \qquad y_p = A\cos t + B\sin t$$

(3) Ask homogeneous question: Are any terms of y_p solutions to the homogeneous equation?

No — Proceed to step (4)

Yes — Multiply those terms by t and repeat step (3)

(4) Insert y_p in (1)(2) and solve for coefficients

Answer

Yes, every term in y_p is a term in y_H

Make new $y_p = t \cdot (\text{old } y_p)$

$$y_p = At\cos t + Bt\sin t$$

Repeat step (3)

Are any terms of y_p solutions to the homogeneous equation?

Answer No - proceed to step (4)

Step (4)

$$\ddot{y}_p + y_p = \cos t$$

$$y_p = At\cos t + Bt\sin t$$

$$\dot{y}_p = A\cos t - At\sin t + B\sin t + Bt\cos t$$

$$\ddot{y}_p = -A\sin t - A\sin t - At\cos t$$
$$+ B\cos t + B\cos t - Bt\sin t$$

Organize terms

$$\ddot{y}_p = -At\cos t - Bt\sin t - 2A\sin t + 2B\cos t$$

$$\ddot{y}_p + y_p = -At\cos t - Bt\sin t - 2A\sin t + 2B\cos t$$
$$+ At\cos t + Bt\sin t$$

Simplify

$$= -2A\sin t + 2B\cos t$$

So we must find A and B so that

$$-2A\sin t + 2B\cos t = \cos t$$

$$A = 0 \text{ and } B = \tfrac{1}{2}$$

$$\boxed{y_p = \tfrac{1}{2} t\sin(t)}$$

Where does the "t" come from?

$$\ddot{y} + y = \cos(1.0001t)$$

$$y_p = A\cos(1.0001t) + B\sin(1.0001t)$$

But

$$\ddot{y} + y = \cos t$$

$$y_p = At\cos t + Bt\sin t$$

To see why we solve

$$\ddot{y} + y = \cos \omega t$$

with initial conditions

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

(These initial conditions make the calculation simpler, but any initial conditions will work)

We will write down a formula for the solution for $\omega \neq 1$, and then take the limit as $\omega \to 1$.

$$\ddot{y} + y = \cos \omega t \qquad \text{(DE)}$$

$$y(0) = 0 \qquad \dot{y}(0) = 0 \qquad \text{(IC)}$$

Homogeneous Solution

$$\ddot{y}_H + y_H = 0 \qquad y_H = c_1 \cos t + c_2 \sin t$$

Seek particular solution

$$y_p = A \cos \omega t + B \sin \omega t$$

$$\ddot{y}_p = -\omega^2 A \cos \omega t - \omega^2 B \sin \omega t$$

Insert y_p in (DE)

$$(1-\omega^2) A \cos \omega t + (1-\omega^2) B \sin \omega t = \ddot{y}_p + y_p = \cos \omega t$$

Conclude: $\quad A = \dfrac{1}{1-\omega^2} \qquad B = 0$

so $\quad y(t) = \dfrac{\cos \omega t}{1-\omega^2} + c_1 \cos t + c_2 \sin t$

Now use (IC)

$$0 = y(0) = \frac{1}{1-\omega^2} + c_1 \Longrightarrow c_1 = \frac{-1}{1-\omega^2}$$

$$0 = \dot{y}(0) = c_2 \Longrightarrow c_2 = 0$$

$$y(t) = \frac{\cos \omega t - \cos t}{\omega^2 - 1}$$

The solution to

$$\ddot{y} + y = \cos\omega t \qquad \text{(DE)}$$

$$y(0) = 0 \qquad \dot{y}(0) = 0 \qquad \text{(IC)}$$

is $$y(t) = \frac{\cos\omega t - \cos t}{\omega^2 - 1}$$

Now we can let $\omega \to 1$

$$\lim_{\omega \to 1} \frac{\cos\omega t - \cos t}{\omega^2 - 1} = \frac{\cos t - \cos t}{1 - 1} = \text{"}\frac{0}{0}\text{"}$$

so we use L'Hôpital's rule

$$\lim_{\omega \to 1} \frac{\cos\omega t - \cos t}{\omega^2 - 1} = \lim_{\omega \to 1} \frac{\frac{d}{d\omega}(\cos\omega t - \cos t)}{\frac{d}{d\omega}(\omega^2 - 1)}$$

$$= \lim_{\omega \to 1} \frac{t \sin\omega t}{2\omega} = \frac{t \sin t}{2}$$

so we see that the forcing function $\cos t$ results in a solution $y(t)$ of the form $A\underbrace{t}\sin t$. This is one way to see why we multiply the forcing term by t when the forcing term matches a term from the homogeneous solution.

Another Example

Find the correct form for y_p

$$\ddot{y} + 5\dot{y} + 6y = t^2 e^{-2t}$$

① $y_h = C_1 e^{-2t} + C_2 e^{-3t}$

② Seek y_p as sum of forcing term and derivatives

$$y_p = A t^2 e^{-2t} + B t e^{-2t} + C e^{-2t}$$

③ Homogeneous Question - Yes

Multiply y_p by t

$$y_p = A t^3 e^{-2t} + B t^2 e^{-2t} + C t e^{-2t}$$

③ Homogeneous Question - No

Proceed to step ④

Steady State and Transients

$$\ddot{y} + 0.2\dot{y} + y = \cos 2t$$

1. $$r^2 + 0.2r + 1 = 0$$

$$(r+0.1)^2 = -0.99$$

$$r = -0.1 \pm i\sqrt{0.99}$$

$$y_H = c_1 e^{-0.1t}\cos(0.995t) + c_2 e^{-0.1t}\sin(0.995t)$$

2. $$y_p = A\cos 2t + B\sin 2t$$

$$\dot{y}_p = -2A\sin 2t + 2B\cos 2t$$

$$\ddot{y}_p = -4A\cos 2t - 4B\sin 2t$$

$$\ddot{y}_p + 0.2\dot{y}_p + y_p = (-3A + 0.4B)\cos 2t + (-3B - 0.4A)\sin 2t$$

$$\cos 2t =$$

$$\ddot{y}_p + 0.2\,\dot{y}_p + y_p = (-3A + 0.4\,B)\cos 2t$$
$$+ (-3B - 0.4\,A)\sin 2t$$

$$\cos 2t = (-3A + 0.4\,B)\cos 2t + (-3B - 0.4\,A)\sin 2t$$

$$\left.\begin{aligned} 1 &= -3A + 0.4B \\ 0 &= -0.4A - 3B \end{aligned}\right\} \quad \begin{aligned} A &= -0.375 \\ B &= 0.0437 \end{aligned}$$

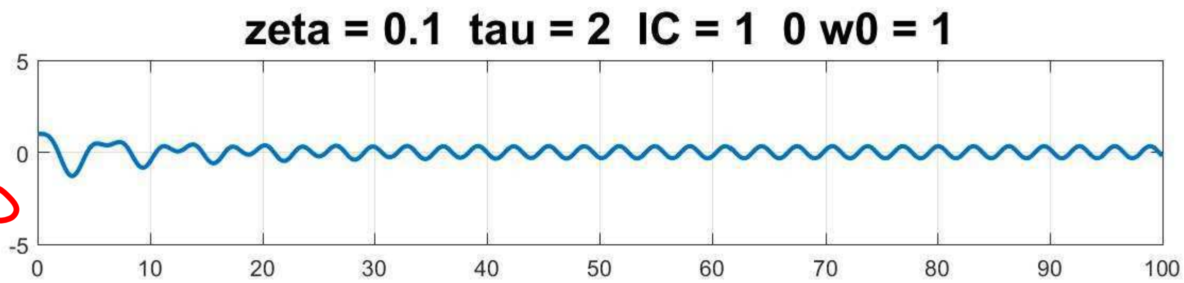
$$\ddot{y} + 0.2\dot{y} + y = \cos 2t$$

$$y = -0.375\cos 2t + 0.0437\sin 2t$$

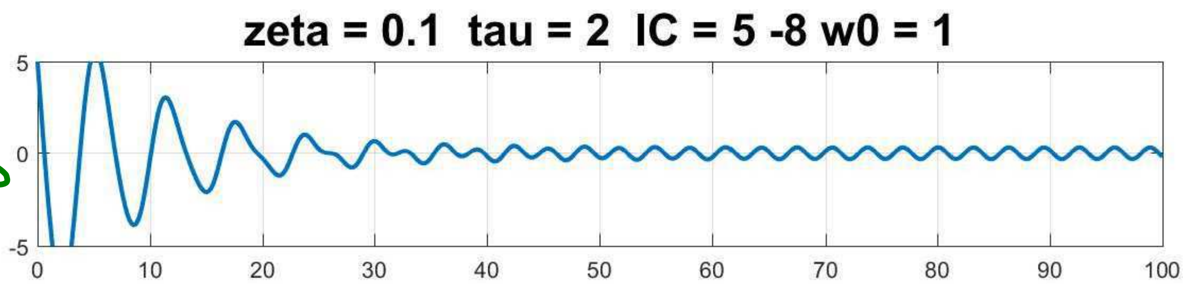
$$+ \; c_1 e^{-0.1t}\cos(0.995t) + c_2 e^{-0.1t}\sin(0.995t)$$

Initial Conditions

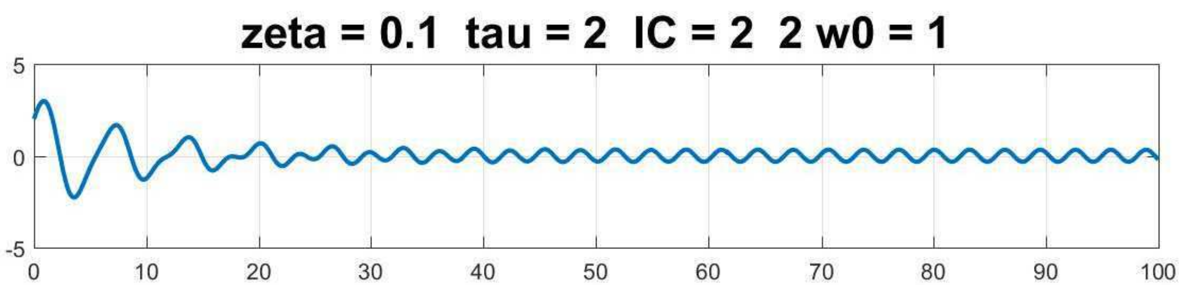
$y(0) = 1$
$\dot{y}(0) = 0$



$y(0) = 5$
$\dot{y}(0) = -8$



$y(0) = 2$
$\dot{y}(0) = 2$



All solutions are essentially the same for $t > 50$. Why?

All solutions are essentially the same for $t > 50$. Why?

We want to compare the sizes of y_p and y_H so we write them in amplitude-phase form.

$$y_p = -0.375\cos 2t + 0.0437\sin 2t$$

$$= 0.3775\cos(2t - 0.116)$$

$$y_H = c_1 e^{-0.1t}\cos(0.995t) + c_2 e^{-0.1t}\sin(0.995t)$$

$$= e^{-0.1t}(c_1^2+c_2^2)^{1/2}\cos(0.995t + \operatorname{atan2}(c_2,c_1))$$

$$e^{-0.1\cdot 50} = e^{-5} = 0.067$$

For $t \geq 50$ the amplitude of y_p is 0.3775, while the amplitude of y_H is less than 0.067 times the initial amplitude.

$$y_P = 0.3775\cos(2t - 0.116)$$

constant (steady) amplitude 0.3775

$$y_H = \underbrace{e^{-0.1t}(c_1^2+c_2^2)^{1/2}}_{\text{decaying amplitude}}\cos(0.995t + \operatorname{atan2}(c_2,c_1))$$

Something which decays with time is called transient. If you wait long enough, its gone.

When the homogeneous solution decays, we call y_p the steady state solution, and y_H the transient.

$$y = 0.3775\cos(2t - 0.116) + e^{-0.1t}(c_1^2+c_2^2)^{1/2}\cos(0.995t + \operatorname{atan2}(c_2,c_1))$$

$$y = y_{ss} + y_{tr}$$

$$y = 0.3775\cos(2t - 0.116)$$

$$+ \; e^{-0.1t}(c_1^2+c_2^2)^{1/2}\cos(0.995t + \operatorname{atan2}(c_2, c_1))$$

$$y = y_{ss} + y_{tr}$$

The steady state does not depend on the initial conditions. It only depends on the forcing term.

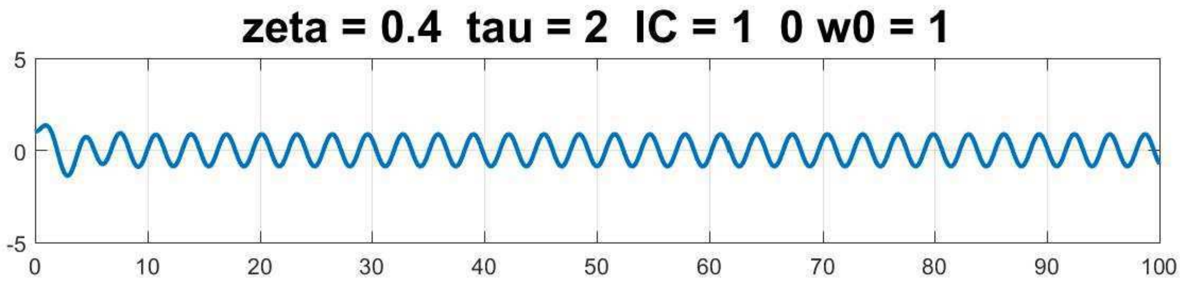
Only the transient part depends on the initial conditions.

The more damping, the faster the transient part decays

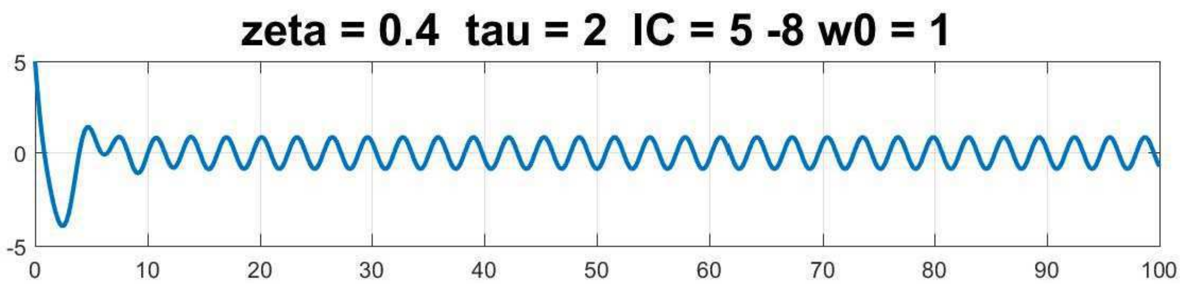
An Example with more damping

$$\ddot{y} + 0.8\,\dot{y} + y = 3\cos 2t$$

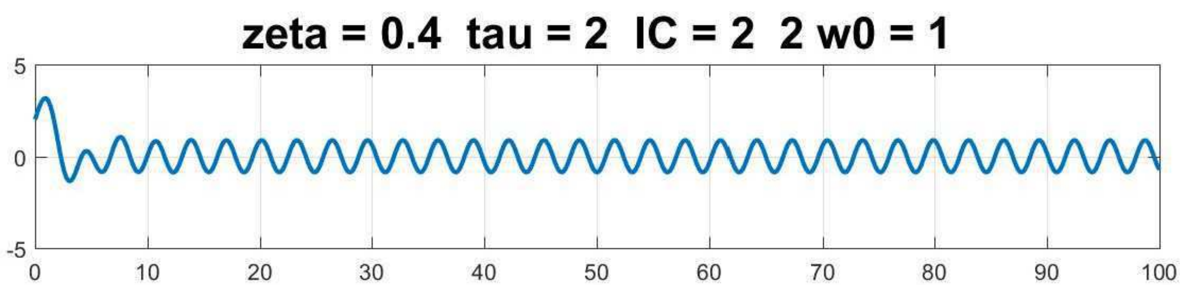
$y(0) = 1$
$\dot{y}(0) = 0$



$y(0) = 5$
$\dot{y}(0) = -8$



$y(0) = 2$
$\dot{y}(0) = 2$



$$y_H = y_{tr} = e^{-0.4t}\, A\cos(0.6t - \phi_1)$$

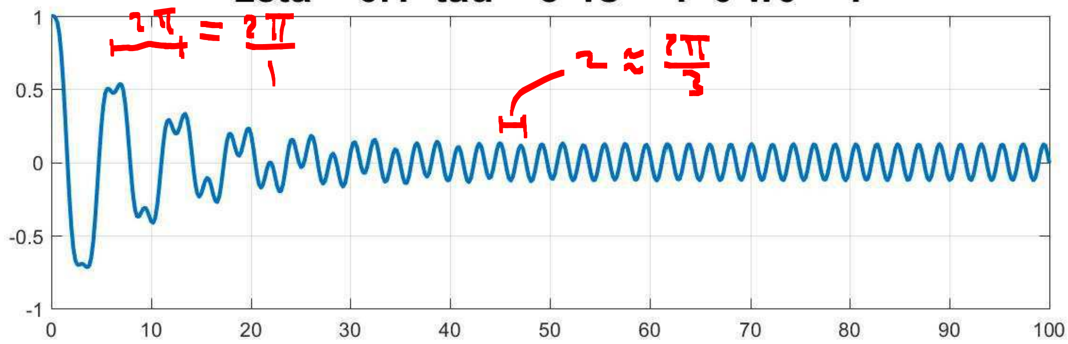
These depend on IC's

$$y_p = y_{ss} = 0.8824\cos(2t - 2.65)$$

$$\ddot{y} + 0.1\dot{y} + y = \cos 3t$$

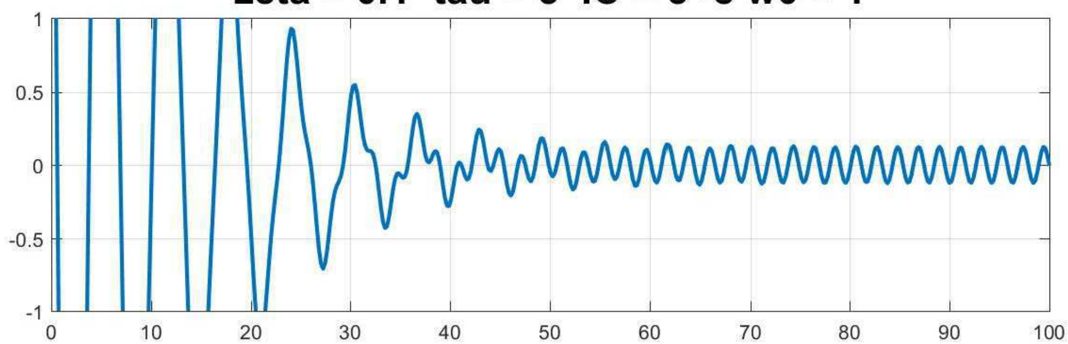
$y(0)=1 \qquad \dot{y}(0)=0$

zeta = 0.1 tau = 3 IC = 1 0 w0 = 1



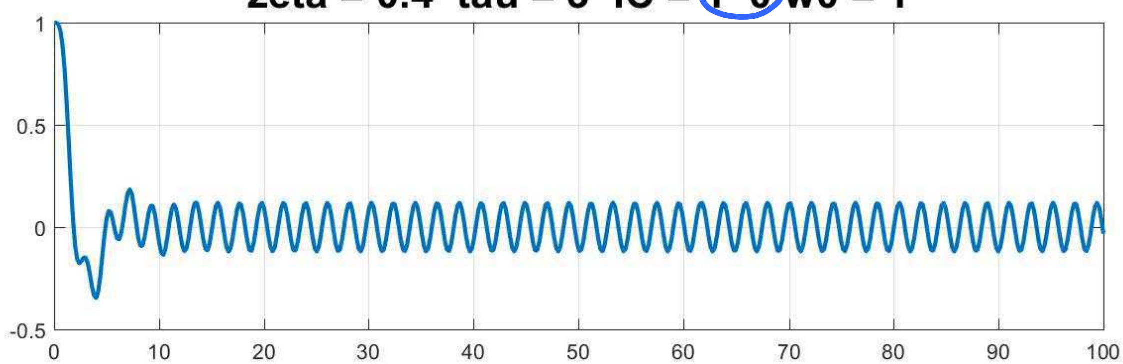
$y(0)=5 \qquad \dot{y}(0)=-8$

zeta = 0.1 tau = 3 IC = 5 -8 w0 = 1

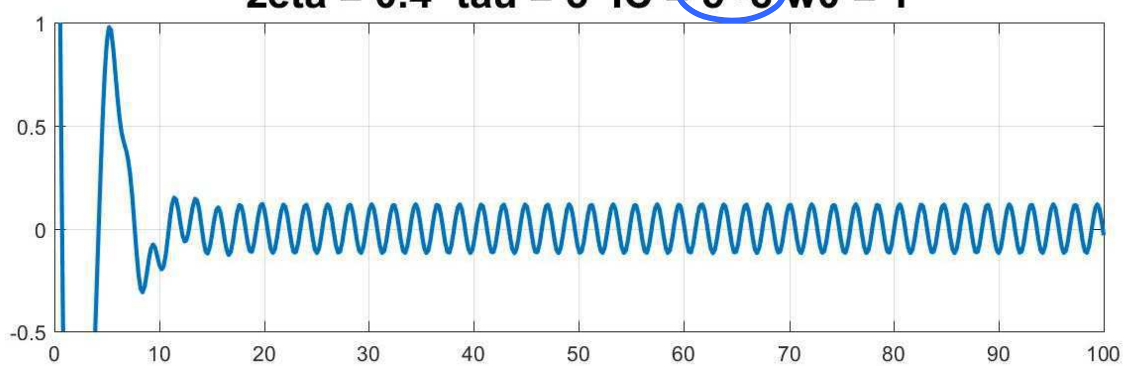


$$\ddot{y} + 0.4\dot{y} + y = \cos 3t$$

zeta = 0.4 tau = 3 IC = 1 0 w0 = 1 IC's



zeta = 0.4 tau = 3 IC = 5 -8 w0 = 1 IC's



Note Title 2/12/2020

Two Formulas from Last Time

$$\ddot{y} + y = \cos \omega t \qquad y(0) = 0 \qquad \dot{y}(0) = 0$$

$$\omega \neq 1 \quad y(t) = \frac{\cos \omega t - \cos t}{1 - \omega^2}$$

$$\omega = 1 \quad y(t) = \frac{t \sin t}{2}$$

Slightly more general

$$\ddot{y} + \omega_0^2 y = \cos \omega t \qquad y(0) = 0 \qquad \dot{y}(0) = 0$$

$$\omega \neq \omega_0 \qquad y(t) = \frac{\cos \omega t - \cos \omega_0 t}{\omega_0^2 - \omega^2}$$

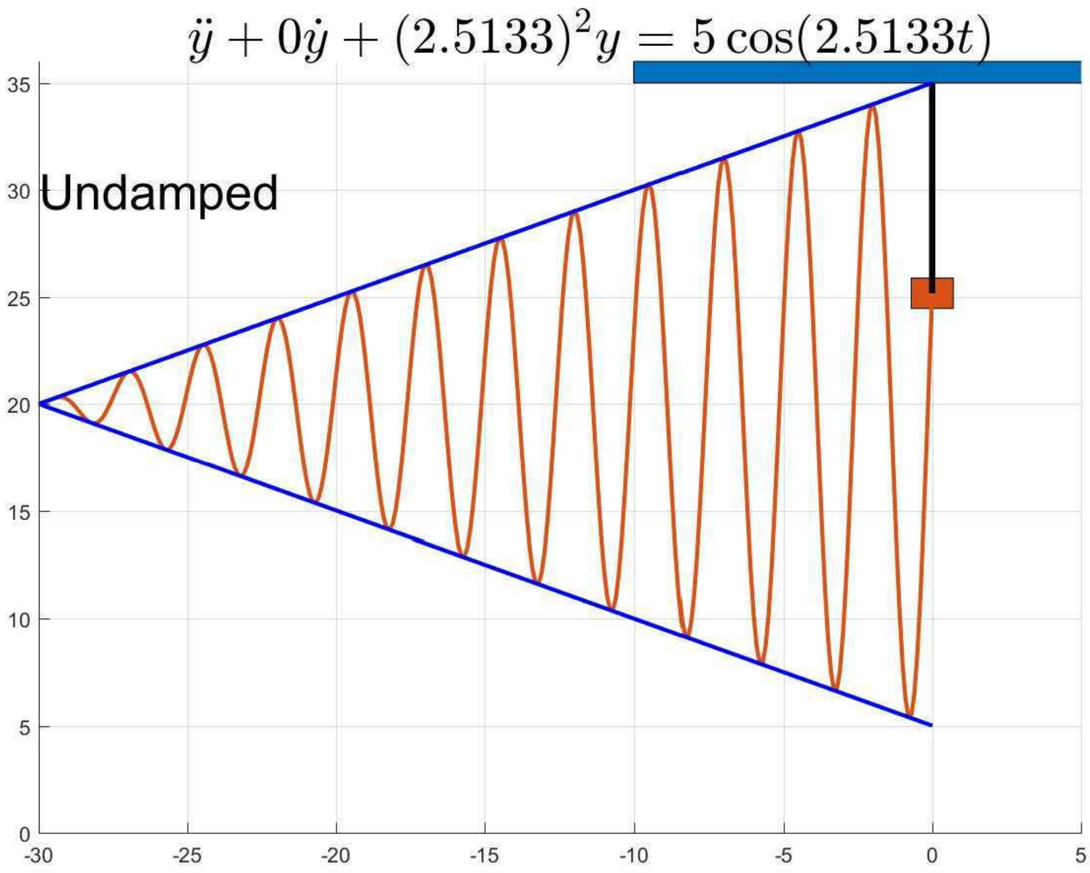
$$\omega = \omega_0 \qquad y(t) = \frac{t \sin \omega_0 t}{2 \omega_0}$$

ω_0 = Natural Frequency

ω = Forcing frequency

What do they look like?

$\omega_0 = \omega$ Resonance

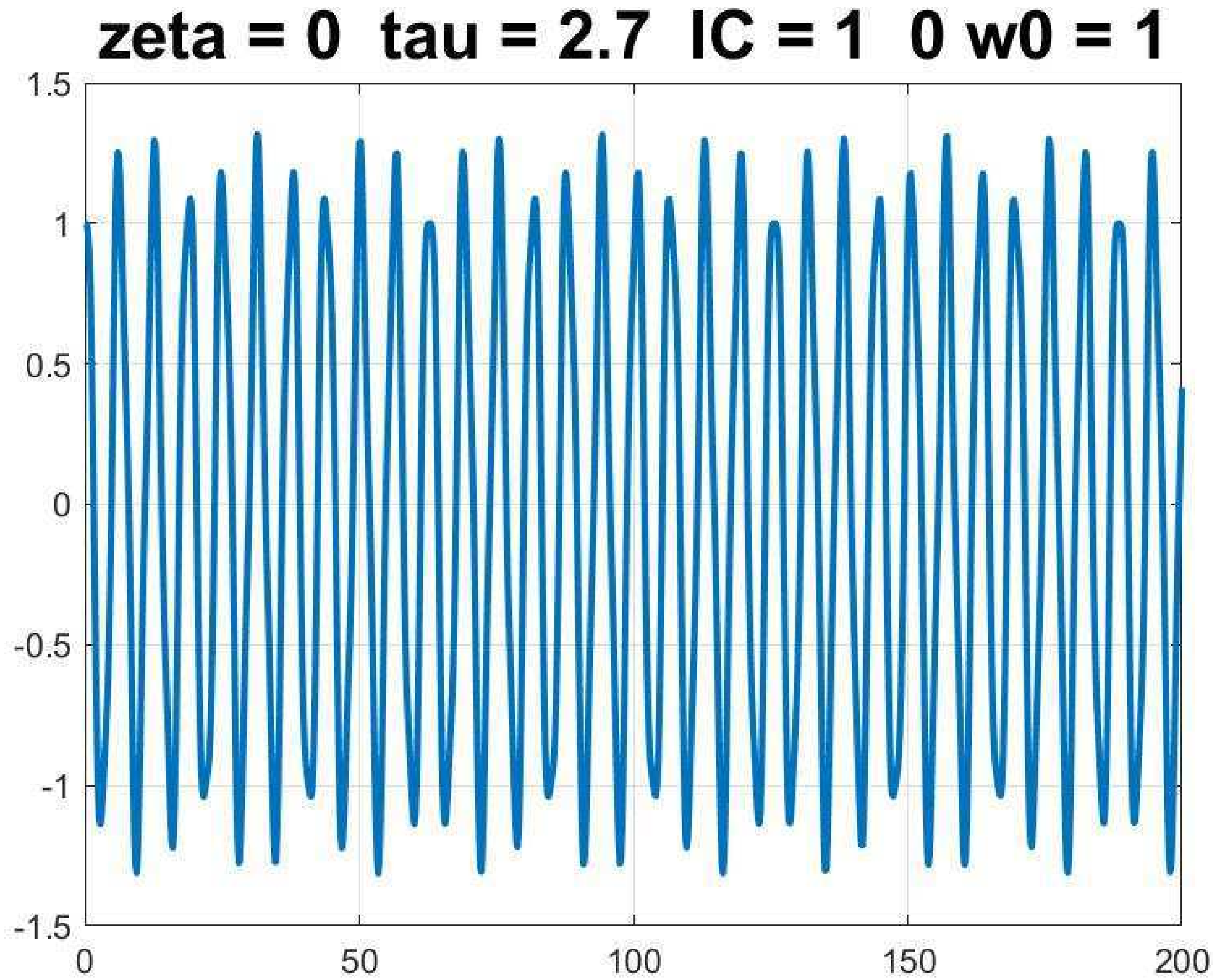


$$y(t) = \frac{t\sin(2.5133t)}{2 \cdot 2.5133}$$

Resonance - displacement $y(t)$ grows larger and larger - this can lead to disaster

ω_0 far from ω - Nothing Special

$$y(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}$$

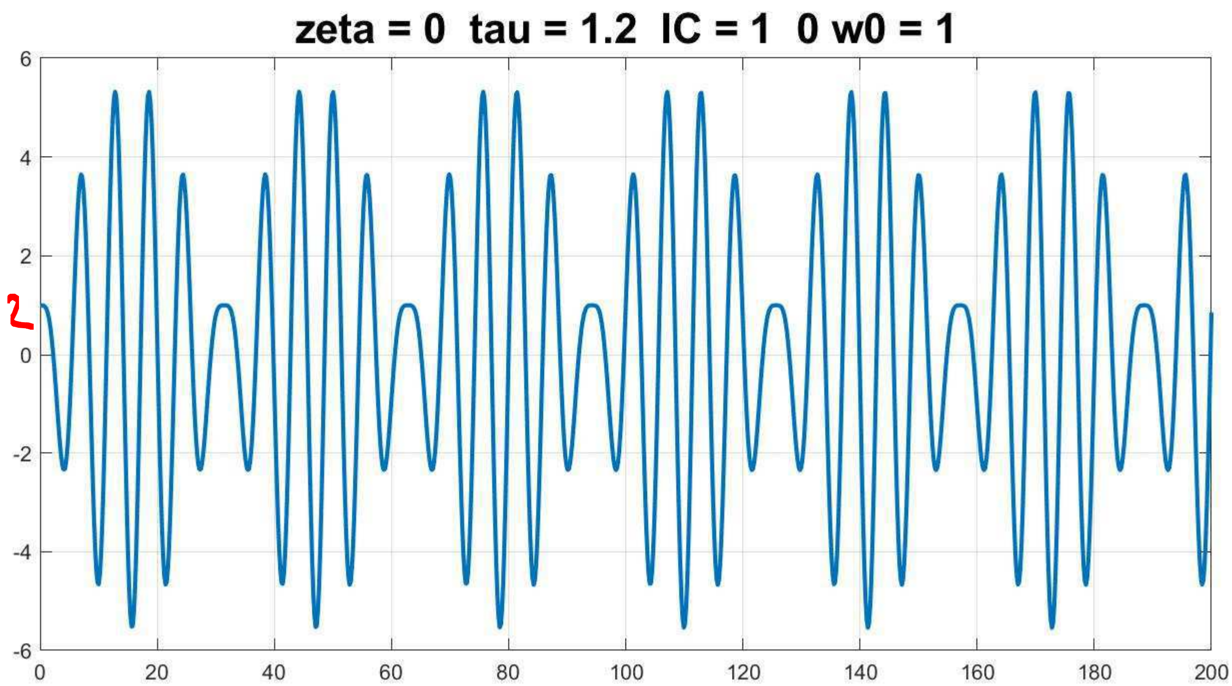


ω_0 close to ω - Beats

$$y(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}$$

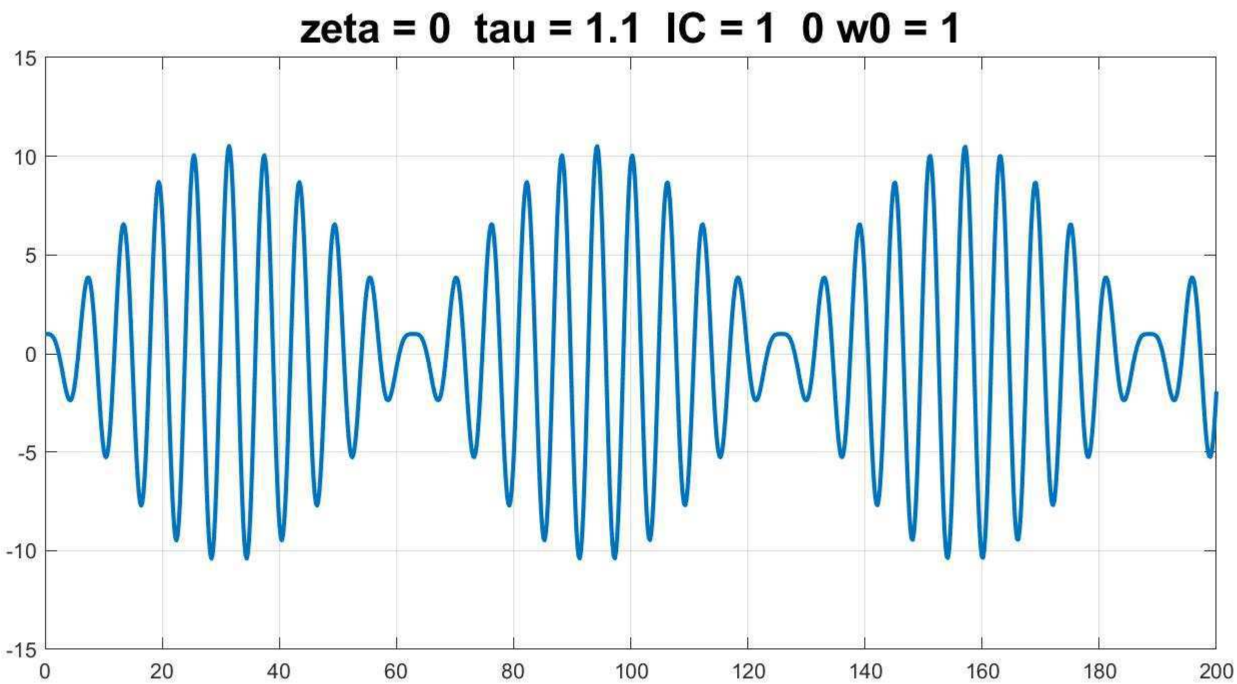
$\omega_0 = 1$

$\omega = 1.2$



$\omega_0 = 1$

$\omega = 1.1$

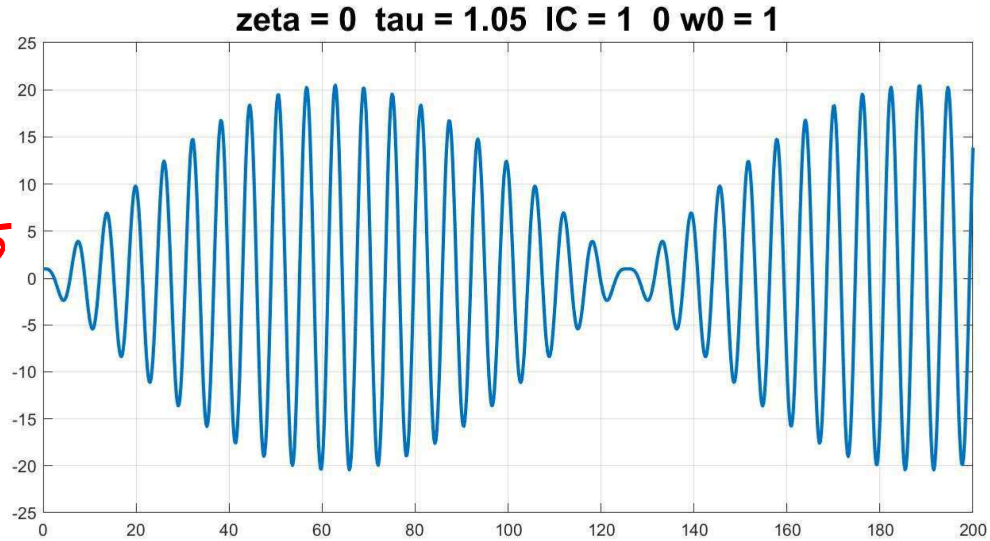


ω_0 close to ω - Beats

$$y(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}$$

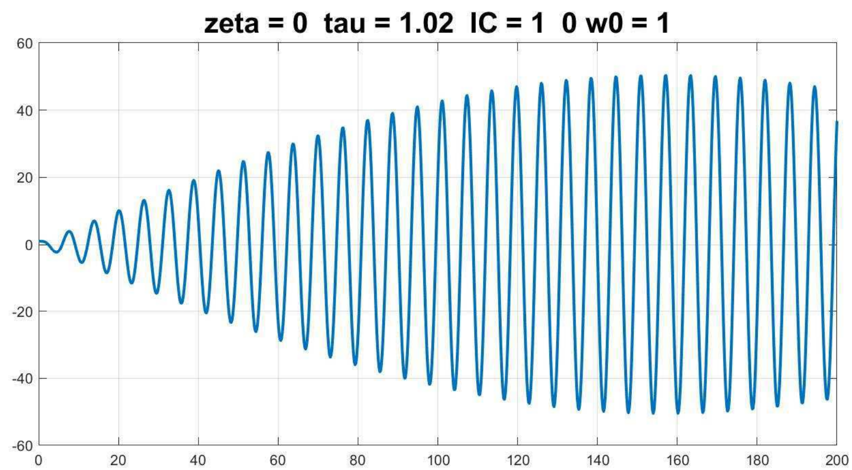
$\omega_0 = 1$

$\omega = 1.05$



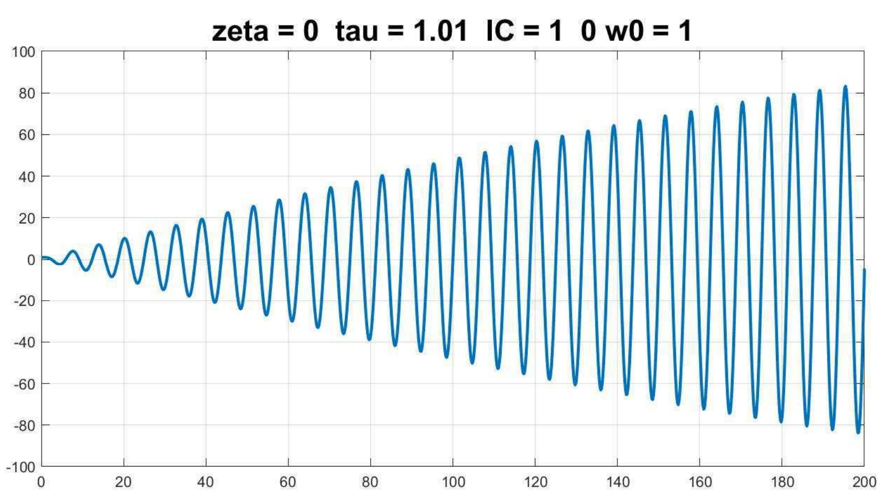
$\omega_0 = 1$

$\omega = 1.02$



$\omega_0 = 1$

$\omega = 1.01$

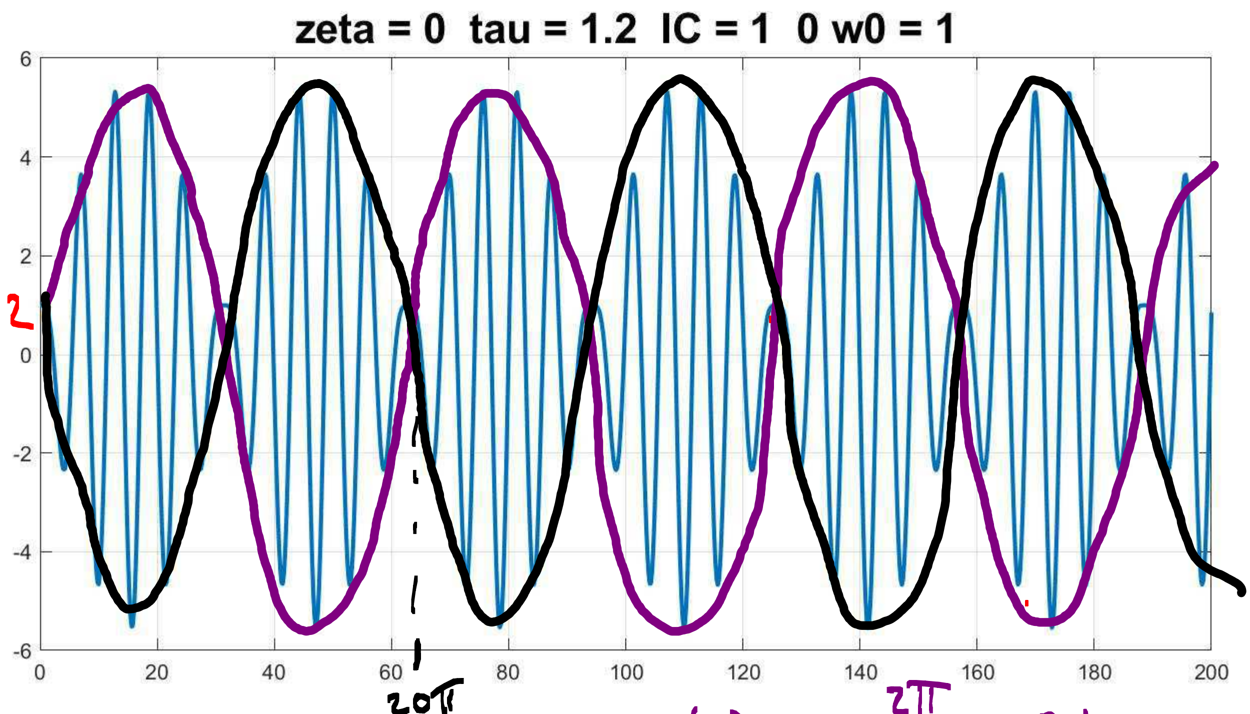


ω_0 close to ω - Beats

$$y(t) = \frac{\cos \omega t - \cos \omega_0 t}{\omega_0^2 - \omega^2}$$

$\omega_0 = 1$

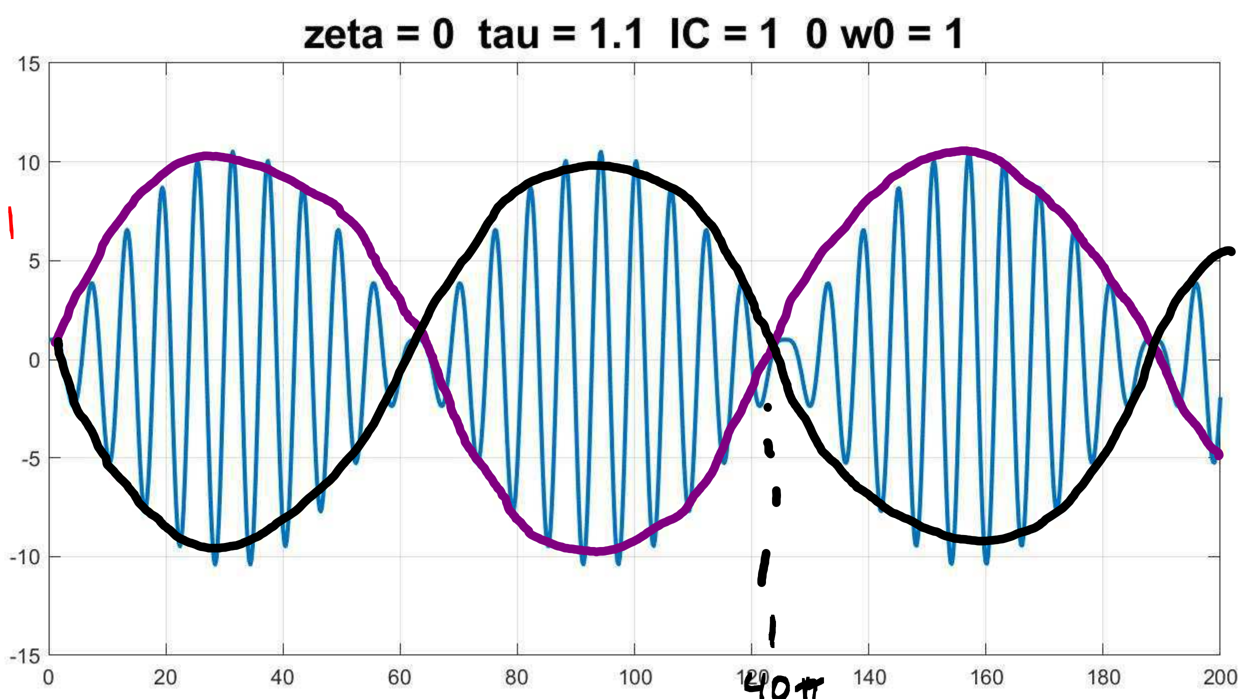
$\omega = 1.2$



Beat period $= 20\pi$ $\qquad \omega_{Beat} = \frac{2\pi}{20\pi} = 0.1$

$\omega_0 = 1$

$\omega = 1.1$



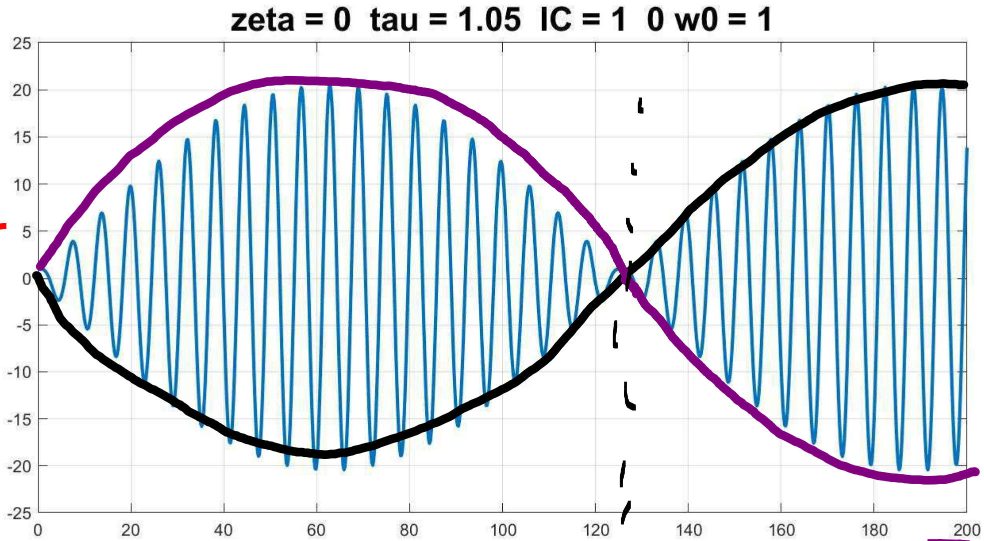
Beat period $= 40\pi$ $\qquad \omega_{Beat} = \frac{2\pi}{40\pi} = 0.05$

ω_0 close to ω - Beats

$$y(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}$$

$\omega_0 = 1$

$\omega = 1.05$

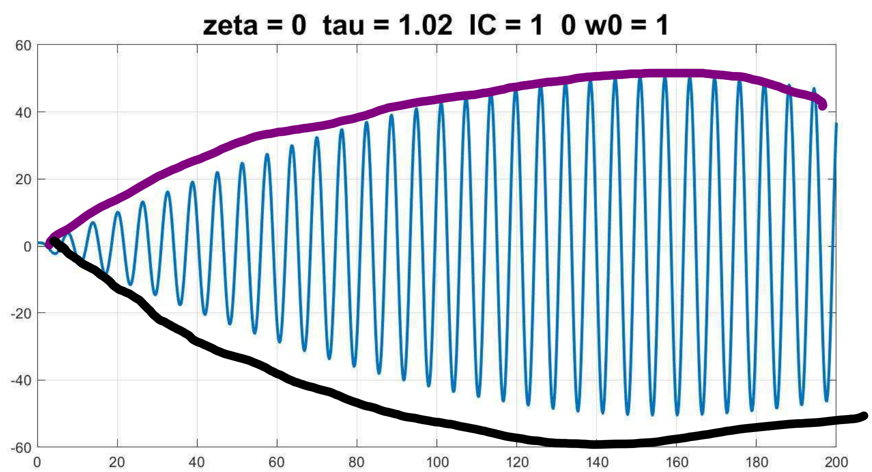


40π

Beat period $= 80\pi$ $\quad \omega_{Beat} = \frac{2\pi}{80\pi} = 0.025$

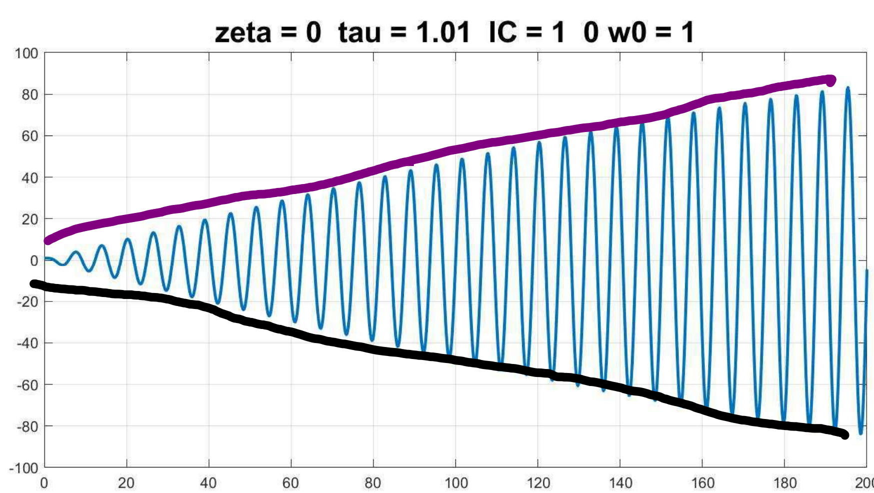
$\omega_0 = 1$

$\omega = 1.02$



$\omega_0 = 1$

$\omega = 1.01$



Observation

$$\omega_{Beat} = \frac{\omega_0 - \omega}{2}$$

We can't easily see this from the formula:

$$y(t) = \frac{\cos \omega t - \cos \omega_0 t}{\omega_0^2 - \omega^2}$$

but this formula can be rewritten in a way that makes the beats phenemonon visible

$$y(t) = \frac{-2 \sin\left(\frac{\omega_0 - \omega}{2} t\right)}{\omega_0 - \omega} \cdot \frac{\sin\left(\frac{\omega_0 + \omega}{2} t\right)}{\omega_0 + \omega}$$

Product of Sines with

$\frac{\omega_0 + \omega}{2}$ average frequency $\quad \frac{\omega_0 - \omega}{2}$ half difference

$$\omega = \frac{\omega+\omega_0}{2} + \frac{\omega-\omega_0}{2}$$

$$\omega_0 = \frac{\omega+\omega_0}{2} - \frac{\omega-\omega_0}{2}$$

$\frac{\omega+\omega_0}{2}$	Average
$\frac{\omega-\omega_0}{2}$	Half Difference

Sum of angles formula for cosine

$$\cos\omega t = \cos\left(\frac{\omega+\omega_0}{2}t + \frac{\omega-\omega_0}{2}t\right) =$$

$$\cos\left(\frac{\omega+\omega_0}{2}t\right)\cos\left(\frac{\omega-\omega_0}{2}t\right) - \sin\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right) \quad (1)$$

$$\cos\omega_0 t = \cos\left(\frac{\omega+\omega_0}{2}t - \frac{\omega-\omega_0}{2}t\right) =$$

$$\cos\left(\frac{\omega+\omega_0}{2}t\right)\cos\left(\frac{\omega-\omega_0}{2}t\right) + \sin\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right) \quad (2)$$

<u>subtract</u> (2) from (1)

$$\cos\omega t - \cos\omega_0 t = -2\sin\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right)$$

$$y(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}$$

$$= \frac{-2\sin\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right)}{(\omega_0-\omega)(\omega_0+\omega)}$$

$$y(t) = \frac{-2\sin\left(\frac{\omega-\omega_0}{2}t\right)}{\omega-\omega_0} \cdot \frac{\sin\left(\frac{\omega+\omega_0}{2}t\right)}{\omega+\omega_0}$$

Example

$$\ddot{y} + 4^2 y = \cos 5t$$

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

Solution (after some calculations)

$$y = \frac{\cos 4t - \cos 5t}{9}$$

We write y as a product to visualize the "beats".

$$\omega_{AV} = \frac{4+5}{2} \qquad \omega_{HD} = \frac{5-4}{2}$$

[use positive difference for convenience]

$$\cos 4t = \cos\left(\tfrac{9}{2}t - \tfrac{1}{2}t\right) = \cos\left(\tfrac{9}{2}t\right)\cos\left(\tfrac{t}{2}\right) + \sin\left(\tfrac{9}{2}t\right)\sin\tfrac{t}{2}$$

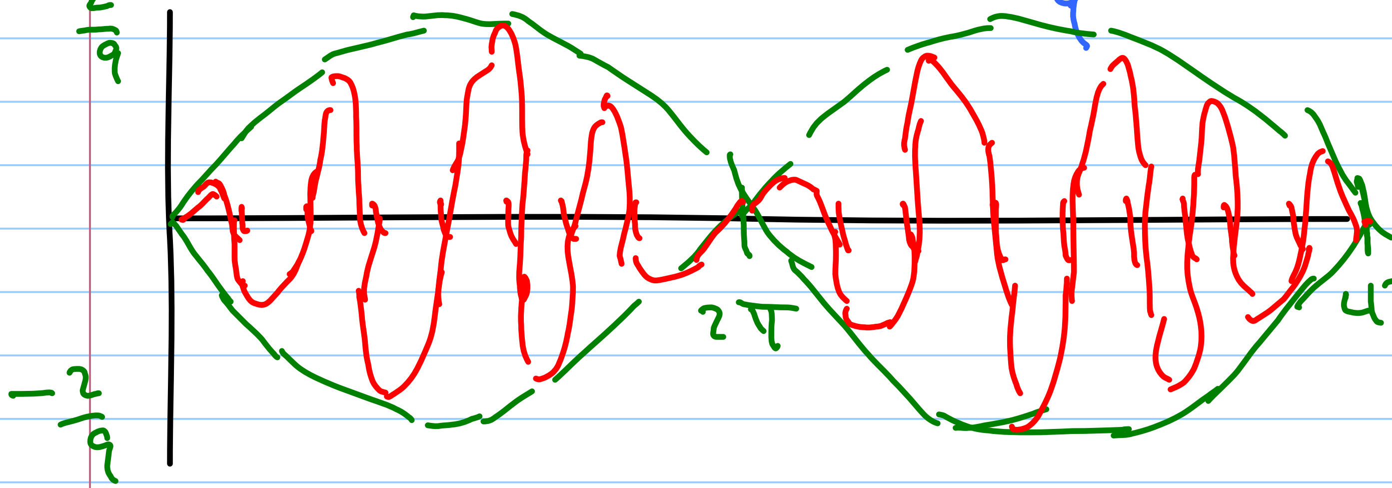
$$\cos 5t = \cos\left(\tfrac{9}{2}t + \tfrac{1}{2}t\right) = \cos\left(\tfrac{9}{2}t\right)\cos\left(\tfrac{t}{2}\right) - \sin\left(\tfrac{9}{2}t\right)\sin\tfrac{t}{2}$$

$$\cos 4t - \cos 5t = -2\sin\tfrac{9}{2}t\,\sin\tfrac{t}{2}$$

$$y(t) = \frac{-2\sin\frac{9}{2}t\,\sin\frac{t}{2}}{9}$$

Sketch

$$y(t) = \frac{2\sin\frac{9}{2}t\,\sin\frac{t}{2}}{9}$$



$\sin\frac{t}{2}$ $\quad P = \frac{2\pi}{\frac{1}{2}} = 4\pi \approx 12$

$\sin\frac{9}{2}t$ period $= \frac{2\pi}{9/2} = \frac{4\pi}{9} \approx \frac{12}{9}$

oscillates 9 times faster than $\sin\frac{t}{2}$

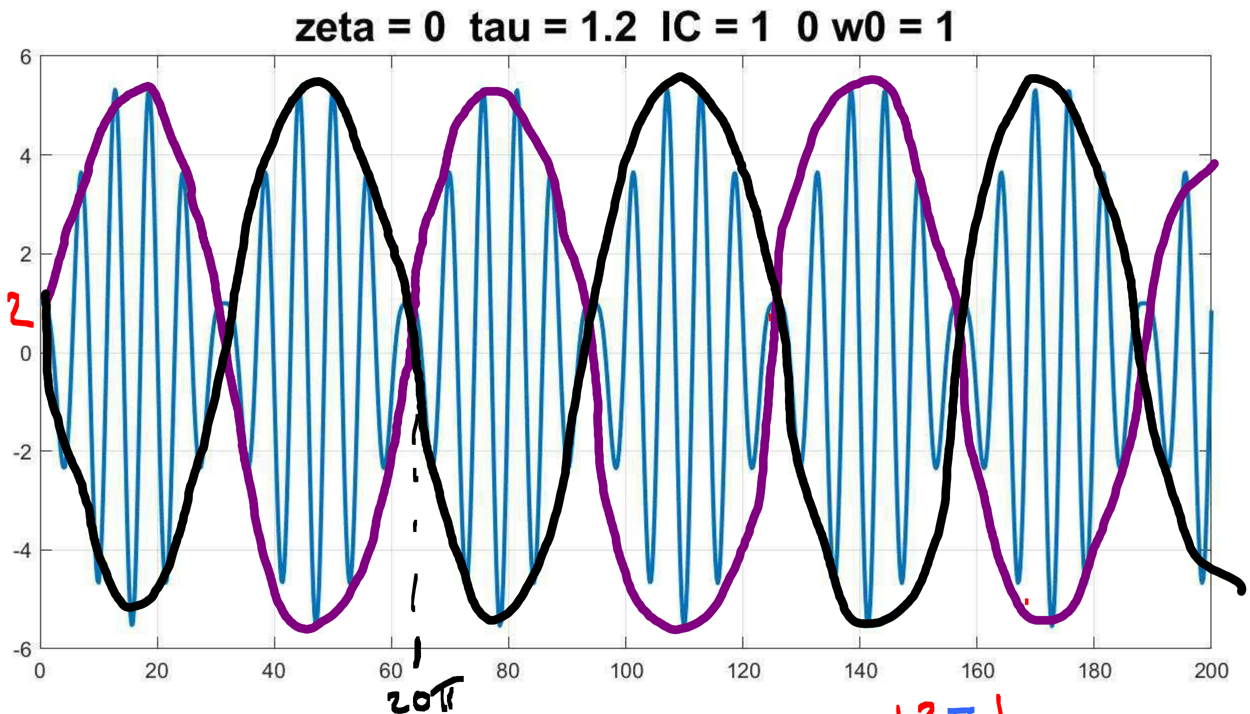
$\sin\frac{9}{2}t$ has 9 zeroes for every zero of $\sin\frac{t}{2}$

ω_0 close to ω - Beats

$$y(t) = \frac{-2\sin\left(\frac{\omega_0-\omega}{2}t\right)}{\omega_0-\omega}\cdot\frac{\sin\left(\frac{\omega_0+\omega}{2}t\right)}{\omega_0+\omega}$$

$\omega_0 = 1$

$\omega = 1.2$

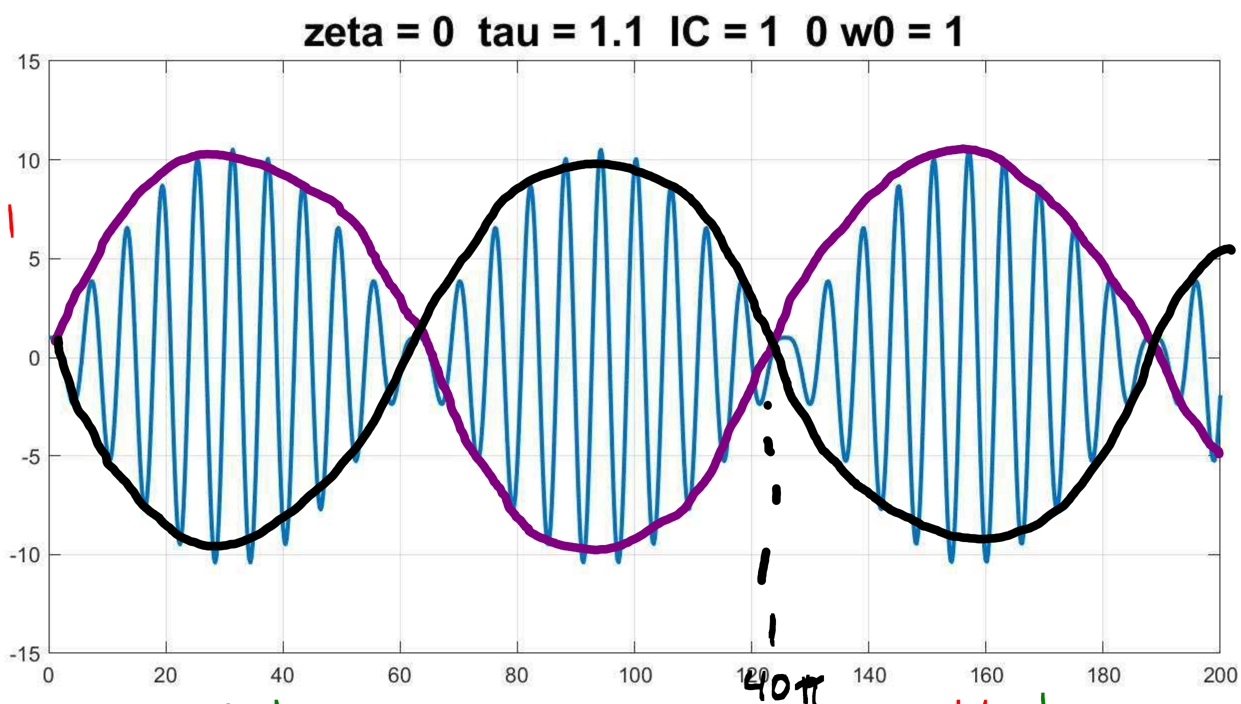


$\omega_{AV} = \frac{1+1.2}{2} = 1.1$

$\omega_{Beat} = \frac{1.2-1}{2} = 0.1$

$\omega_0 = 1$

$\omega = 1.1$



$\omega_{AV} = \frac{1.1+1}{2} = 1.05$

$\omega_{Beat} = \frac{1.1-1}{2} = .05$

Lecture 17 Damped Forced

Note Title 11/2/2017

Damped Force Oscillations (Filters)

Example $\ddot{y} + \rho\dot{y} + y = \cos(\omega t)$

① Find the steady state solution

② Find the amplitude and phase as functions of ρ and ω

③ At what frequency ω is the amplitude largest?

Answer

Seek $y = A\cos\omega t + B\sin\omega t$

$$\rho\dot{y} = -\rho A\omega\sin\omega t + \rho B\omega\cos\omega t$$

$$+\quad \ddot{y} = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

$$\ddot{y} + \rho\dot{y} + y = \left[(1-\omega^2)A + \rho\omega B\right]\cos\omega t + \left[(1-\omega^2)B - \rho\omega A\right]\sin\omega t$$

$\cos\omega t =$

so

$$\left[(1-\omega^2)A + \rho\omega B\right] = 1$$

$$\left[(1-\omega^2)B - \rho\omega A\right] = 0$$

$$(1-\omega^2)A + \rho\omega B = 1$$

$$(1-\omega^2)B - \rho\omega A = 0 \implies A = \frac{(1-\omega^2)}{\rho\omega} B$$

$$\frac{(1-\omega^2)^2}{\rho\omega} B + \rho\omega B = 1 \implies \left[(1-\omega^2)^2 + (\rho\omega)^2\right] B = \rho\omega$$

$$B = \frac{\rho\omega}{(1-\omega^2)^2 + (\rho\omega)^2} \qquad A = \frac{(1-\omega^2)}{(1-\omega^2)^2 + (\rho\omega)^2}$$

$$y_{ss} = \frac{(1-\omega^2)}{(1-\omega^2)^2 + (\rho\omega)^2} \cos\omega t + \frac{\rho\omega}{(1-\omega^2)^2 + (\rho\omega)^2} \sin\omega t$$

We want to know how big the steady state solution will become, so we write y_{ss} in amplitude phase form

$$y_{ss} = A\cos(\omega t - \varphi)$$

Amplitude Phase Form

$$A\cos(\omega t-\varphi) = A\cos\varphi\cos\omega t + A\sin\varphi\sin\omega t$$

$$y_{ss} = \frac{(1-\omega^2)}{(1-\omega^2)^2+(\zeta\omega)^2}\cos\omega t + \frac{\zeta\omega}{(1-\omega^2)^2+(\zeta\omega)^2}\sin\omega t$$

$$A\cos\varphi = \frac{(1-\omega^2)}{(1-\omega^2)^2+(\zeta\omega)^2} \qquad A\sin\varphi = \frac{\zeta\omega}{(1-\omega^2)^2+(\zeta\omega)^2}$$

$$A^2 = \frac{(1-\omega^2)^2+(\zeta\omega)^2}{\left[(1-\omega^2)^2+(\zeta\omega)^2\right]^2}$$

$$A = \frac{1}{\left[(1-\omega^2)^2+(\zeta\omega)^2\right]^{1/2}} \qquad \tan\varphi = \frac{\zeta\omega}{(1-\omega^2)}$$

$$\boxed{y_{ss} = \frac{\cos\left(\omega t - \operatorname{atan}\left(\frac{\zeta\omega}{1-\omega^2}\right)\right)}{\left[(1-\omega^2)^2+(\zeta\omega)^2\right]^{1/2}}}$$

$$\boxed{y_{ss} = \frac{\cos\left(\omega t - \operatorname{atan}\left(\frac{\zeta\omega}{1-\omega^2}\right)\right)}{\left[(1-\omega^2)^2 + (\zeta\omega)^2\right]^{1/2}}}$$

This formula answers many important questions.

$$A(\omega, \zeta) = \frac{1}{\left[(1-\omega^2)^2 + (\zeta\omega)^2\right]^{1/2}}$$

The amplitude tells us how much the forcing amplitude is magnified or damped.

<u>Example</u> $\zeta = 0.1$

$$\ddot{y} + 0.1\,\dot{y} + y = \cos(1.1t) + \cos(3t)$$

Because the equation is linear, y_{ss} can be written as a sum of two solutions:

$$\ddot{y}_1 + 0.1\,\dot{y}_1 + y_1 = \cos(1.1t)$$

$$\ddot{y}_3 + 0.1\,\dot{y}_3 + y_3 = \cos(3t)$$

$$y_1 = \frac{1}{\left[(1-1.1^2)^2 + (0.11)^2\right]^{1/2}} \cos\left(1.1t + \operatorname{atan}\left(\frac{0.1}{1-1.1^2}\right)\right)$$

$$y_2 = \frac{1}{\left[(1-3^2)^2 + (0.33)^2\right]^{1/2}} \cos\left(3t + \operatorname{atan}\left(\frac{0.1}{1-3^2}\right)\right)$$

$$y_1 = 4.2\cos(1.1t + 2.7)$$

$$y_2 = 0.125\cos(3t + 3.10)$$

$$\frac{4.2}{0.125} = 33.7$$

The response to $\cos(1.1t)$ is 34 times bigger than the response to $\cos(3t)$.

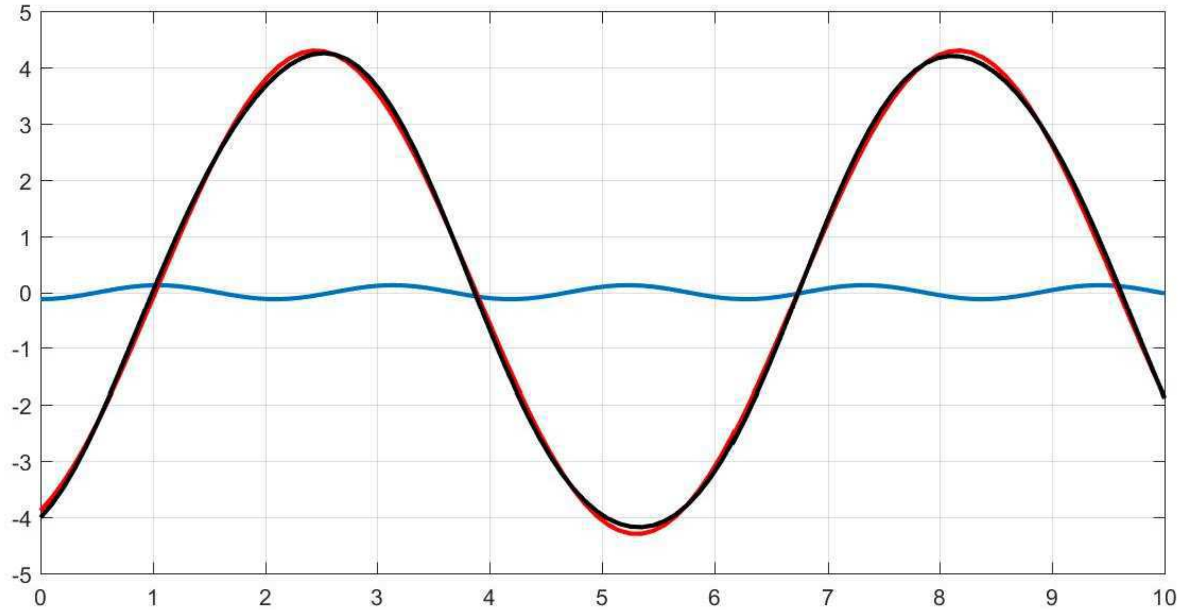
Summary

$$\ddot{y} + 0.1\,\dot{y} + y = \cos(1.1t) + \cos(3t)$$

$$y_{ss} = 4.2\cos(1.1t + 2.7) + 0.125\cos(3t + 3.10)$$

$4.2\cos(1.1t+2.7)$ $\quad$ $4.2\cos(1.1t+2.7) + 0.125\cos(3t+3.10)$

$0.125\cos(3t+3.10)$



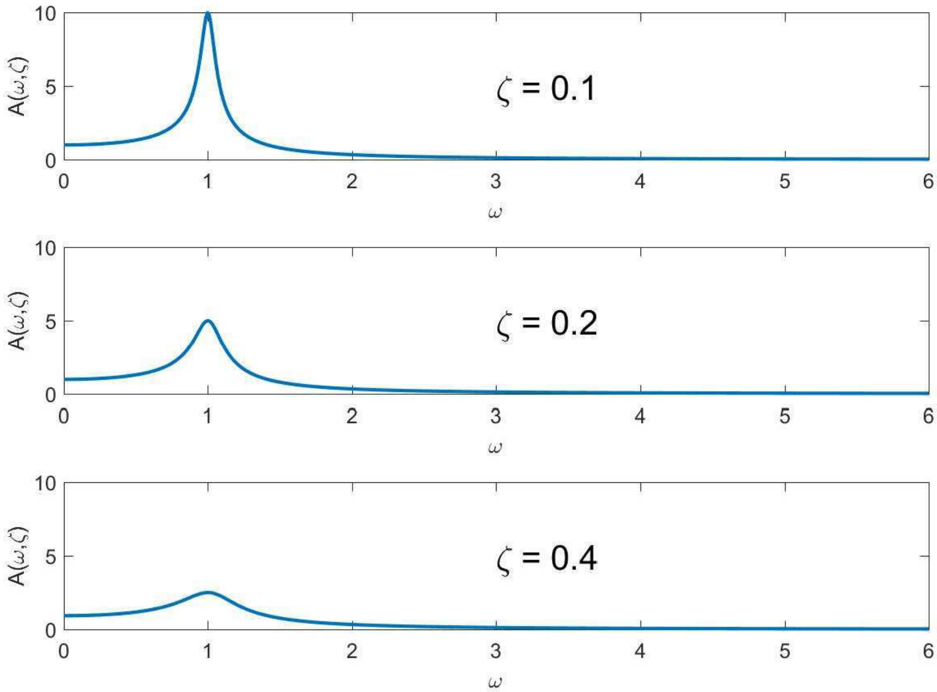
The steady state solution magnifies frequencies near $\omega=1$ and suppresses frequencies far from $\omega=1$. It "filters out" frequencies away from $\omega=1$.

$$\ddot{y} + \zeta \dot{y} + y = \cos \omega t$$

$$y_{ss} = A(\omega, \zeta) \cos(\omega t - \varphi(\omega, \zeta))$$

$$A(\omega, \zeta) = \frac{1}{\left[(1-\omega^2)^2 + (\zeta\omega)^2\right]^{1/2}}$$

The smaller the value of ζ, the higher and narrower the peak.



If we replace the DE with

$$\ddot{y} + \omega_0 \zeta \dot{y} + \omega_0^2 y = \cos \omega t$$

then the frequency peak moves to ω_0

③ At what frequency ω is the amplitude largest?

$$A(\omega,\varsigma) = \frac{1}{\left[(1-\omega^2)^2+(\varsigma\omega)^2\right]^{1/2}}$$

A is largest when $(1-\omega^2)^2+(\varsigma\omega)^2$ is smallest

$$0 = \frac{d}{d\omega}(1-\omega^2)^2+(\varsigma\omega)^2 = -\cancel{2}\cancel{\omega}2(1-\omega^2) + \cancel{2}\varsigma^2\cancel{\omega}$$

$$0 = 2-2\omega^2+\varsigma^2$$

$$\omega_{max} = \sqrt{1-\frac{\varsigma^2}{2}}$$

④ What is the maximum amplitude?

$$A(\omega_{max},\varsigma) = \frac{1}{\sqrt{\left(1-\left(1-\frac{\varsigma^2}{2}\right)\right)^2+\varsigma^2\left(1-\frac{\varsigma^2}{2}\right)}}$$

$$= \frac{1}{\sqrt{\frac{\varsigma^4}{4}+\varsigma^2-\frac{\varsigma^4}{2}}}$$

$$= \frac{1}{\varsigma\sqrt{1-\frac{\varsigma^2}{4}}}$$

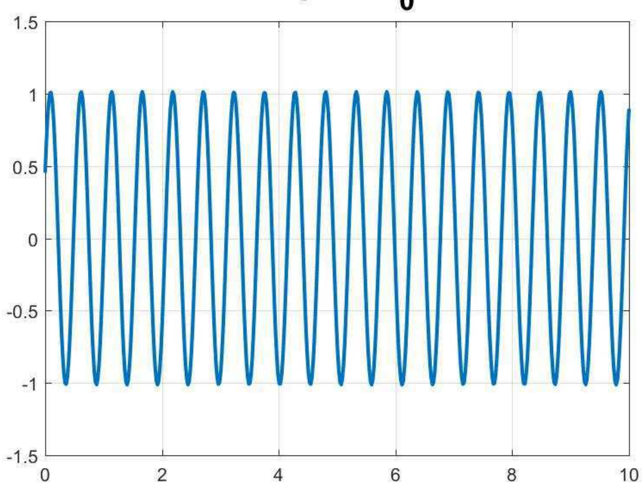
Note Title 11/8/2017

Summary -

Second Order Constant Coefficient ODE
Harmonic Motion

Unforced Motion

Undamped ω_0 = 12



Undamped

$\ddot{y} + \frac{k}{m} y = 0$ mass-spring

$\ddot{y} + \omega_0^2 y = 0$ harmonic oscillator

$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$

$= A \cos(\omega_0 t - \varphi)$

Damped

$\ddot{y} + 2\zeta\omega_0 \dot{y} + \omega_0^2 y = 0$ harmonic oscillator

$\ddot{y} + \frac{\gamma}{m}\dot{y} + \frac{k}{m} y = 0$ mass-spring

Overdamped

$y = c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$

$r^2 + \frac{\gamma}{m} r + \frac{k}{m} = 0$

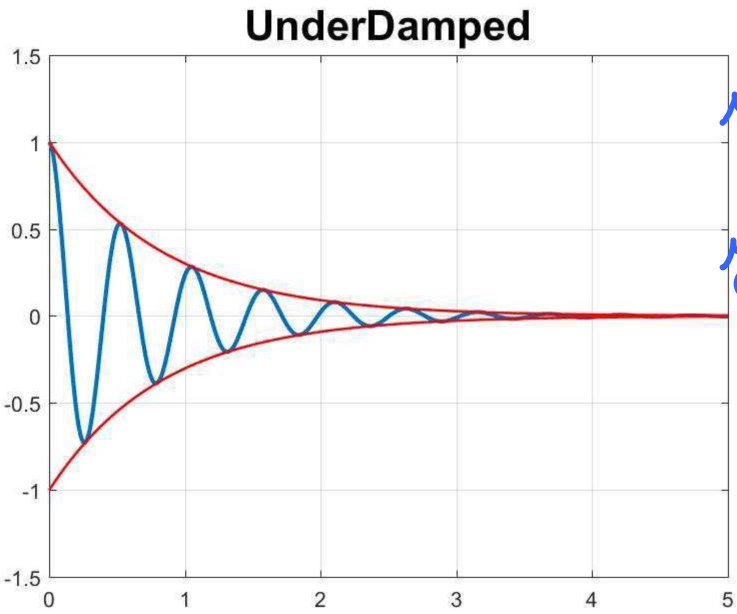
has 2 real roots

Critically Damped

$r^2 + \frac{\gamma}{m} r + \frac{k}{m} = 0$

has 1 (real) root

$y = c_1 e^{-rt} + c_2 t e^{-rt}$



$$y = c_1 e^{-\frac{r}{2m}t}\cos\omega_d t + c_2 e^{-\frac{r}{2m}t}\sin\omega_d t$$

$$y = A e^{-\frac{r}{2m}t}\cos(\omega_d t - \alpha)$$

$$r^2 + \frac{r}{m} r + \frac{k}{m} = 0$$

has complex roots

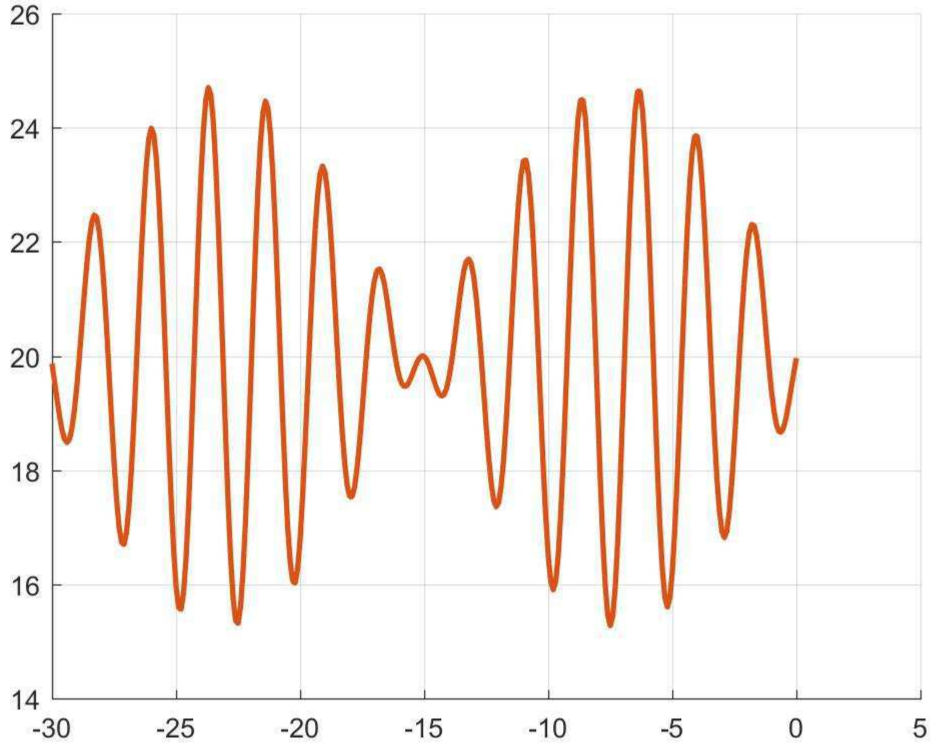
Forced Motion

Undamped

Beats $\ddot{y} + \omega_0^2 y = \cos\omega t$

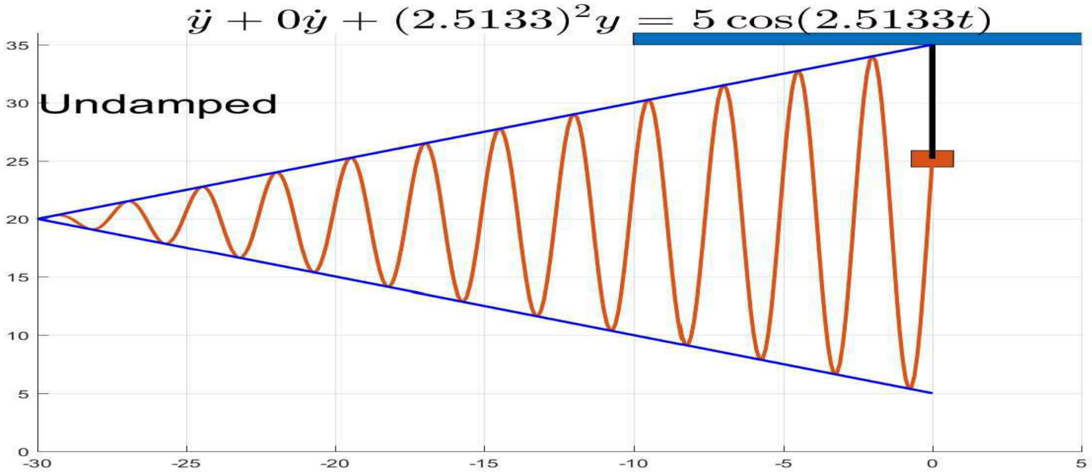
$$y = \frac{2\sin\left(\frac{\omega-\omega_0}{2}t\right)\sin\left(\frac{\omega+\omega_0}{2}t\right)}{\omega^2-\omega_0^2}$$

average freq.
half difference



Resonance $\ddot{y} + \omega_0^2 y = \cos\omega_0 t \qquad y(0) = 0 \quad \dot{y}(0) = 1$

$$y = t\cos\omega_0 t$$



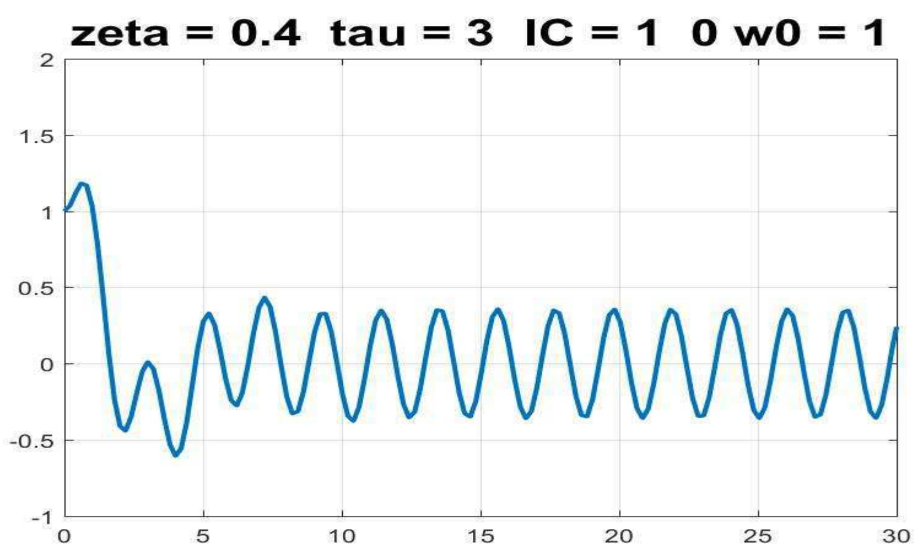
Damped Forced $\ddot{y} + \frac{\gamma}{m}\dot{y} + \frac{k}{m}y = \cos\omega t$

$$\left[\text{or } \ddot{y} + 2\zeta\omega_0\dot{y} + \omega_0^2 y = \cos\omega t\right]$$

$$y_{ss} = A\cos(\omega t - \varphi)$$

we can calculate A and φ in terms of ζ, ω_0, ω

$$A = \frac{1}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\zeta^2\omega_0^2\omega^2}}$$



I don't expect you to memorize any of these formulas. I expect you to be able to work them out in specific cases.

Mathematical Methods for Second Order Constant Coefficient ODE

1. Find homogeneous solutions as linear combination of functions e^{rt} or te^{rt}.

2. Particular Solutions are linear combinations of forcing functions and their derivatives.

If forcing terms satisfy homogeneous equation, particular solution must be multiplied by t. Repeat until no term satisfies homogeneous equation.

Note Title 2/18/2020

$$\ddot{x} + \frac{\gamma}{m}\dot{x} + \frac{k}{m}x = 0$$ Mass-Spring

Often rewritten as

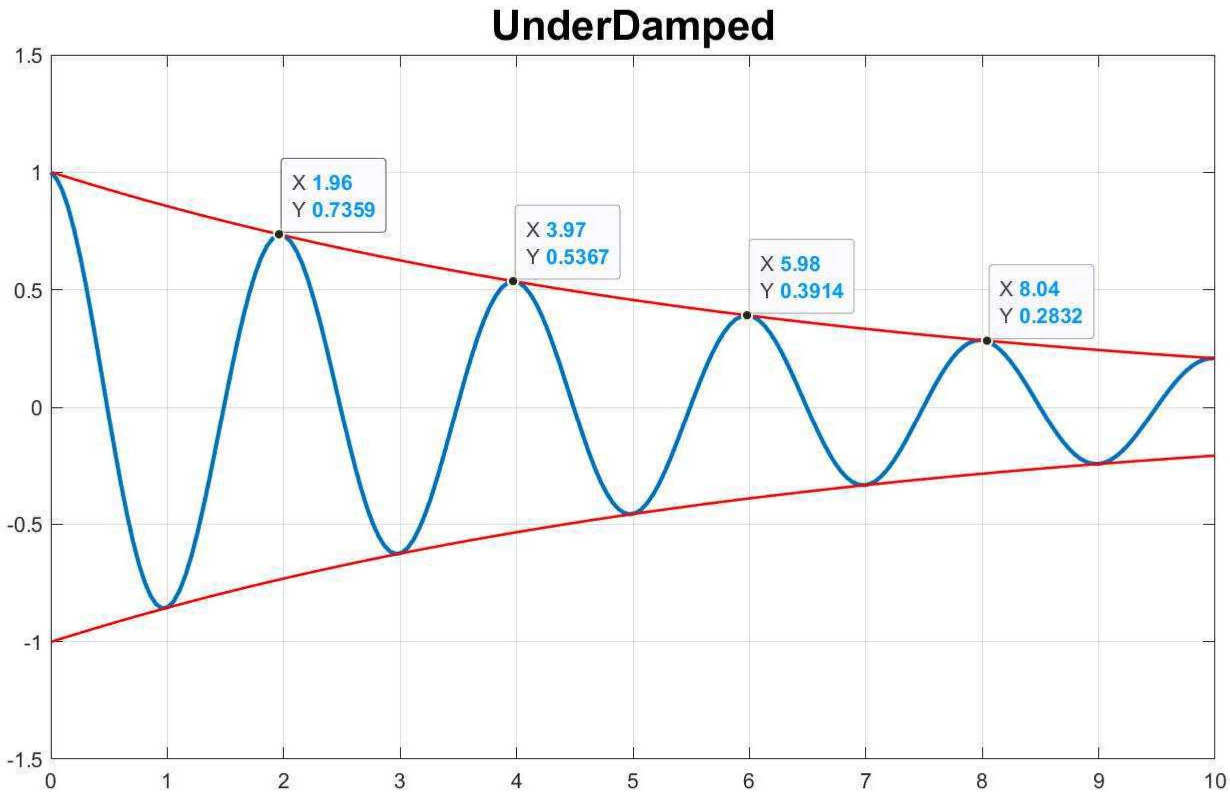
$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0 \qquad \text{(DE)}$$

Solution in polar form

$$x(t) = A\,e^{-\zeta\omega_0 t}\cos(\omega_d t - \phi)$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_0$$

(DE) is written this way because its easy to find ω_d, ζ, ω_0 from the graph.



$$\ddot{x} + 2\zeta\omega_0 \dot{x} + \omega_0^2 x = 0$$

Solution in polar form

$$x(t) = A e^{-\zeta\omega_0 t} \cos(\omega_d t - \varphi)$$

Derivation of Solution

Seek $\quad x(t) = e^{rt}$

$$r^2 + 2\zeta\omega_0 r + \omega_0^2 = 0$$

$$(r + \zeta\omega_0)^2 = -\omega_0^2(1-\zeta^2)$$

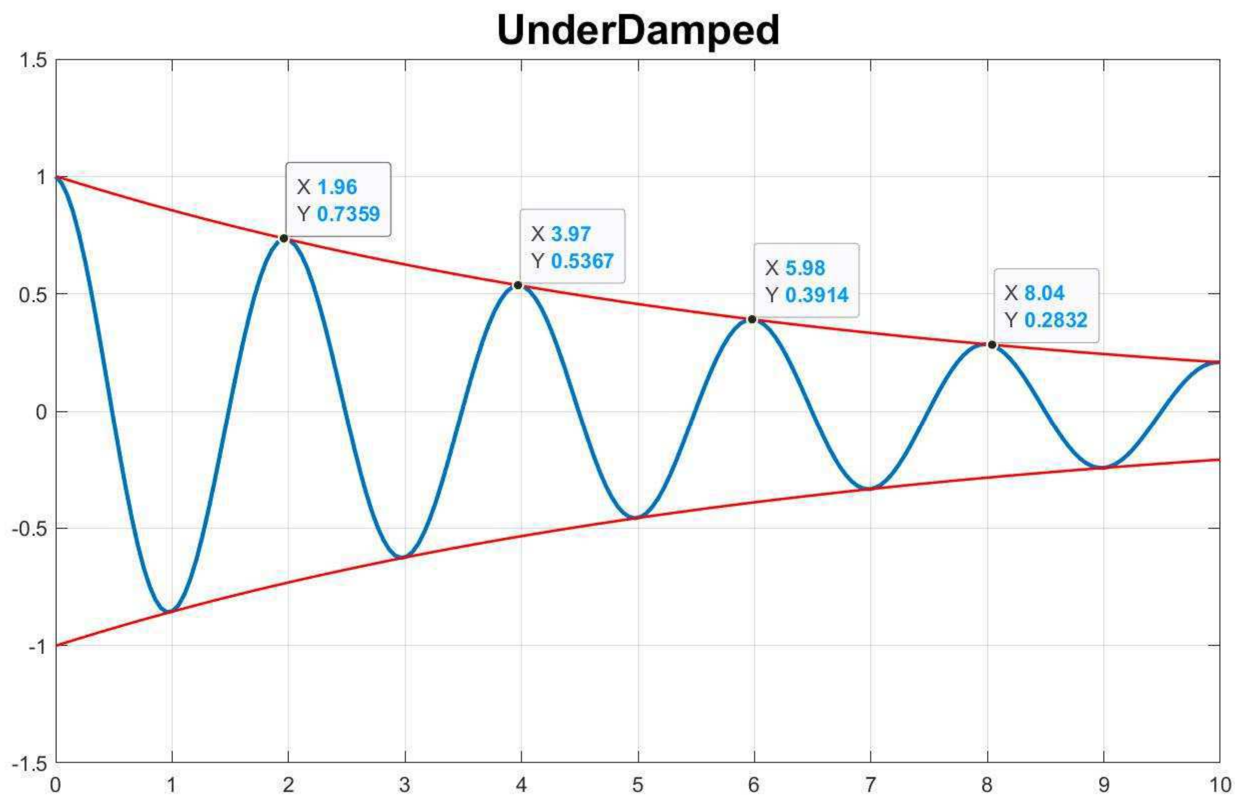
$$r = -\zeta\omega_0 \pm i\omega_0\sqrt{1-\zeta^2}$$

Definition $\quad \omega_d = \sqrt{1-\zeta^2}\,\omega_0$

$$x(t) = e^{-\zeta\omega_0 t}(c_1 \cos\omega_d t + c_1 \sin\omega_d t)$$

$$= e^{-\zeta_0\omega t} A\cos(\omega_d t - \varphi)$$

Problem: Find ω_d and ζ $\left(\omega_0 = \frac{\omega_d}{\sqrt{1-\zeta^2}}\right)$

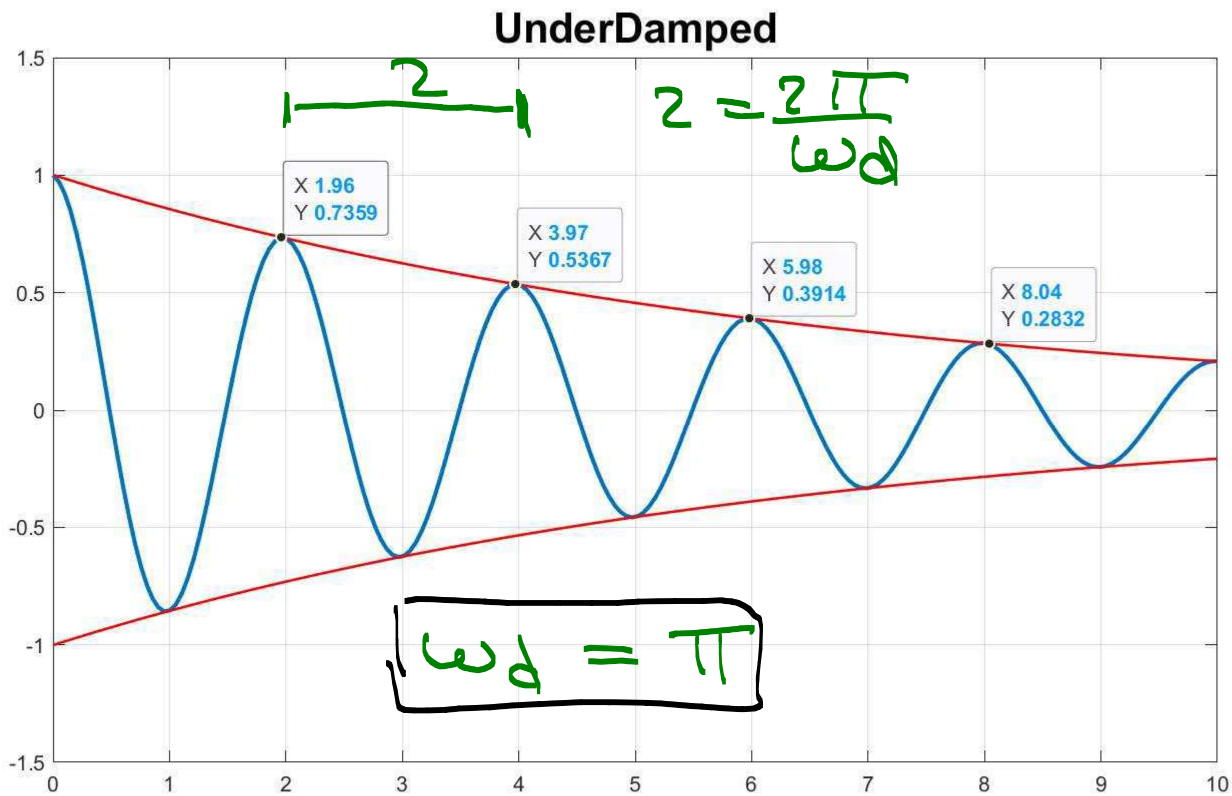


Relevant Facts

① The time between peaks (local maxima) is $\frac{2\pi}{\omega_d}$

② The cosine term $\cos(\omega_d t - \alpha)$ has the same value at all peaks

① The time between peaks is $\frac{2\pi}{\omega_d}$



2

$2 = \frac{2\pi}{\omega_d}$

$\boxed{\omega_d = \pi}$

$e^{-\zeta\omega_0(1.96)} \cos(\omega_d \cdot 1.96 - \alpha) = 0.7359$

$e^{-\zeta\omega_0(3.97)} \cos(\omega_d \cdot 3.97 - \alpha) = 0.5367$

② The cosine term $\cos(\omega_d t - \alpha)$ has the same value at all peaks

$$\frac{e^{-\zeta\omega_0(1.96)}}{e^{-\zeta\omega_0(3.97)}} = \frac{0.7359}{0.5367}$$

$$\frac{e^{-\zeta\omega_o(1.96)}}{e^{-\zeta\omega_o(3.97)}} = \frac{0.7359}{0.5367}$$

$$e^{\zeta\omega_o \cdot 2} = 1.3712 \quad \swarrow \omega_d$$

$$\zeta\omega_o = \frac{\ln(1.3712)}{2} = 0.1578$$

<u>Usual Approximation</u> $\omega_o \approx \omega_d$

$$\omega_o = \omega_d = \pi$$

$$\zeta = \frac{0.1578}{\pi} = 0.0502$$

<u>Exact Calculation</u>

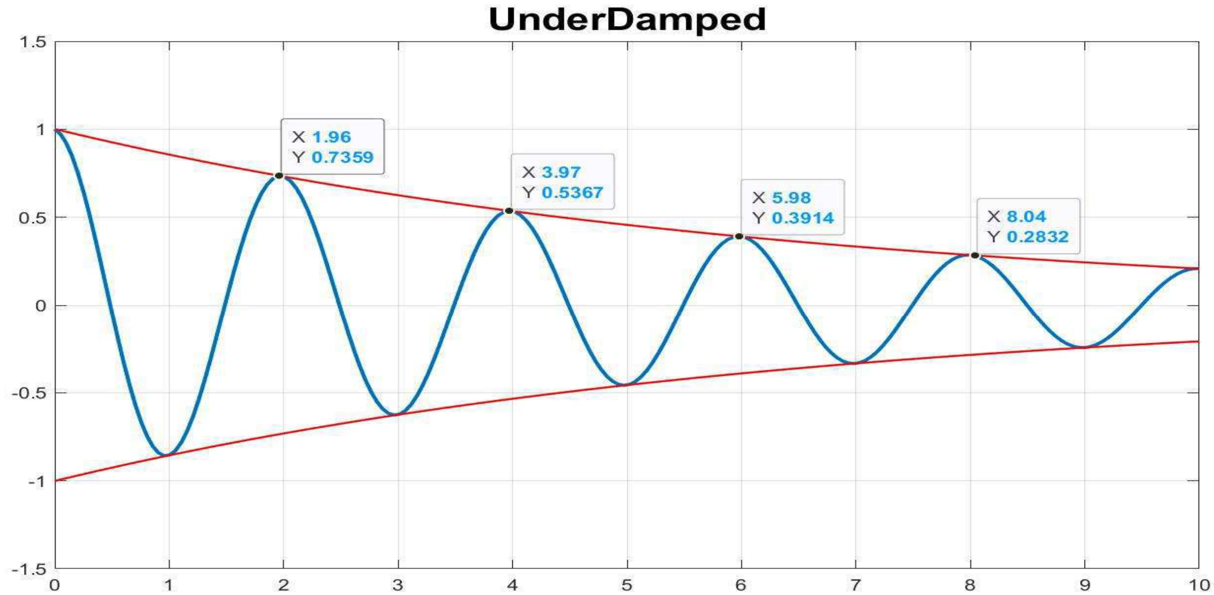
$$(\zeta\omega_o)^2 + \omega_d^2 = (\zeta\omega_o)^2 + (1-\zeta^2)\omega_o^2$$

$$= \omega_o^2$$

$$(0.1578)^2 + \pi^2 = \omega_o^2$$

$$3.1455 = \omega_o$$

$$0.0502 = \frac{0.1578}{3.1455} = \zeta$$



$$x(t) = A\, e^{-\zeta \omega_0 t} \cos(\omega_d t - \phi)$$

① The time between peaks is $\frac{2\pi}{\omega_d}$

② The cosine term $\cos(\omega_d t - \phi)$ has the same value at all peaks

"Approximate Proof" of ① and ②

$\cos(\omega_D t - \phi) \approx 1$ at the peaks *

Both cosines are 1 at peaks proves ②

so $\omega_D t_0 - \phi = 0$ at first peak

$\omega_D t_1 - \phi = 2\pi$ at next peack

subtract $\boxed{\omega_d (t_1 - t_0) = 2\pi}$ proves ①

* This is only approximately true

"Exact Proof"

Max occurs at t_0 where $\frac{dy}{dt} = 0$

$$\frac{d}{dt}\left(e^{-\zeta\omega_0 t}\cos(\omega_d t - \varphi)\right)$$

$$-\zeta\omega_0 \cancel{e^{-\zeta\omega_0 t}} \cos(\omega_d t - \varphi) = \cancel{e^{-\zeta\omega_0 t}} \omega_d \sin(\omega_d t - \varphi)$$

$$-\frac{\zeta\omega_0}{\omega_d} = \tan(\omega_d t - \varphi)$$

So at 2 consecutive peaks t_0 & t_1

$$\tan(\omega_d t_0 - \varphi) = \tan(\omega_d t_1 - \varphi)$$

$$\omega_d t_0 - \cancel{\varphi} = \omega_d t_1 - \cancel{\varphi} + N\pi$$

At consecutive maxima

$$\omega_d t_0 - \omega_d t_1 = 2\pi$$

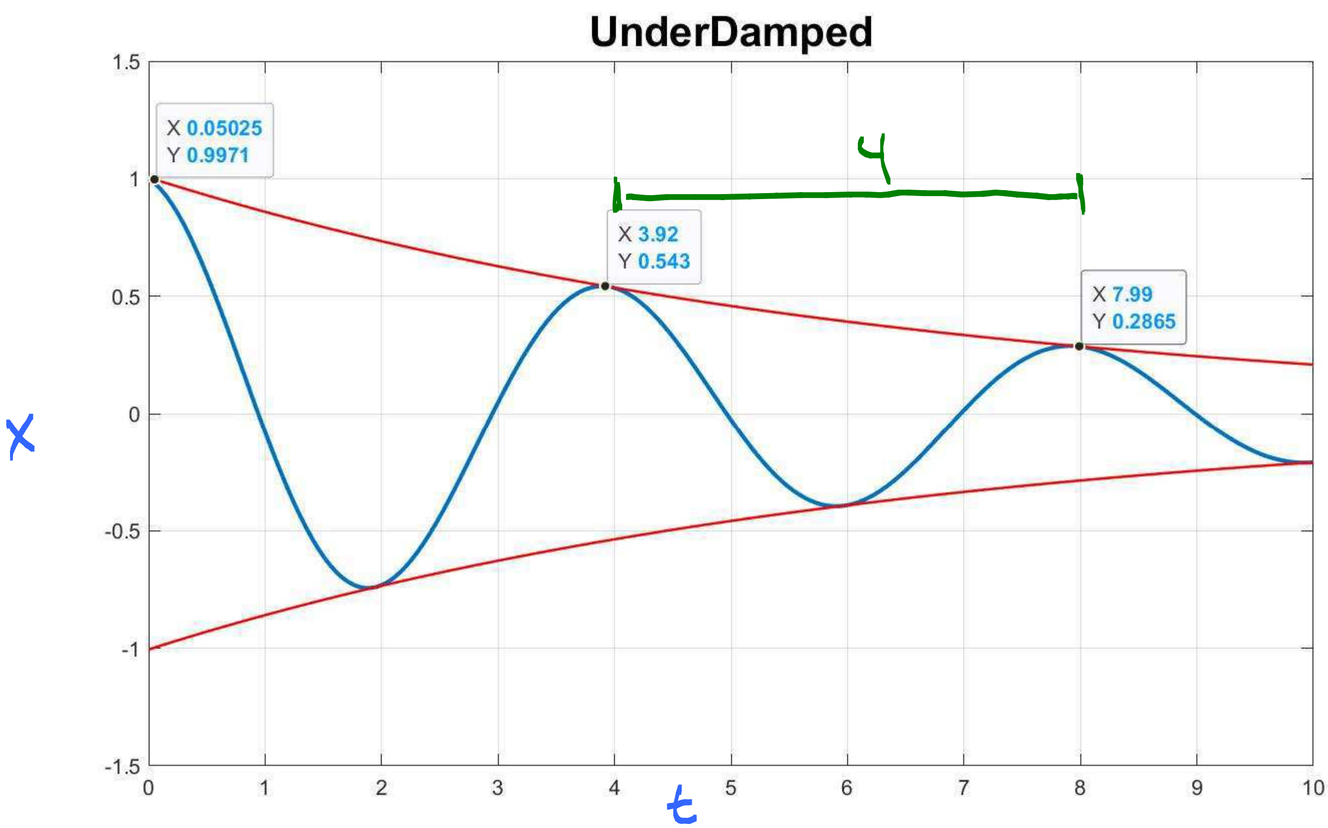
② If $\tan(\omega_d t_0 - \varphi) = \tan(\omega_d t_1 - \varphi)$

then $\cos(\omega_d t_0 - \varphi) = \pm \cos(\omega_d t_1 - \varphi)$

at maxima $\cos(\omega_d t_0 - \varphi) = \cos(\omega_d t_1 - \varphi)$

Problem Find ζ and ω_o

$$x(t) = A\, e^{-\zeta \omega_o t} \cos(\omega_d t - \phi)$$



$$\frac{2\pi}{4} = \omega_d \approx \omega_o$$

$$4 \zeta \omega_o = \ln\left(\frac{0.543}{0.286}\right) = 0.64$$

$$\zeta = \frac{0.64}{2\pi} = 0.102$$

Note Title 11/9/2017

Laplace Transform - a method for solving constant coefficient DE's / IVP's

Definition:

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Notation: The Laplace transform of $f(t)$ is $F(s)$. Also written as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

These two statements mean the same thing:

$\mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$ $\quad f(t) = e^{-t} \quad F(s) = \frac{1}{s+1}$

The Laplace transform changes a function of t into a function of s.

<u>Problem</u> - Calculate $\mathcal{L}\{e^{-t}\}$

$$\mathcal{L}\{e^{-t}\} = \int_0^\infty e^{-st} e^{-t}\,dt = \int_0^\infty e^{-(s+1)t}\,dt$$

$$= \left.\frac{e^{-(s+1)t}}{-(s+1)}\right|_0^\infty = \frac{0}{-(s+1)} - \frac{1}{-(s+1)} = \frac{1}{s+1}$$

Problem - Calculate $\mathcal{L}\{e^{-t}\}$

$$\mathcal{L}\{e^{-t}\} = \int_0^\infty e^{-st} e^{-t}\,dt = \int_0^\infty e^{-(s+1)t}\,dt$$

Some technical details

$$\int_0^\infty e^{-(s+1)t}\,dt = \lim_{M\to\infty} \int_0^M e^{-(s+1)t}\,dt$$

$$= \lim_{M\to\infty} \left.\frac{e^{-(s+1)t}}{-(s+1)}\right|_0^M$$

$$= \lim_{M\to\infty} \left(\frac{1}{s+1} - \frac{e^{-M(s+1)}}{s+1} \right)$$

we always assume s is big enough so that this limit $= 0$

$$= \frac{1}{s+1} - 0 \qquad [\text{for } s > -1]$$

Calculate $\mathcal{L}\{e^{rt}\}$ where r is a constant.

$$\mathcal{L}\{e^{rt}\} = \int_0^\infty e^{-st} e^{rt}\,dt = \int_0^\infty e^{-(s-r)t}\,dt$$

$$= \left.\frac{e^{-(s-r)t}}{-(s-r)}\right|_0^\infty$$

$$= 0 + \frac{1}{(s-r)}$$

$$\boxed{\mathcal{L}\{e^{rt}\} = \frac{1}{(s-r)}}$$

Why we use Laplace Transform

The Laplace Transform of the derivative

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = s\,\mathcal{L}\{y\} - y(0)$$

Proof

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = \int_0^\infty e^{-ts}\frac{dy}{dt}\,dt$$

$$= e^{-ts}y(t)\Big|_0^\infty - \int_0^\infty \frac{d}{dt}e^{-ts}\,y(t)\,dt$$

$$= 0 - y(0) - \int_0^\infty (-s)\,e^{-ts}\,y(t)\,dt$$

$$= -y(0) + s\int_0^\infty e^{-ts}\,y(t)\,dt$$

$$= s\,\mathcal{L}\{y\} - y(0)$$

$\blacksquare$

Note Title 11/17/2017

Laplace Transform Table

	$y(t)$	$Y(s) = \mathcal{L}\{y(t)\}$
①	e^{rt}	$\frac{1}{s-r}$
②	$\frac{dy}{dt}$	$sY(s) - y(0)$
③	$f(t) + g(t)$	$F(s) + G(s)$

$Y(s)$ is notation for $\mathcal{L}\{y(t)\}$

Table summarizes these facts

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$

Laplace transform is linear

$$\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \qquad \mathcal{L}\{0\} = 0$$

One Additional Fact

Laplace transform is one to one.

If $\mathcal{L}\{y(t)\} = 0$ then $y(t) = 0$.

Solve IVP $\begin{cases} \dfrac{dy}{dt} + 4y = 0 & \text{(DE)} \\ y(0) = 1 & \text{(IC)} \end{cases}$ using Laplace transforms

$$\frac{dy}{dt} + 4y = 0$$

$$\mathcal{L}\left\{ \frac{dy}{dt} + 4y \right\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + 4\mathcal{L}\{y\} = 0$$ uses 3 from table

$$sY(s) - y(0) + 4Y(s) = 0$$ uses 2 from table

$$(s+4)Y(s) - 1 = 0$$

$$Y(s) = \frac{1}{s+4}$$

$$Y(s) = \mathcal{L}\{e^{-4t}\}$$ uses 1 from table

$$\boxed{y(t) = e^{-4t}}$$

We have solved (IVP) without integrating or differentiating. We only used algebra.

Laplace Transform Approach to IVP

$$y'' + 2y' + y = \cos\omega t$$
$$y(0) = 0$$
$$y'(0) = 0$$

$\mathcal{L}$ Laplace Transform

$$s^2 Y + 2sY + Y = \frac{s}{s^2+\omega^2}$$

where

$$Y(s) = \mathcal{L}\{y\}$$
$$\frac{s}{s^2+\omega^2} = \mathcal{L}\{\cos\omega t\}$$

$$Y(s) = \frac{s}{(s^2+2s+1)(s^2+\omega^2)}$$

$\mathcal{L}^{-1}$ "Inverse" Laplace Transform

I haven't explained this step yet

$$\frac{2\omega \sin\omega t - (\omega^2-1)\cos\omega t - e^{-t} - (2\omega^2+1)\, t e^{-t}}{(\omega^2+1)^2}$$

Laplace Transforms $\mathcal{L}\{y(t)\} := \int_0^\infty e^{-st} y(t)\, dt$

Specific Transforms

Notation:

$Y(s) := \mathcal{L}\{y(t)\}$

e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
$e^{at}\cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$e^{at}\sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\frac{t^n}{n!}$	$\frac{1}{(s-a)^{n+1}}$
$\dot{y}(t)$	$sY(s) - y(0)$
$\ddot{y}(t)$	$s^2Y(s) - s\,y(0) - \dot{y}(0)$

Note Title 11/10/2017

Laplace Transforms $\mathcal{L}\{y(t)\} := \int_0^\infty e^{-st}\, y(t)\, dt$

Specific Transforms

e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	?
$\cos \omega t$	?
1	?
t	?
$\frac{t^n}{n!}$	?

We will fill in the rest of the table using "general rules" rather than by computing integrals.

Notation:

$Y(s) := \mathcal{L}\{y(t)\}$

Rule 1 $\mathcal{L}\{\dot{y}\} = sY(s) - y(0) \quad [= s\,\mathcal{L}\{y\} - y(0)]$

Rule 2 $\mathcal{L}\{e^{at}\, y(t)\} = Y(s-a)$

Rule 3 $\mathcal{L}\{t\, y(t)\} = -\frac{d}{ds} Y(s)$

Rule 4 $\mathcal{L}\{y(at)\} = \frac{1}{a} Y\left(\frac{s}{a}\right)$

General Rules

$$\mathcal{L}\{\dot{y}\} = sY(s) - y(0)$$

Proof
$$\int_0^\infty \dot{y}\, e^{-st}\,dt = y\, e^{-st}\Big|_0^\infty + s\int_0^\infty y(t)\, e^{-st}\,dt$$

$$= -y(0) + s\,Y(s)$$

Combine $\mathcal{L}\{0\} = 0$ and $\mathcal{L}\{\dot{y}\} = s\,\mathcal{L}\{y\} - y(0)$

to calculate $\mathcal{L}\{1\}$ using the fact that $\frac{d}{dt}1 = 0$

$$0 = \mathcal{L}\{0\} = \mathcal{L}\{\tfrac{d}{dt}1\} = s\,\mathcal{L}\{1\} - 1$$

$$0 = s\,\mathcal{L}\{1\} - 1$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}}$$

Compute $\mathcal{L}\{t\}$ using $\dot{t} = 1$

General Rule $\mathcal{L}\{\dot{y}\} = s\,\mathcal{L}\{y\} - y(0)$

$$\mathcal{L}\{\dot{t}\} = s\,\mathcal{L}\{t\} - 0$$

$$\frac{1}{s} = \mathcal{L}\{1\} = s\,\mathcal{L}\{t\}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

Compute $\mathcal{L}\{t^n\}$

using $(t^n)' = n t^{n-1}$

$$\mathcal{L}\{\dot{y}\} = s\mathcal{L}\{y\} - y(0)$$

$$\mathcal{L}\{n t^{n-1}\} = s\mathcal{L}\{t^n\} - 0 \quad \text{for } n \geq 1$$

$$n\mathcal{L}\{t^{n-1}\} = s\mathcal{L}\{t^n\}$$

$$\boxed{\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}}$$

$$\mathcal{L}\{t^2\} = \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}$$

$$\mathcal{L}\{t^3\} = \frac{3}{s}\mathcal{L}\{t^2\} = \frac{3}{s}\cdot\frac{2}{s^3} = \frac{3!}{s^4}$$

$$\boxed{\mathcal{L}\left\{\frac{t^n}{n!}\right\} = \frac{1}{s^{n+1}}}$$

Compute Laplace Transform of Sine and cosine

$(\sin t)' = \cos t$ so we may apply Rule 1

$$\mathcal{L}\{(\cos t)\} = \mathcal{L}\{(\sin t)'\} = s\,\mathcal{L}\{\sin t\} - 0 \qquad (1)$$

$(-\cos t)' = \sin t$ so we may apply Rule 1

$$\mathcal{L}\{\sin t\} = \mathcal{L}\{-(\cos t)'\} = -(s\,\mathcal{L}\{\cos t\} - 1)$$
$$= -s\,\mathcal{L}\{\cos t\} + 1 \qquad (2)$$

$$\mathcal{L}\{(\cos t)\} \overset{(1)}{=} s\,\mathcal{L}\{\sin t\} \overset{(2)}{=} s\left(-s\,\mathcal{L}\{\cos t\} + 1\right)$$

so $\mathcal{L}\{(\cos t)\} = -s^2\,\mathcal{L}\{\cos t\} + s$

$$(1+s^2)\,\mathcal{L}\{\cos t\} = s$$

$$\boxed{\mathcal{L}\{\cos t\} = \frac{s}{(1+s^2)}}$$

$\mathcal{L}\{\sin t\} = -s\,\mathcal{L}\{\cos t\} + 1$ from (2)

$$= \frac{-s^2}{1+s^2} + 1 = \frac{-\cancel{s^2} + 1 + \cancel{s^2}}{1+s^2}$$

$$\boxed{\mathcal{L}\{\sin t\} = \frac{1}{1+s^2}}$$

More general Rules

Compute $\mathcal{L}\{e^{at}y(t)\}$

$$\mathcal{L}\{e^{at}y(t)\} = \int_0^\infty e^{-st}e^{at}y(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t}y(t)\,dt$$

$$= Y(s-a) = \mathcal{L}\{y(t)\}\Big|_{s-a}$$

$$\boxed{\mathcal{L}\{e^{at}y(t)\} = Y(s-a)} \quad \underline{\text{Rule 2}}$$

Compute $\mathcal{L}\{e^{at}\cos\omega t\}$

$$\mathcal{L}\{e^{at}\cos\omega t\} = \mathcal{L}\{\cos\omega t\}\Big|_{s\to s-a}$$

$$= \frac{(s-a)}{(s-a)^2+\omega^2}$$

$$\mathcal{L}\{e^{at}\sin\omega t\} = \mathcal{L}\{\sin\omega t\}\Big|_{s\to s-a}$$

$$= \frac{\omega}{(s-a)^2+\omega^2}$$

One more rule $\mathcal{L}\{t\,y(t)\} = ?$

Calculate $\frac{d}{ds} Y(s)$

$$\frac{d}{ds} Y(s) = \frac{d}{ds} \int_0^\infty e^{-st}\, y(t)\, dt$$

$$= \int_0^\infty \left(\frac{d}{ds} e^{-st}\right) y(t)\, dt$$

$$= \int_0^\infty \left(-t e^{-st}\right) y(t)\, dt$$

$$= -\int_0^\infty e^{-st}\, t\, y(t)\, dt$$

$$= -\mathcal{L}\{t\,y(t)\}$$

$$\mathcal{L}\{t\,y(t)\} = -\frac{d}{ds} Y(s)$$ <u>Rule 3</u>

Compute $\mathcal{L}\{t e^{at}\}$

$$\mathcal{L}\{t e^{at}\} = -\frac{d}{ds} \mathcal{L}\{e^{at}\} = -\frac{d}{ds}\left(\frac{1}{s-a}\right)$$

$$\boxed{\mathcal{L}\{t e^{at}\} = \frac{1}{(s-a)^2}}$$

Rule 4 $\quad \mathcal{L}\{y(at)\} = \frac{1}{a} Y\left(\frac{s}{a}\right)$

Use Rule 4 to calculate $\mathcal{L}\{\sin\omega t\}$

$$\mathcal{L}\{\sin\omega t\} = \frac{1}{\omega} \mathcal{L}\{\sin t\}\Big|_{s \to \frac{s}{\omega}}$$

$$= \frac{1}{\omega} \frac{1}{1+\left(\frac{s}{\omega}\right)^2}$$

$$= \frac{1}{\omega} \frac{\omega^2}{s^2+\omega^2} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{\cos\omega t\} = \frac{1}{\omega} \mathcal{L}\{\cos t\}\Big|_{s \to \frac{s}{\omega}}$$

$$= \frac{1}{\omega} \frac{s/\omega}{1+\left(\frac{s}{\omega}\right)^2}$$

$$= \frac{1}{\omega} \frac{\omega s}{s^2+\omega^2} = \frac{s}{s^2+\omega^2}$$

Alternative Calculation of $\mathcal{L}\{te^{at}\}$

$$\mathcal{L}\{te^{at}\} = -\frac{d}{ds}\mathcal{L}\{e^{at}\}$$

$$= -\frac{d}{ds}\left(\frac{1}{s-a}\right)$$

$$= \frac{1}{(s-a)^2}$$

Proof of Rule 4

$$\mathcal{L}\{y(at)\} = \int_0^\infty e^{-st}\, y(at)\, dt$$

$$= \int_0^\infty e^{-\frac{s}{a}(at)}\, y(at)\, dt$$

Let $\tau = at$

$$= \int_0^\infty e^{-\left(\frac{s}{a}\right)\tau}\, y(\tau)\, \frac{d\tau}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)\tau}\, y(\tau)\, d\tau$$

$$= \frac{1}{a}\, \mathcal{L}\{y(t)\}\Big|_{s \mapsto \frac{s}{a}} = \frac{1}{a}\, Y\left(\frac{s}{a}\right)$$

Laplace Transforms $\mathcal{L}\{y(t)\} := \int_0^\infty e^{-st} y(t)\,dt$

Specific Transforms

e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$

Notation:

$Y(s) := \mathcal{L}\{y(t)\}$

Rule 1 $\mathcal{L}\{\dot{y}\} = sY(s) - y(0)$ $\left[= s\mathcal{L}\{y\} - y(0)\right]$

Rule 2 $\mathcal{L}\{e^{at} y(t)\} = Y(s-a)$

Rule 3 $\mathcal{L}\{t\, y(t)\} = -\frac{d}{ds} Y(s)$

Rule 4 $\mathcal{L}\{y(at)\} = \frac{1}{a} Y\left(\frac{s}{a}\right)$

Table of Laplace Transforms

Note Title 11/11/2017

General

	$\frac{dy}{dt}$	$sY(s) - y(0)$
	$\frac{d^2y}{dt^2}$	$s^2Y(s) - s\,y(0) - y'(0)$
	$e^{at}\,y(t)$	$Y(s-a)$
	$t\,y(t)$	$-\frac{d}{ds}Y(s)$
Specific	e^{at}	$1/(s-a)$
	$\sin\omega t$	$\omega/(s^2+\omega^2)$
	$\cos\omega t$	$s/(s^2+\omega^2)$
	t^n	$\frac{n!}{s^{n+1}}$

Inverse Laplace Transform

There is a formula ($Y(s) = \int_0^\infty e^{-st}y(t)\,dt$) to compute the Laplace Transform, but there is no comparable formula for the Inverse Laplace Transform. We compute the inverse Laplace transform using the table above.

https://en.wikipedia.org/wiki/Post's_inversion_formula

https://www.rose-hulman.edu/~bryan/invlap.pdf

Table of Laplace Transforms

General	$\frac{dy}{dt}$	$sY(s) - y(0)$
	$\frac{d^2y}{dt^2}$	$s^2Y(s) - s\,y(0) - y'(0)$
	$e^{at}\,y(t)$	$Y(s-a)$
	$e^{at}\,y(t)$	$= \mathcal{L}^{-1}\{Y(s-a)\}$
	$y(t)$	$= \mathcal{L}^{-1}\{Y(s)\}$
	$t\,y(t)$	$-\frac{d}{ds}Y(s)$
Specific	e^{at}	$1/(s-a)$
	$\sin\omega t$	$\omega/(s^2+\omega^2)$
	$\cos\omega t$	$s/(s^2+\omega^2)$
	t^n	$\frac{n!}{s^{n+1}}$
	1	$\frac{1}{s}$

Inverse Laplace Transform

Examples

Compute $\mathcal{L}^{-1}\{Y(s)\}$ for each of the following

$$Y(s) = \frac{1}{s-4}$$

$$Y(s) = \frac{1}{(s-4)^2}$$

$$Y(s) = \frac{3s+4}{s^2+9}$$

$$Y(s) = \frac{1}{s^2+2s+5}$$

$$Y(s) = \frac{3s+4}{s^2+2s+5}$$

$$Y(s) = \frac{1}{s^2+2s-8}$$

$$Y(s) = \frac{1}{s-4}$$

This is $\frac{1}{s-a}$ with $a=4$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = e^{4t}$$

Alternatively

Table Entry

$$\left.\begin{aligned} \mathcal{L}^{-1}\{Y(s)\} &= y(t) \\ \mathcal{L}^{-1}\{Y(s-a)\} &= e^{at}y(t) \end{aligned}\right\} \Longleftrightarrow \quad e^{at}y(t) \qquad Y(s-a)$$

$$Y(s) = \frac{1}{s-4}$$

$$Y(s+4) = \frac{1}{s}$$

$$\mathcal{L}^{-1}\{Y(s+4)\} = 1$$

$$e^{-4t}y(t) = 1$$

$$y(t) = e^{4t}$$

$$Y(s) = \frac{1}{(s-4)^2}$$

$$\frac{1}{(s-4)^2} \text{ `` }= \frac{1}{s^2}\Big|_{s \mapsto s-4} \text{ ''}$$

" $= \frac{1}{s^2}$ with s replaced by $(s-4)$ "

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\} = e^{4t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^{4t}\cdot t$$

Alternatively,

$$Y(s+4) = \frac{1}{s^2}$$ This one is in table

$$\mathcal{L}^{-1}\{Y(s+4)\} = t$$

Now use $e^{at}y(t) = \mathcal{L}^{-1}\{Y(s-a)\}$

$$e^{-4t}y(t) = t$$

$$\boxed{y(t) = t\,e^{4t}}$$

$$Y(s) = \frac{1}{(s-4)^2}$$

This is minus the derivative of $\frac{1}{s-4}$

$$\frac{d}{ds}\frac{1}{s-4} = \frac{-1}{(s-4)^2}$$

so

$$\frac{1}{(s-4)^2} = -\frac{d}{ds}\frac{1}{(s-4)}$$

so

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\} = t\,\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = te^{4t}$$

$$Y(s) = \frac{5s+4}{s^2+9}$$

Two tabe entries

$$\cos 3t \qquad \frac{s}{s^2+9}$$

$$\sin 3t \qquad \frac{3}{s^2+9}$$

$$Y(s) = 5 \cdot \frac{s}{s^2+9} + \frac{4}{3} \cdot \frac{3}{s^2+9}$$

$$\mathcal{L}^{-1}\{Y\} = 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$\boxed{y(t) = 5\cos 3t + \frac{4}{3}\sin 3t}$$

$$Y(s) = \frac{1}{s^2 + 2s + 5}$$

What are the roots?

$$s^2 + 2s + 5 = 0$$

$$s^2 + 2s + 1 = -4$$

$$(s+1)^2 = -4$$

$$s = -1 \pm 4i$$

<u>Complex Roots</u> - Complete the square

$$Y(s) = \frac{1}{(s+1)^2 + 4} = \text{"} \left.\frac{1}{s^2+4}\right|_{s \to s+1} \text{"}$$

$$\mathcal{L}^{-1}\{Y\} = e^{-t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{e^{-t}}{2}\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{e^{-t}}{2}\sin 2t$$

$$\boxed{y(t) = \frac{e^{-t}}{2}\sin 2t}$$

Alternatively, use

$$\left.\begin{aligned} \mathcal{L}^{-1}\{Y(s)\} &= y(t) \\ \mathcal{L}^{-1}\{Y(s-a)\} &= e^{at} y(t) \end{aligned}\right\} \iff$$

	Table Entry
$e^{at} y(t)$	$Y(s-a)$

$$Y(s) = \frac{1}{(s+1)^2+4}$$

$$Y(s-1) = \frac{1}{(s-1+1)^2+4} = \frac{1}{s^2+4}$$

This one is in the table.

$$e^{1t} y(t) = \mathcal{L}^{-1}\{Y(s-1)\} = \frac{1}{2}\sin 2t$$

$$y(t) = \frac{1}{2} e^{-t} \sin 2t$$

$$Y(s) = \frac{3s+4}{s^2+2s+5}$$

$$= \frac{3s+4}{(s+1)^2+4}$$

I want to recognize $Y(s)$ as something with s replaced by s+1

$$= \frac{3(s+1)-3+4}{(s+1)^2+4}$$

$$= \frac{3(s+1)+1}{(s+1)^2+4} \text{ "}= \left.\frac{3s+1}{s^2+4}\right|_{s \mapsto s+1}$$

$$\mathcal{L}^{-1}\{Y\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{3s+1}{s^2+4}\right\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{3\frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}\right\}$$

$$= e^{-t}\left[3\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\right]$$

$$\boxed{y(t) = 3e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t}$$

Alternatively, use

$\mathcal{L}^{-1}\{Y(s)\} = y(t)$	$\Longleftrightarrow$	Table Entry	
$\mathcal{L}^{-1}\{Y(s-a)\} = e^{at}y(t)$		$e^{at}y(t)$	$Y(s-a)$

I want the denominator to be in the table. Not worried about numerator

$$Y(s) = \frac{3s+4}{(s+1)^2+4}$$

$$Y(s-1) = \frac{3(s-1)+4}{s^2+4}$$

$$= \frac{3s}{s^2+4} + \frac{1}{s^2+4}$$

$$e^{t}y(t) = \mathcal{L}^{-1}\{Y(s-1)\} = 3\cos 2t + \frac{1}{2}\sin 2t$$

$$y(t) = e^{-t}\left(3\cos 2t + \frac{1}{2}\sin 2t\right)$$

$$Y(s) = \frac{1}{s^2+2s-8}$$

Roots of Denominator

$$s^2+2s+1 = 9 \implies (s+1)^2 = 9$$

$$s = -1 \pm 3 = 2, -4$$

$$Y(s) = \frac{1}{(s-2)(s+4)}$$

$\frac{1}{s-2}$ and $\frac{1}{s+4}$ are in the table

Partial Fractions

$$\frac{1}{(s-2)(s+4)} = \frac{A}{s-2} + \frac{B}{s-4}$$

Cover Up - multiply by $(s-2)$

$$\frac{1}{s+4} = A + \frac{B(s-2)}{s+4}$$

Set $s = 2$ $\frac{1}{2+4} = A = \frac{1}{6}$

Now multiply by $s+4$ and set $s = -4$

to find $B = \frac{-1}{6}$

$$Y(s) = \frac{1}{(s-2)(s+4)} = \frac{1/6}{s-2} - \frac{1/6}{s+4}$$

$$\mathcal{L}^{-1}\left\{\frac{1/6}{s-2} - \frac{1/6}{s-4}\right\} = 1/6\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 1/6\,\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= \frac{1}{6}e^{2t} - \frac{1}{6}e^{-4t}$$

Solve the IVP using Laplace Transforms

$$y'' + 2y' + 10y = 0$$

$$y(0) = 1 \qquad y'(0) = 2$$

$$\mathcal{L}\{y'' + 2y' + 10y\} = \mathcal{L}\{0\} = 0$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 10\mathcal{L}\{y\} = 0$$

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 10[Y(s)] = 0$$

$$[s^2Y(s) - s\cdot 1 - 2] + 2[sY(s) - 1] + 10[Y(s)] = 0$$

$$[s^2 + 2s + 10]\,Y(s) - s - 4 = 0$$

$$Y(s) = \frac{s+4}{s^2+2s+10}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+4}{s^2+2s+10}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+4}{(s+1)^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+9}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+4}{(s+1)^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+9}\right\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + e^{-t}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$\boxed{y(t) = e^{-t}\cos 3t + e^{-t}\sin 3t}$$

Alternatively, use

$$\left.\begin{aligned} \mathcal{L}^{-1}\{Y(s)\} &= y(t) \\ \mathcal{L}^{-1}\{Y(s-a)\} &= e^{at}y(t) \end{aligned}\right\} \iff e^{at}y(t) \qquad Y(s-a)$$

$$Y(s) = \frac{s+4}{(s+1)^2+9}$$

$$Y(s-1) = \frac{s-1+4}{s^2+9} = \frac{s}{s^2+9} + \frac{3}{s^2+9}$$

$$\mathcal{L}^{-1}\{Y(s-1)\} = \cos 3t + \sin 3t$$

$$e^{t}\,y(t) =$$

$$y(t) = e^{-t}(\cos 3t + \sin 3t)$$

lecture

Note Title 3/4/2020

Inhomogeneous DE's via Laplace Transform

Solve using Laplace Transform

$$\ddot{y} + y = e^{-2t}$$

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

<u>Solution</u>

$$\mathcal{L}\{\ddot{y} + y\} = \mathcal{L}\{e^{-2t}\}$$

$$\mathcal{L}\{\ddot{y}\} + \mathcal{L}\{y\} = \frac{1}{s+2}$$

$$s^2 Y - s\,y(0) - \dot{y}(0) + Y = 1/(s+2)$$

$$s^2 Y - 0 - 0 + Y = \frac{1}{s+2}$$

$$(s^2+1)\,Y = \frac{1}{s+2}$$

$$Y = \frac{1}{(s+2)(s^2+1)}$$

Find $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s^2+1)}\right\}$ - Partial Fractions

General Partial Fractions

$$\frac{Q(s)}{P(s)}$$ degree Q < degree P

$P(s)$ factors $P(s) = P_1(s)\,P_2(s)$

In General

$$\frac{Q(s)}{P_1(s)\,P_2(s)} = \frac{Q_1}{P_1(s)} + \frac{Q_2}{P_2(s)}$$

$\deg Q_1 = \deg P_1 - 1$ $\deg Q_2 = \deg P_2 - 1$

IF P_1 and P_2 have no common factor.

Example:

degree < 3 degree 0 degree 1

$$\frac{1}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+2}$$

degree 3 degree 1 polynomial degree 2 polynomial

---

degree < 3 Example

$$\frac{6s^2+3s+4}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+2}$$

degree 3

---

degree < 3 Bad Example

$$\frac{6s^2+3s+4}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs}{s^2+2}$$

Not good enough!!
Need most general degree 1 polynomial
$Bs + C$

degree 3

Examples

$$\frac{1}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4}$$ ✓

$$\frac{6s^3+4s^2+3s+6}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4}$$ ✓

$$\frac{1}{(s+3)^2(s+2)^2} = \frac{As+B}{(s+3)^2} \quad \frac{Cs+D}{(s+2)^2}$$

Correct, but not as useful as the one below

$$\frac{1}{(s+3)^2(s+2)^2} = \frac{A}{s+3} + \frac{\tilde{B}}{(s+3)^2} + \frac{C}{s+2} + \frac{\tilde{D}}{(s+2)^2}$$

If you check $\tilde{B} = B-3A$

$\tilde{D} = D-2C$

We use this one because $\frac{1}{s+3}$, $\frac{1}{(s+3)^2}$, $\frac{1}{s+2}$, $\frac{1}{(s+2)^2}$ are easier to find in the transform table.

Example

$$\frac{1}{(s+1)(s+2)(s+3)(s+4)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} + \frac{D}{s+4}$$

Back to Initial Value Problem on page 1

Find $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s^2+1)}\right\}$ - Partial Fractions

$$\frac{1}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+1}$$

Clear denominators and write as polynomial equality

$$1 = A(s^2+1) + (Bs+C)(s+2)$$

$$1 = As^2 + A + Bs^2 + Cs + 2Bs + 2C$$

$$1 = (A+B)s^2 + (C+2B)s + (A+2C)$$

$$0s^2 + 0s + 1 = (A+B)s^2 + (C+2B)s + (A+2C)$$

$A + 2C = 1 \qquad A + B = 0 \qquad C + 2B = 0$

Eliminate A

$$\left.\begin{array}{r} A + 2C = 1 \\ - \quad A + B = 0 \\ \hline 2C - B = 1 \end{array}\right\}$$

Combine with $C + 2B = 0 \Rightarrow C = -2B$

$$2C - B = 1 \Rightarrow -4B - B = 1 \Rightarrow B = -\frac{1}{5}$$

$$A = -B = \frac{1}{5} \text{ and } C = -2B = \frac{2}{5}$$

$$\frac{1}{(s+2)(s^2+1)} = \frac{1/5}{s+2} + \frac{-\frac{1}{5}s + \frac{2}{5}}{s^2+1}$$

$$= \frac{1}{5}\,\frac{1}{s+2} - \frac{1}{5}\,\frac{s}{s^2+1} + \frac{2}{5}\,\frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s^2+1)}\right\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t) = \frac{1}{5}e^{-2t} - \frac{1}{5}\cos t + \frac{2}{5}\sin t}$$

How to find the constants?

Example

$$\frac{1}{(s-1)(s-2)(s-3)} = \frac{A_1}{(s-1)} + \frac{A_2}{s-2} + \frac{A_3}{(s-3)}$$

$$1 = A_1(s-2)(s-3) + A_2(s-1)(s-3) + A_3(s-1)(s-2)$$

Always works, but lots of calculating

$$1 = (A_1+A_2+A_3)s^2 + (-5A_1-4A_2-3A_3)s + (6A_1+4A_2+2A_3)$$

$$A_1 + A_2 + A_3 = 0$$
$$-5A_1 - 4A_2 - 3A_3 = 0$$
$$6A_1 + 4A_2 + 2A_3 = 1$$

Three equations in three unknowns. Solve them.

Faster, especially for simple roots

$$1 = A_1(s-2)(s-3) + A_2(s-1)(s-3) + A_3(s-1)(s-2)$$

Set $s=1$

$$1 = A_1(1-2)(1-3) + 0 \qquad + 0$$

$$\boxed{\frac{1}{2} = A_1}$$

Set $s=2$

$$1 = 0 + A_2(2-1)(2-3)$$

$$\boxed{-1 = A_2}$$

Set $s=3$

$$1 = 0 + 0 + A_3(3-1)(3-2)$$

$$\boxed{\frac{1}{2} = A_3}$$

Example

$$\frac{s+2}{(s-1)(s-2)^2} = \frac{A_1}{s-1} + \frac{A_2}{s-2} + \frac{B_2}{(s-2)^2}$$

$$s+2 = A_1(s-2)^2 + A_2(s-2)(s-1) + B_2(s-1)$$

Set $s=1$

$$1+2 = A_1(1-2)^2 + 0 + 0$$

$$\boxed{3 = A_1}$$

Set $s=2$

$$2+2 = 0 + 0 + B_2(2-1)$$

$$\boxed{4 = B_2}$$

set $s=0$ (or anything else

$$0+2 = 3(0-2)^2 + A_2(0-2)(0-1) + 4(0-1)$$

$$2-12+4 = 2A_2$$

$$\boxed{7 = A_2}$$

Example

$$\frac{1}{(s-1)(s^2+4)} = \frac{A_1}{s-1} + \frac{A_2 s + B_2}{s^2+4}$$

$$1 = A_1(s^2+4) + (A_2 s + B_2)(s-1)$$

set $s=1$ $\boxed{A_1 = \frac{1}{5}}$

Now multiply it out

$$1 = (A_1 + A_2)s^2 + (B_2 - A_2)s + (4A_1 - B_2)$$

$A_1 + A_2 = 0$ so $\boxed{A_2 = -\frac{1}{5}}$

$B_2 - A_2 = 0$ so $\boxed{B_2 = -1/5}$

$4\left(\frac{1}{5}\right) - \left(\frac{-1}{5}\right) = 1$ ✓

Note Title 11/11/2017

One More Example

$$y'' + 9y = \cos 2t \qquad y(0) = 0 \qquad y'(0) = 0$$

Laplace Transform

$$s^2 Y + 9Y = \frac{s}{s^2+4}$$

$$(s^2+1)\,Y = \frac{s}{s^2+4}$$

$$Y = \frac{s}{(s^2+9)(s^2+4)}$$

Inverse Laplace Transform

Partial Fractions

$$\frac{s}{(s^2+9)(s^2+4)} = \frac{As+B}{(s^2+9)} + \frac{Cs+D}{(s^2+4)}$$

$$s = (As+B)(s^2+4) + (Cs+D)(s^2+9)$$

$$= As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2 + 9Cs + 9D$$

$$s = (A+C)s^3 + (B+D)s^2 + (4A+9C)s + (4B+9D)$$

$$\underline{0}\, s^3 + \underline{0}\, s^2 + \underline{1}\, s + 0 =$$

$$s = (A+C)s^3 + (B+D)s^2 + (4A+9C)s + (4B+9D)$$

$$A + C = 0 \Rightarrow A = -C$$

$$B + D = 0 \Rightarrow B = -D$$

$$4B + 9D = 0 \Rightarrow -4D + 9D = 0 \Rightarrow D = 0 \Rightarrow B = 0$$

$$4A + 9C = 1 \Rightarrow -4C + 9C = 1 \Rightarrow C = \frac{1}{5} \Rightarrow A = -\frac{1}{5}$$

$$Y(s) = \frac{-1}{5} \frac{s}{s^2+9} + \frac{1}{5} \frac{s}{s^2+4}$$

$$y(t) = \frac{-1}{5} \cos 3t + \frac{1}{5} \cos 2t$$

Alternatively

$$\frac{s}{(s^2+9)(s^2+4)} = s\left(\frac{1}{(s^2+9)(s^2+4)}\right) =$$

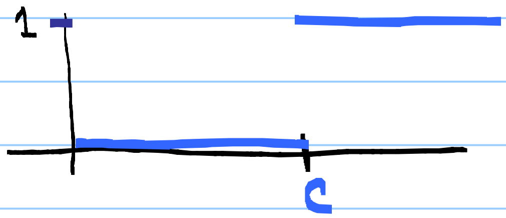
$$= s\left(\frac{A}{(s^2+9)} + \frac{C}{(s^2+4)}\right)$$

why

$$= s\left(\frac{-1/5}{s^2+9} + \frac{1/5}{s^2+4}\right)$$

The Heaviside Function and Time Delay

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$



Multiplying a function by $u_c(t)$ turns it on at time $t = c$. Multiplying by $(1 - u_c(t))$ turns it off at $t = c$.

Example

$$f(t) = \begin{cases} t & 0 \leq t \leq 1 \\ e^{(t-1)} & 1 < t \end{cases}$$

f equals t until time 1, then it becomes e^{t-1}

$$f(t) = t \cdot \underbrace{(1 - u_1(t))}_{\text{turn off } t} + e^{(t-1)} \underbrace{u_1(t)}_{\text{turn on } e^{(t-1)}}$$

Example

$$f(t) = \begin{cases} 6 & 0 \le t \le 1 \\ e^t & 1 < t \le 2 \\ \frac{t-1}{2} & 2 < t \le 3 \\ 4 & 3 < t \end{cases}$$

$$f(t) = 6\left(1-u_1(t)\right) + e^t\left(u_1(t) - u_2(t)\right) + \left(\frac{t-1}{2}\right)\left(u_2(t) - u_3(t)\right) + 4\,u_3(t)$$

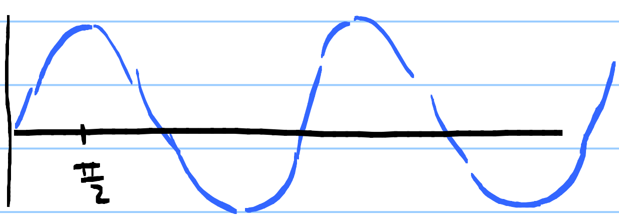
Turn on the 6 term — Turn it off — Turn on the e^t term — Turn off the e^t term — Turn on — Turn off — Turn on

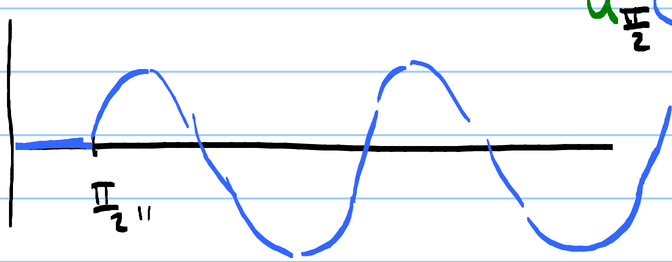
$$1 - u_1(t) = \begin{cases} 1 & t < 1 & \text{on} \\ 0 & t > 1 & \text{off} \end{cases}$$

$$\left(u_1(t) - u_2(t)\right) = \begin{cases} 0 & t < 1 & \text{off} \\ 1 & 1 < t < 2 & \text{on} \\ 0 & 2 < t & \text{off} \end{cases}$$

Interpretation of u_c as a delay

$\sin t$





$u_{\frac{\pi}{2}}(t)\sin\left(t - \frac{\pi}{2}\right) =$ "$\sin t$ delayed by $\frac{\pi}{2}$"

$$= \begin{cases} 0 & t < \frac{\pi}{2} \\ \sin\left(t - \frac{\pi}{2}\right) & \frac{\pi}{2} < t \end{cases}$$

Laplace Transform of u_c

$$\mathcal{L}\{u_c(t)\,y(t-c)\} = e^{-cs}\,Y(s)$$ ⟸ use this one for calculating inverse transform

Alternatively,

$$\mathcal{L}\{u_c(t)\,y(t)\} = e^{-cs}\,\mathcal{L}\{y(t+c)\}$$ ⟸ use this one for calculating transform

Calculating $\mathcal{L}\{u_c(t)\}$

$$\mathcal{L}\{u_c(t)\,y(t)\} = \int_0^\infty e^{-st}\,u_c(t)\,y(t)\,dt$$

$$= \int_c^\infty e^{-st}\,y(t)\,dt$$

Change variables $t = \tau + c$

$$= \int_0^\infty e^{-s(\tau+c)}\,y(\tau+c)\,d\tau$$

$$= e^{-sc}\int_0^\infty e^{-s\tau}\,y(\tau+c)\,d\tau$$

(1) $$\mathcal{L}\{u_c(t)\,y(t)\} = e^{-sc}\,\mathcal{L}\{y(t+c)\}$$

To see the other formulation $\left(\mathcal{L}\{u_c(t)\,y(t-c)\} = e^{-cs}\,Y(s)\right)$

Define a new function $h(t) = y(t-c)$

then $y(t) = h(t+c)$ and (1) becomes

$$\mathcal{L}\{u_c(t)\,h(t+c)\} = e^{-sc}\,\mathcal{L}\{h(t)\} = e^{-cs}\,H(s)$$

Example 1

$$f(t) = \begin{cases} 0 & 0 \le t \le 1 \\ e^{(t-1)} & 1 < t \end{cases} \qquad f(t) = u_1(t)\, e^{(t-1)}$$

$$F(s) = \mathcal{L}\{u_1(t)\, e^{(t-1)}\} = e^{-1s}\, \mathcal{L}\{e^{t}\} = e^{-s} \cdot \frac{1}{s-1}$$

Example 2

$$f(t) = \begin{cases} 0 & 0 \le t \le 2 \\ e^{t} & 2 < t \end{cases} \qquad f(t) = u_2(t)\, e^{t}$$

$$F(s) = \mathcal{L}\{u_2(t)\, e^{t}\} = e^{-2s}\, \mathcal{L}\{e^{t+2}\}$$

$$= e^{-2s}\, \mathcal{L}\{e^{2} \cdot e^{t}\}$$

$$= e^{-2s} \cdot e^{2}\, \mathcal{L}\{e^{t}\}$$

$$= e^{-2s+2} \cdot \frac{1}{s-1}$$

Example 3

$$f(t) = \begin{cases} \sin t & 0 < t < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < t \end{cases}$$

$$f(t) = \sin t - u_{\pi/2}(t)\sin t$$

$$\mathcal{L}\{f\} = \mathcal{L}\{\sin t\} - \mathcal{L}\{u_{\pi/2}(t)\sin t\}$$

$$= \frac{1}{s^2+1} - e^{-\frac{\pi}{2}s}\,\mathcal{L}\{\sin(t+\tfrac{\pi}{2})\}$$

$$= \frac{1}{s^2+1} - e^{-\frac{\pi}{2}s}\,\mathcal{L}\{\cos t\}$$

$$= \frac{1}{s^2+1} - e^{-\frac{\pi}{2}s}\left(\frac{s}{s^2+1}\right)$$

Lecture 26

Note Title 11/29/2017

Inverse Laplace Transforms involving $u_c(t)$

Table Entry

use for inverse $\quad u_c(t)\,y(t-c) \qquad e^{-cs}\,Y(s)$

or

use for fwd $\quad \mathcal{L}\{u_c(t)\,y(t)\} = e^{-cs}\,\mathcal{L}\{y(t+c)\}$

Problem - Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2}\right\}$

$$\mathcal{L}^{-1}\left\{e^{-2s}\,\frac{1}{s^2}\right\} = u_2(t)\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\Big|_{t \mapsto t-2}\right]$$

$$= u_2(t)\left(t\,\Big|_{t \mapsto t-2}\right)$$

$$= u_2(t)\,(t-2)$$

Find $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+4)}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+4)}\right\} = u_2(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s(s+4)}\right\}\Bigg|_{\text{"}t \mapsto t-2\text{"}}$$

Step 1 Compute $\mathcal{L}^{-1}\left\{\frac{1}{s(s+4)}\right\}$

$$\frac{1}{s(s+4)} = \frac{1/4}{s} - \frac{1/4}{s+4}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+4)}\right\} = 1/4 \cdot 1 - 1/4\, e^{-4t}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+4)}\right\} = u_2(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s(s+4)}\right\}\Bigg|_{\text{"}t \mapsto t-2\text{"}}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+4)}\right\} = u_2(t)\left(\frac{1}{4} - \frac{e^{-4t}}{4}\right)\Bigg|_{\text{"}t \mapsto t-2\text{"}}$$

$$= u_2(t)\left(\frac{1}{4} - \frac{e^{-4(t-2)}}{4}\right)$$

Find $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+9)}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+9)}\right\} = u_2(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\}\Bigg|_{t \mapsto t-2}$$

Compute $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\}$

$$\frac{1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}$$

$$1 = A(s^2+9) + (Bs+C)s$$

$$1 = (A+B)s^2 + Cs + 9A$$

so $\quad 9A = 1 \qquad C = 0 \qquad A+B = 0$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/9}{s} \quad \frac{-1/9\, s}{s^2+9}\right\}$$

$$= \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}$$

$$= 1/9 - \frac{1}{9}\cos 3t$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+9)}\right\} = u_2(t)\left(\frac{1}{9} - \frac{1}{9}\cos 3t\right)\Bigg|_{t \mapsto t-2}$$

$$= \frac{u_2(t)}{9}\Big(1 - \cos 3(t-2)\Big)$$

Problem 2 Solve the IVP

$$y'' + y = g(t) \qquad g(t) = \begin{cases} e^{-t} & t<2 \\ 0 & 2<t \end{cases}$$

$$y(0) = 0 \qquad y'(0) = 0$$

Laplace Transforms

$$s^2 Y + Y = \mathcal{L}\{g\}$$

$$Y = \frac{1}{s^2+1} \mathcal{L}\{g\}$$

Compute $\mathcal{L}\{g\}$

$$g(t) = e^{-t}(1 - u_2(t))$$

$$\mathcal{L}\{g\} = \mathcal{L}\{e^{-t}\} - \mathcal{L}\{u_2(t)\, e^{-t}\}$$

version of Table Entry for Fwd Transform

$$\mathcal{L}\{u_c(t)\, y(t)\} = e^{-cs}\mathcal{L}\{y(t+c)\}$$

$$= \frac{1}{s+1} - e^{-2s}\mathcal{L}\{e^{-(t+2)}\}$$

$$= \frac{1}{s+1} - e^{-2s}\mathcal{L}\{e^{-2}e^{-t}\}$$

$$= \frac{1}{s+1} - e^{-2s}e^{-2}\mathcal{L}\{e^{-t}\}$$

$$= \frac{1}{s+1} - e^{-2s-2}\frac{1}{s+1}$$

$$s^2 Y + Y = \mathcal{L}\{g\}$$

$$Y = \frac{1}{s^2+1}\mathcal{L}\{g\}$$

$$\mathcal{L}\{g\} = \frac{1}{s+1} - e^{-2s-2}\frac{1}{s+1}$$

so

$$Y = \frac{1}{(s^2+1)(s+1)} - e^{-2} e^{-2s}\frac{1}{(s^2+1)(s+1)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} - e^{-2}\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{(s^2+1)(s+1)}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} - e^{-2} u_2(t)\, \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\}\Bigg|_{t=t-2}$$

Calculate $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\}$

$$\frac{1}{(s^2+1)(s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$1 = (s^2+1)A + (Bs+C)(s+1)$$

$$= (A+B)s^2 + (B+C)s + (A+C)$$

$A+B=0$ and $B+C=0$ and $A+C=1$

so $A=-B=C$ and $2A=1$

$$\frac{1}{(s^2+1)(s+1)} = \frac{1/2}{s+1} + \frac{-\frac{1}{2}s+\frac{1}{2}}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

$$= \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right)$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} - e^{-2}u_2(t)\,\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\}\Bigg|_{t=t-2}$$

$$y(t) = \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right) - e^{-2}u_2(t)\frac{1}{2}\left(e^{-(t-2)} - \cos(t-2) + \sin(t-2)\right)$$

1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$u_a(t)$	$\frac{e^{-as}}{s}$
$\delta_a(t)$	e^{-as}
$y'(t)$	$sY(s) - y(0)$
$y''(t)$	$s^2Y(s) - y(0)s - y'(0)$
$e^{at}y(t)$	$Y(s-a)$
$ty(t)$	$-\frac{d}{ds}Y(s)$
$u_a(t)y(t-a)$	$e^{-as}Y(s)$
$u_a(t)y(t)$	$e^{-as}\mathcal{L}\{y(t+a)\}$
$y(at)$	$\frac{1}{a}Y(\frac{s}{a})$

Note Title 11/25/2017

The transfer function

$$m\ddot{y} + \gamma\dot{y} + ky = f(t)$$

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

$$(ms^2 + \gamma s + k)Y(s) = F(s)$$

$$Y(s) = \frac{1}{ms^2 + \gamma s + k} \cdot F(s)$$

The Laplace transform of the solution $Y(s)$ is the Laplace transform of the forcing function $F(s)$ times $\frac{1}{ms^2+\gamma s+k}$ (this part comes from the DE)

$$Y(s) = G(s)F(s) \qquad G(s) = \frac{1}{ms^2 + \gamma s + k}$$

In control theory, signal processing, and engineering

$F(s)$ is called the input

$Y(s)$ is called the output or system response

$G(s)$ is called the transfer function

$g(t) = \mathcal{L}^{-1}\{G(s)\}$ is called the impulse response

Convolution Theorem

The solution to $m\ddot{y} + \gamma\dot{y} + ky = f(t)$

$y(0) = 0 \quad \dot{y}(0) = 0$

is $y(t) = \int_0^t g(t-\tau) f(\tau)\, d\tau$

$\int_0^t g(t-\tau) f(\tau)\, d\tau$ is called the convolution of g and f.

Example $\ddot{y} + 25y = \cos t$

$y(0) = 0 \quad \dot{y}(0) = 0$

$G = 1/(s^2 + 25)$

$g = \mathcal{L}^{-1}\{G\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\} = \frac{\sin 5t}{5}$

$$y(t) = \int_0^t \frac{\sin 5(t-\tau)}{5} \cos\tau \, d\tau$$

Example $\ddot{y} + 25y = f(t)$

$y(0) = 0 \quad \dot{y}(0) = 0$

$$G = s^2 + 25$$

$$g = \mathcal{L}^{-1}\{G\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\} = \frac{\sin 5t}{5}$$

$$y(t) = \int_0^t \sin 5(t-\tau)\, f(\tau)\, d\tau \;\text{''}= \frac{\sin 5t}{5} * f\;\text{''}$$

Special Case $f(t) = e^{-t}$

$$y(t) = \int_0^t \sin 5(t-\tau)\, e^{-\tau}\, d\tau$$

Convolution theorem in words

The response (of a linear time invariant system, i.e. a DE modelling a physical system) is the convolution of the input and the impulse response.

* I want you to be able to write down the convolution integral. We won't worry about evaluating the integral.

Full Example

(with integration)

$$\ddot{y} - 25y = e^t$$

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

$$G(s) = \frac{1}{s^2 - 25} = \frac{1/10}{s-5} - \frac{1/10}{s+5}$$

$$g(t) = \frac{e^{5t}}{10} - \frac{e^{-5t}}{10}$$

$$y(t) = g(t) * e^t = \int_0^t \frac{e^{5(t-\tau)} - e^{-5(t-\tau)}}{10} e^{\tau} d\tau$$

$$= \frac{1}{10}\left[\int_0^t e^{5(t-\tau)} e^{\tau} d\tau - \int_0^t e^{-5(t-\tau)} e^{\tau} d\tau\right]$$

$$= \frac{1}{10}\left[e^{5t}\int_0^t e^{-4\tau} d\tau - e^{-5t}\int_0^t e^{6\tau} d\tau\right]$$

$$= \frac{1}{10}\left[e^{5t} \cdot \left.\frac{e^{-4\tau}}{-4}\right|_0^t - e^{-5t} \left.\frac{e^{6\tau}}{6}\right|_0^t\right]$$

$$= \frac{1}{10}\left[e^{5t}\left(\frac{1}{4} - \frac{e^{-4t}}{4}\right) - e^{-5t}\left(\frac{e^{6t}}{6} - \frac{1}{6}\right)\right]$$

$$= \frac{e^{5t}}{40} - \frac{e^t}{40} + \frac{e^{-5t}}{60} - \frac{e^t}{60}$$

$$y(t) = -\frac{e^t}{24} + \frac{e^{5t}}{40} + \frac{e^{-5t}}{60}$$

Question

① What is the impulse responce of the system

$$\ddot{y} + 4\dot{y} + 5y$$

② Use the impulse response to write the solution to

$$\ddot{y} + 4\dot{y} + 5y = f(t)$$

$$y(0) = 0 \quad \dot{y}(0) = 0$$

Answer

The tranfer function is $G(s) = \dfrac{1}{s^2 + 4s + 5}$

$$= \frac{1}{(s+2)^2 + 1}$$

so the impulse response is

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{-2t} \sin t$$

②

$$y(t) = \int_0^t e^{-2(t-\tau)} \sin(t-\tau)\, f(\tau)\, d\tau$$

<u>Remark</u> - If $f(t)$ is a measured input, rather than a formula, ② can be evaluated numerically.

If $f(t)$ is given by a formula, then ② gives the same answer as the other methods we know.

Two Homework Problems on Convolution

Express the solution as a convolution integral:

① $y'' + \omega^2 y = 2e^{-t}\cos 2t$

$y(0) = 0 \quad y'(0) = 0$

② $y'' + 2y' + 2y = \sin t$

$y(0) = 0 \quad y'(0) = 0$

Answers

① $y(t) = \int_0^t \frac{\sin\omega(t-\tau)}{\omega} e^{-\tau}\cos 2\tau \, d\tau$

② $y(t) = \int_0^t e^{-(t-\tau)} \sin(t-\tau) \sin\tau \, d\tau$

The Dirac-delta function

Question Does the impulse response satisfy a DE?

or

$g(t)$ is the system response to what input?

Example

$$\ddot{y} + 25y \iff \text{system (e.g. mass-spring)}$$

$$G(s) = \frac{1}{s^2+25} = \text{transfer function}$$

$$g(t) = \frac{\sin 5t}{5} = \text{impulse response}$$

$$\ddot{g} + 25g = f(t) \qquad g(0) = 0 \quad \dot{g}(0) = 0$$

What is $f(t)$?

Its easy to see what $F(s)$ is.

$$s^2G(s) + 25G(s) = F(s)$$

$$G(s) = \frac{F(s)}{s^2+25}$$

so $F(s) = 1$

What function $f(t)$ has Laplace transform $= 1$?

Not Useful Answer - there is no such function?

Top Hat function comes close

$$h_\varepsilon(t) = \begin{cases} 1/\varepsilon & t \le \varepsilon \\ 0 & \varepsilon < t \end{cases}$$

$$\int_0^\infty e^{-st} h_\varepsilon(t)\, dt = 1/\varepsilon \int_0^\varepsilon e^{-st}\, dt$$

$$= \frac{1}{\varepsilon}\left(\frac{e^{-st}}{-s} \right)\Bigg|_0^\varepsilon = \frac{1 - e^{-\varepsilon s}}{\varepsilon s}$$

$$\mathcal{L}\{h_\varepsilon(t)\} = H_\varepsilon(s) = \frac{1 - e^{-\varepsilon s}}{\varepsilon s}$$

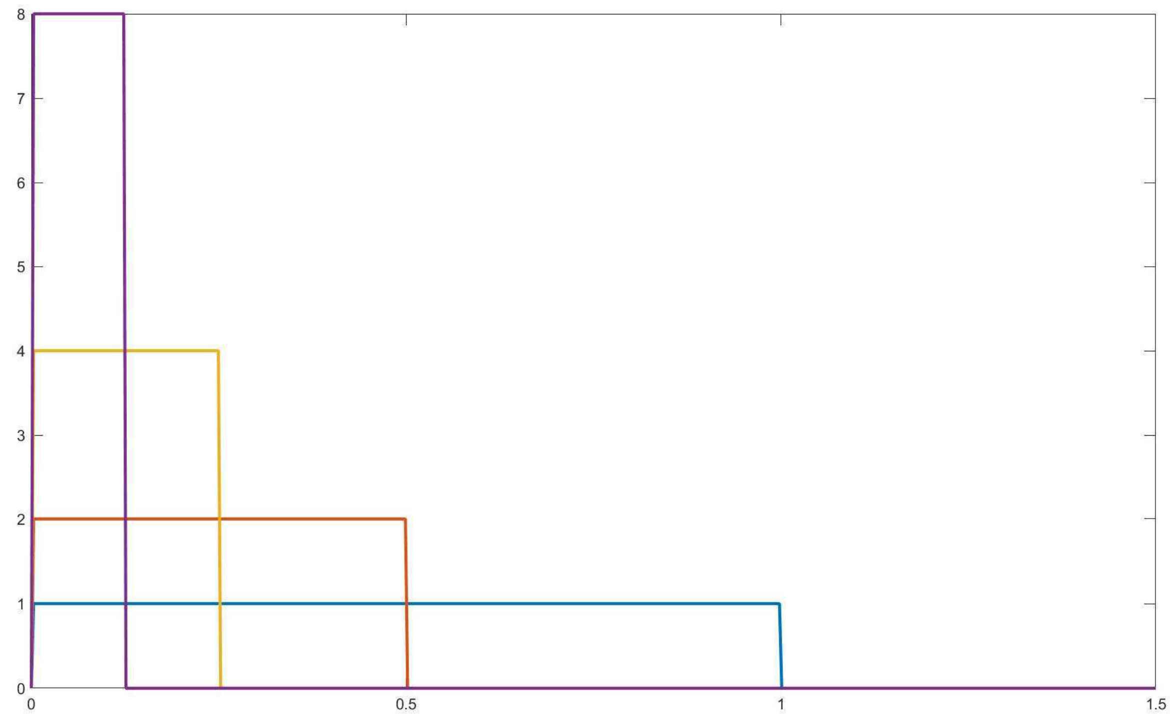
$$\lim_{\varepsilon \to 0^+} H_\varepsilon(s) = \text{“}\frac{0}{0}\text{”} = \frac{\frac{d}{d\varepsilon}\left(1 - e^{-\varepsilon s}\right)}{\frac{d}{d\varepsilon}\varepsilon s}$$

$$= \lim_{\varepsilon \to 0^+} \frac{\cancel{s}\, e^{-\varepsilon s}}{\cancel{s}}$$

$$= 1$$

$$\lim_{\varepsilon \to 0^+} H_\varepsilon(s) = 1$$

What about $\lim_{\varepsilon \to 0^+} h_\varepsilon(t)$?



$$\lim_{\varepsilon \to 0^+} h_\varepsilon(t) = \delta_0(t)$$

$\delta_0(t)$ is not a "real" function

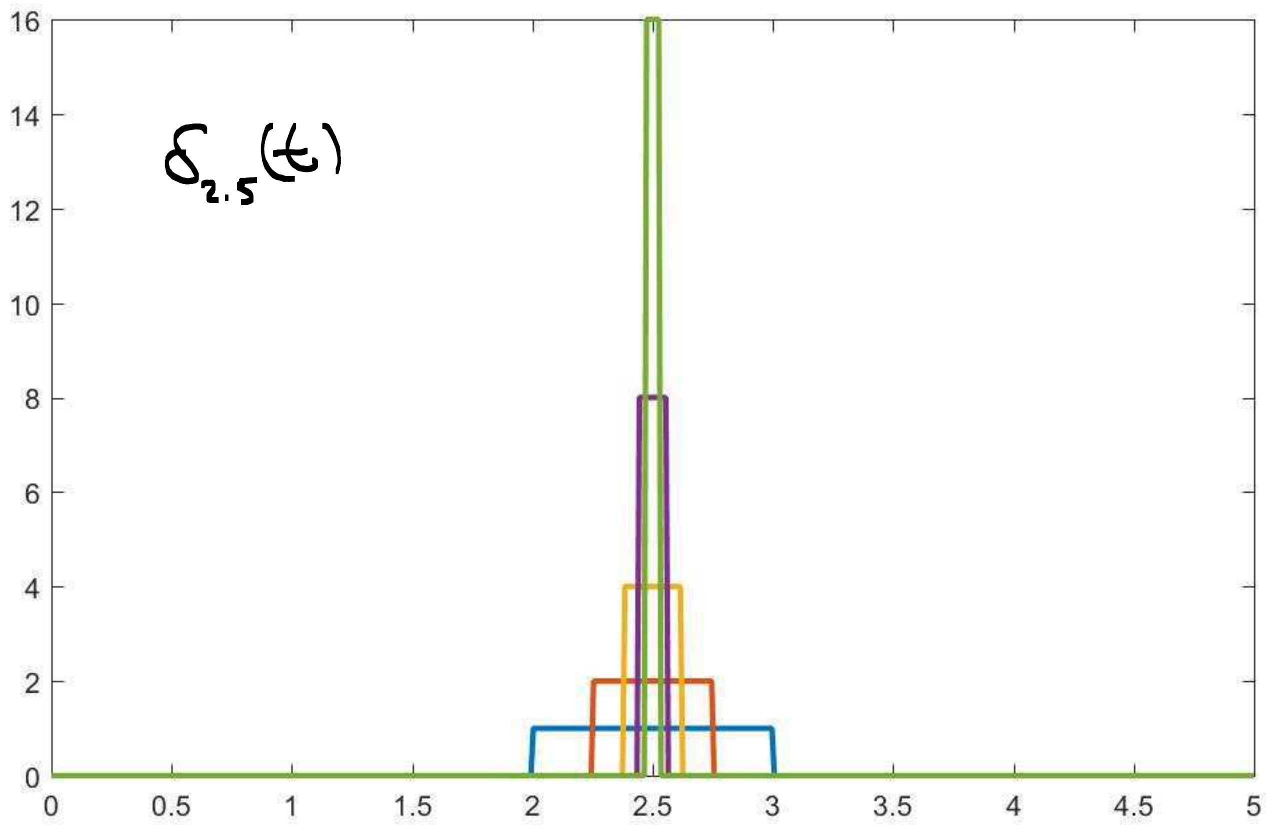
$$\delta_0(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \int_0^\infty \delta_0(t)\, dt = 1 \end{cases}$$

$$\delta_c(t) = \lim_{\varepsilon \to 0^+} h_{\varepsilon/2}(|t-c|)$$

$$\delta_c(t) = \delta_0(t-c)$$

$$\mathcal{L}\{\delta_c(t)\} = e^{-cs}\mathcal{L}\{\delta_0(t)\}$$

$$\mathcal{L}\{\delta_c(t)\} = e^{-cs}$$



You can stop here!! I am including some info below but won't test you on it.

The (Dirac) delta function was introduced by Paul Dirac to model a point mass

https://en.wikipedia.org/wiki/Dirac_delta_function

It was "anticipated" by Fourier in conjunction with the Fourier transform which is similar to the Laplace Transform

Interpretation of the delta

Physical - A force that is applied during a time interval so short that that your equipment can't resolve it, but delivers a unit impulse, i.e. $\int \delta(t)\,dt = 1$

Mathematical Definition of the delta

Fact - Suppose two functions f_1 and f_2 satisfy $\int f_1(t)\, g(t)\, dt = \int f_2(t)\, g(t)\, dt$ for every continuous function $g(t)$. Then $f_1(t) = f_2(t)$.

So we can identify a "generalized" function with how it integrates against conventional continuous functions.

The delta function is defined by

$$\int \delta_a(t)\, g(t)\, dt = g(a)$$

for every continuous function $g(a)$.

It can also be defined as the "derivative" of the Heaviside function.

$$\frac{d}{dt} u_a(t) = \delta_a(t)$$

Convolution and Laplace Transform

Thm $\mathcal{L}^{-1}\{G(s)\,F(s)\} = \int_0^t g(t-\tau)f(\tau)\,d\tau$

equivalently, [because Laplace transform is invertible]

$$\mathcal{L}\left\{\int_0^t g(t-\tau)f(\tau)\,d\tau\right\} = F(s)\,G(s)$$

Proof

$$F(s)\,G(s) = \int_0^\infty e^{-st}f(t)\,dt \int_0^\infty e^{-s\tau}g(\tau)\,d\tau$$

$$= \int_0^\infty \int_0^\infty e^{-s(t+\tau)} f(t)\,g(\tau)\,dt\,d\tau$$

Change Variabl in dt integral

$\omega = t+\tau$

$d\omega = dt$

$$= \int_0^\infty \int_\tau^\infty e^{-s\omega} f(\omega-\tau)\,g(\tau)\,d\omega \quad d\tau$$

Change the order of integration

$$= \int_0^\infty \int_0^t e^{-s\omega} f(\omega-\tau)\,g(\tau)\,d\tau\,d\omega$$

$$= \int_0^\infty e^{-s\omega}\left[\int_0^t f(\omega-\tau)\,g(\tau)\,d\tau\right] d\omega$$

$$= \mathcal{L}\left\{\int_0^t f(\omega-\tau)\,g(\tau)\,d\tau\right\}$$

$\blacksquare$

Convolution Theorem

The solution to
$$m\ddot{y} + \gamma\dot{y} + ky = f(t)$$
$$y(0) = 0 \quad \dot{y}(0) = 0$$

is
$$y(t) = \int_0^t g(t-\tau) f(\tau) d\tau$$

Proof

$$ms^2Y + \gamma sY + kY = F(s)$$

$$(ms^2 + \gamma s + k)Y = F(s)$$

$$Y(s) = \frac{1}{ms^2 + \gamma s + k} F(s)$$

$$Y(s) = G(s)F(s)$$

So
$$y(t) = \int_0^t g(t-\tau) f(\tau) d\tau$$