Note Title 11/25/2017

The transfer function

$$m\ddot{y} + \gamma\dot{y} + ky = f(t)$$

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

$$(ms^2 + \gamma s + k)Y(s) = F(s)$$

$$Y(s) = \frac{1}{ms^2 + \gamma s + k} \cdot F(s)$$

The Laplace transform of the solution $Y(s)$ is the Laplace transform of the forcing function $F(s)$ times $\frac{1}{ms^2+\gamma s+k}$ (this part comes from the DE)

$$Y(s) = G(s)F(s) \qquad G(s) = \frac{1}{ms^2+\gamma s+k}$$

In control theory, signal processing, and engineering

$F(s)$ is called the input

$Y(s)$ is called the output or system response

$G(s)$ is called the transfer function

$g(t) = \mathcal{L}^{-1}\{G(s)\}$ is called the impulse response

Convolution Theorem

The solution to $m\ddot{y} + \gamma\dot{y} + ky = f(t)$

$$y(0) = 0 \quad \dot{y}(0) = 0$$

is $y(t) = \int_0^t g(t-\tau) f(\tau)\, d\tau$

$\int_0^t g(t-\tau) f(\tau)\, d\tau$ is called the convolution of g and f.

Example $\ddot{y} + 25y = \cos t$

$$y(0) = 0 \quad \dot{y}(0) = 0$$

$$G = 1/(s^2 + 25)$$

$$g = \mathcal{L}^{-1}\{G\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\} = \frac{\sin 5t}{5}$$

$$y(t) = \int_0^t \frac{\sin 5(t-\tau)}{5} \cos\tau \, d\tau$$

Example

$$\ddot{y} + 25y = f(t)$$

$$y(0) = 0 \quad \dot{y}(0) = 0$$

$$G = s^2 + 25$$

$$g = \mathcal{L}^{-1}\{G\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\} = \frac{\sin 5t}{5}$$

$$y(t) = \int_0^t \sin 5(t-\tau)\, f(\tau)\, d\tau \;\text{"}=\text{"}\; \frac{\sin 5t}{5} * f \;\text{"}$$

Special Case $\quad f(t) = e^{-t}$

$$y(t) = \int_0^t \sin 5(t-\tau)\, e^{-\tau}\, d\tau$$

Convolution theorem in words

The response (of a linear time invariant system, i.e. a DE modelling a physical system) is the convolution of the input and the impulse response.

∓ I want you to be able to write down the convolution integral. We won't worry about evaluating the integral.

Full Example

(with integration)

$$\ddot{y} - 25y = e^{t}$$

$$y(0) = 0 \quad \dot{y}(0) = 0$$

$$G(s) = \frac{1}{s^2 - 25} = \frac{1/10}{s-5} - \frac{1/10}{s+5}$$

$$g(t) = \frac{e^{5t}}{10} - \frac{e^{-5t}}{10}$$

$$y(t) = g(t) * e^{t} = \int_0^t \frac{e^{5(t-\tau)} - e^{-5(t-\tau)}}{10} e^{\tau} d\tau$$

$$= \frac{1}{10}\left[\int_0^t e^{5(t-\tau)} e^{\tau} d\tau - \int_0^t e^{-5(t-\tau)} e^{\tau} d\tau\right]$$

$$= \frac{1}{10}\left[e^{5t}\int_0^t e^{-4\tau} d\tau - e^{-5t}\int_0^t e^{6\tau} d\tau\right]$$

$$= \frac{1}{10}\left[e^{5t} \cdot \left.\frac{e^{-4\tau}}{-4}\right|_0^t - e^{-5t} \left.\frac{e^{6\tau}}{6}\right|_0^t\right]$$

$$= \frac{1}{10}\left[e^{5t}\left(\frac{1}{4} - \frac{e^{-4t}}{4}\right) - e^{-5t}\left(\frac{e^{6t}}{6} - \frac{1}{6}\right)\right]$$

$$= \frac{e^{5t}}{40} - \frac{e^{t}}{40} + \frac{e^{-5t}}{60} - \frac{e^{t}}{60}$$

$$y(t) = -\frac{e^{t}}{24} + \frac{e^{5t}}{40} + \frac{e^{-5t}}{60}$$

Question

1. what is the impulse responce of the system

$$\ddot{y} + 4\dot{y} + 5y$$

2. Use the impulse response to write the solution to

$$\ddot{y} + 4\dot{y} + 5y = f(t)$$

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

Answer

The tranfer function is $G(s) = \dfrac{1}{s^2 + 4s + 5}$

$$= \frac{1}{(s+2)^2 + 1}$$

so the impulse response is

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{-2t} \sin t$$

2.
$$y(t) = \int_0^t e^{-2(t-\tau)} \sin(t-\tau)\, f(\tau)\, d\tau$$

Remark - If $f(t)$ is a measured input, rather than a formula, 2 can be evaluated numerically.

If $f(t)$ is given by a formula, then 2 gives the same answer as the other methods we know.

Two Homework Problems on Convolution

Express the solution as a convolution integral:

① $y'' + \omega^2 y = 2e^{-t}\cos 2t$

$y(0) = 0 \quad y'(0) = 0$

② $y'' + 2y' + 2y = \sin t$

$y(0) = 0 \quad y'(0) = 0$

Answers

① $y(t) = \int_0^t \frac{\sin \omega(t-\tau)}{\omega} e^{-\tau} \cos 2\tau \, d\tau$

② $y(t) = \int_0^t e^{-(t-\tau)} \sin(t-\tau) \sin \tau \, d\tau$

The Dirac-delta function

Question Does the impulse response satisfy a DE?

or

$g(t)$ is the system response to what input?

Example

$$\ddot{y} + 25y \Longleftrightarrow \text{system (e.g. mass-spring)}$$

$$G(s) = \frac{1}{s^2+25} = \text{transfer function}$$

$$g(t) = \frac{\sin 5t}{5} = \text{impulse response}$$

$$\ddot{g} + 25g = f(t) \qquad g(0) = 0 \quad \dot{g}(0) = 0$$

What is $f(t)$?

Its easy to see what $F(s)$ is.

$$s^2 G(s) + 25G(s) = F(s)$$

$$G(s) = \frac{F(s)}{s^2+25}$$

so $F(s) = 1$

What function $f(t)$ has Laplace transform $= 1$?

Not Useful Answer - there is no such function?

Top Hat Function comes close

$$h_\varepsilon(t) = \begin{cases} 1/\varepsilon & t \le \varepsilon \\ 0 & \varepsilon < t \end{cases}$$

$$\int_0^\infty e^{-st} h_\varepsilon(t)\, dt = \frac{1}{\varepsilon}\int_0^\varepsilon e^{-st}\, dt$$

$$= \frac{1}{\varepsilon}\left(\frac{e^{-st}}{-s}\right\Big|_0^\varepsilon = \frac{1 - e^{-\varepsilon s}}{\varepsilon s}$$

$$\mathcal{L}\{h_\varepsilon(t)\} = H_\varepsilon(s) = \frac{1 - e^{-\varepsilon s}}{\varepsilon s}$$

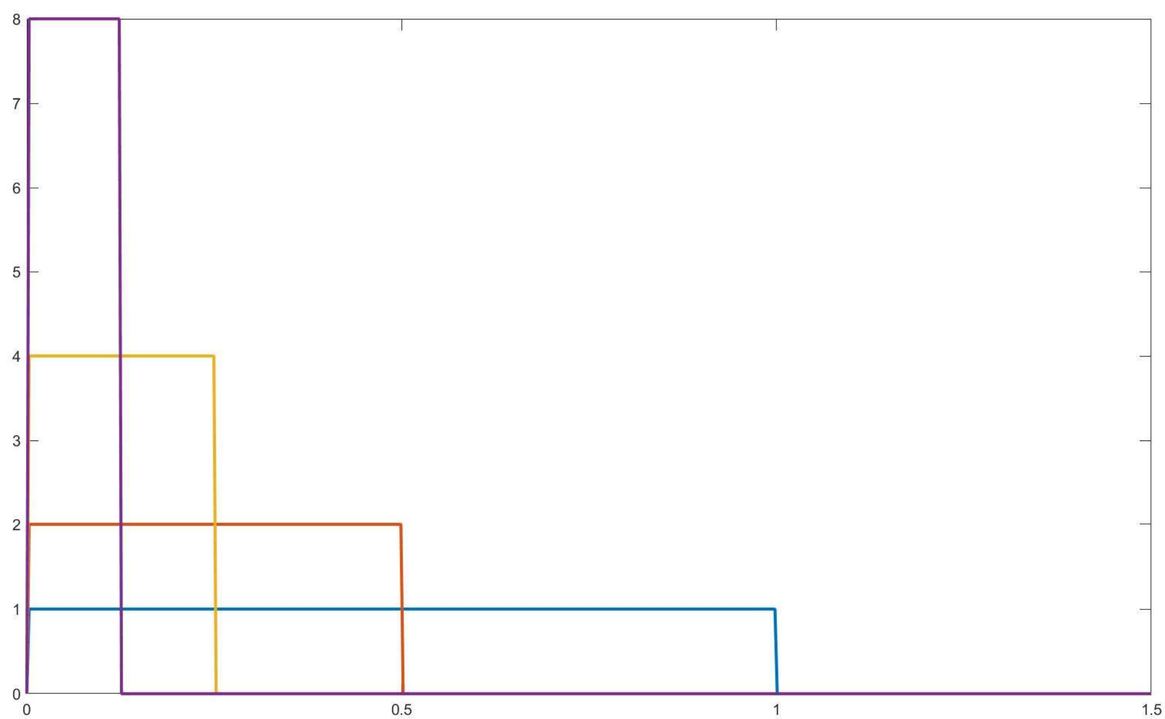
$$\lim_{\varepsilon \to 0^+} H_\varepsilon(s) = \text{"}\frac{0}{0}\text{"} = \frac{\frac{d}{d\varepsilon}\left(1 - e^{-\varepsilon s}\right)}{\frac{d}{d\varepsilon}\varepsilon s}$$

$$= \lim_{\varepsilon \to 0^+} \frac{\cancel{s}\, e^{-\varepsilon s}}{\cancel{s}}$$

$$= 1$$

$$\lim_{\varepsilon \to 0^+} H_\varepsilon(s) = 1$$

What about $\lim_{\varepsilon \to 0^+} h_\varepsilon(t)$?



$$\lim_{\varepsilon \to 0^+} h_\varepsilon(t) = \delta_0(t)$$

$\delta_0(t)$ is not a "real" function

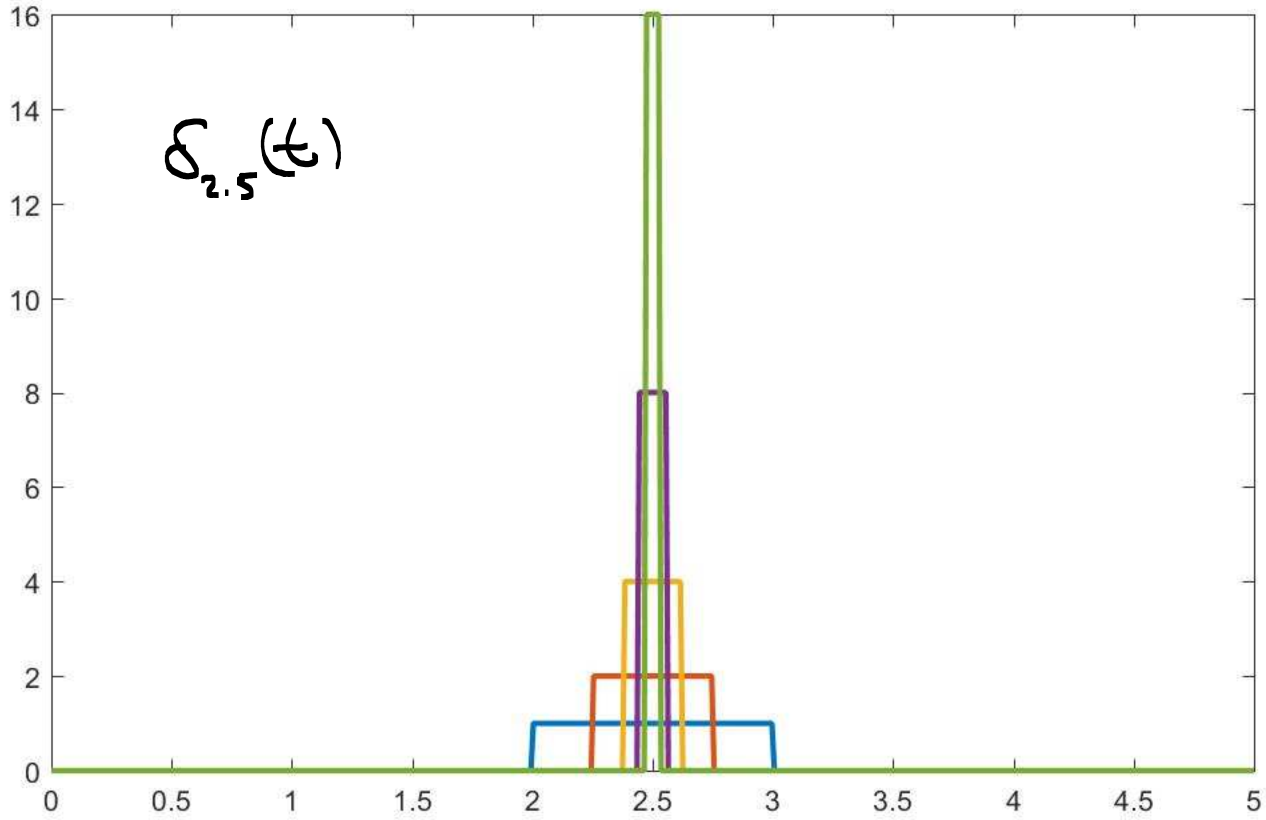
$$\delta_0(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \int_0^\infty \delta_0(t)\, dt = 1 & \end{cases}$$

$$\delta_c(t) = \lim_{\varepsilon \to 0^+} h_{\varepsilon/2}(|t-c|)$$

$$\delta_c(t) = \delta_0(t-c)$$

$$\mathcal{L}\{\delta_c(t)\} = e^{-cs}\mathcal{L}\{\delta_0(t)\}$$

$$\mathcal{L}\{\delta_c(t)\} = e^{-cs}$$



You can stop here!! I am including some info below but won't test you on it.

The (Dirac) delta function was introduced by Paul Dirac to model a point mass

https://en.wikipedia.org/wiki/Dirac_delta_function

It was "anticipated" by Fourier in conjunction with the Fourier transform which is similar to the Laplace Transform

Interpretation of the delta

Physical - A force that is applied during a time interval so short that that your equipment can't resolve it, but delivers a unit impulse, i.e. $\int \delta(t)\,dt = 1$

Mathematical Definition of the delta

Fact - Suppose two functions f_1 and f_2 satisfy $\int f_1(t)\,g(t)\,dt = \int f_2(t)\,g(t)\,dt$ for every continuous function $g(t)$. Then $f_1(t) = f_2(t)$.

So we can identify a "generalized" function with how it integrates against conventional continuous functions.

The delta function is defined by

$$\int \delta_a(t)\,g(t)\,dt = g(a)$$

for every continuous function $g(a)$.

It can also be defined as the "derivative" of the Heaviside function.

$$\frac{d}{dt} u_a(t) = \delta_a(t)$$

Convolution and Laplace Transform

Thm $\mathcal{L}^{-1}\{G(s)F(s)\} = \int_0^t g(t-\tau)f(\tau)\,d\tau$

equivalently, [because Laplace transform is invertible]

$$\mathcal{L}\left\{\int_0^t g(t-\tau)f(\tau)\,d\tau\right\} = F(s)\,G(s)$$

Proof

$$F(s)\,G(s) = \int_0^\infty e^{-st} f(t)\,dt \int_0^\infty e^{-s\tau} g(\tau)\,d\tau$$

$$= \int_0^\infty \int_0^\infty e^{-s(t+\tau)} f(t)\,g(\tau)\,dt\,d\tau$$

Change Variabl in dt integral

$\omega = t+\tau$

$d\omega = dt$

$$= \int_0^\infty \int_\tau^\infty e^{-s\omega} f(\omega-\tau)\,g(\tau)\,d\omega \quad d\tau$$

Change the order of integration

$$= \int_0^\infty \int_0^t e^{-s\omega} f(\omega-\tau)\,g(\tau)\,d\tau\,d\omega$$

$$= \int_0^\infty e^{-s\omega}\left[\int_0^t f(\omega-\tau)\,g(\tau)\,d\tau\right] d\omega$$

$$= \mathcal{L}\left\{\int_0^t f(\omega-\tau)\,g(\tau)\,d\tau\right\}$$

$\blacksquare$

Convolution Theorem

The solution to
$$m\ddot{y} + \gamma\dot{y} + ky = f(t)$$
$$y(0) = 0 \quad \dot{y}(0) = 0$$

is
$$y(t) = \int_0^t g(t-\tau)\, f(\tau)\, d\tau$$

Proof

$$ms^2 Y + \gamma s Y + kY = F(s)$$

$$(ms^2 + \gamma s + k)Y = F(s)$$

$$Y(s) = \frac{1}{ms^2 + \gamma s + k} F(s)$$

$$Y(s) = G(s)\, F(s)$$

So
$$y(t) = \int_0^t g(t-\tau)\, f(\tau)\, d\tau$$