

Lecture 26

Note Title

11/29/2017

Inverse Laplace Transforms involving $u_c(t)$

Table Entry

use for inverse $u_c(t) y(t-c) \quad \bar{e}^{-cs} Y(s)$

use for fwd ^{or} $\mathcal{L}\{u_c(t) y(t)\} = \bar{e}^{-cs} \mathcal{L}\{y(t+c)\}$

Problem - find $\mathcal{L}^{-1}\left\{\frac{\bar{e}^{-2s}}{s^2}\right\}$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\bar{e}^{-2s} \frac{1}{s^2}\right\} &= u_2(t) \left[\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \Big|_{t \mapsto t-2} \right] \\ &= u_2(t) (t \mid_{t \mapsto t-2}) \\ &= u_2(t) (t-2)\end{aligned}$$

$$\text{Find } \mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s(s+4)} \right\}$$

$$\mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s(s+4)} \right\} = u_2(t) \mathcal{L}^{-1} \left\{ \frac{1}{s(s+4)} \right\} \quad \left| \begin{array}{l} \text{"}t \mapsto t-2\text{"} \end{array} \right.$$

$$\text{Step 1} \quad \text{Compute } \mathcal{L}^{-1} \left\{ \frac{1}{s(s+4)} \right\}$$

$$\frac{1}{s(s+4)} = \frac{1/4}{s} - \frac{1/4}{s+4}$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{s(s+4)} \right\} = \frac{1}{4} \cdot 1 - \frac{1}{4} e^{-4t}$$

$$\mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s(s+4)} \right\} = u_2(t) \mathcal{L}^{-1} \left\{ \frac{1}{s(s+4)} \right\} \quad \left| \begin{array}{l} \text{"}t \mapsto t-2\text{"} \end{array} \right.$$

$$\mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s(s+4)} \right\} = u_2(t) \left(\frac{1}{4} - \frac{e^{-4t}}{4} \right) \quad \left| \begin{array}{l} \text{"}t \mapsto t-2\text{"} \end{array} \right.$$

$$= u_2(t) \left(\frac{1}{4} - \frac{e^{-4(t-2)}}{4} \right)$$

Find $\mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s(s^2+9)} \right\}$

$$\mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s(s^2+9)} \right\} = u_2(t) \mathcal{L}^{-1} \left\{ \frac{1}{s(s^2+9)} \right\} \Big|_{t \mapsto t-2}$$

Compute $\mathcal{L}^{-1} \left\{ \frac{1}{s(s^2+9)} \right\}$

$$\frac{1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}$$

$$1 = A(s^2+9) + (Bs+C)s$$

$$1 = (A+B)s^2 + Cs + 9A$$

so $9A=1$ $C=0$ $A+B=0$

$$\mathcal{L}^{-1} \left\{ \frac{1}{s(s^2+9)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1/9}{s} - \frac{1/9 s}{s^2+9} \right\}$$

$$= \frac{1}{9} \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} - \frac{1}{9} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+9} \right\}$$

$$= \frac{1}{9} - \frac{1}{9} \cos 3t$$

$$\mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s(s^2+9)} \right\} = u_2(t) \left(\frac{1}{9} - \frac{1}{9} \cos 3t \right) \Big|_{t \mapsto t-2}$$

$$= \frac{u_2(t)}{9} \left(1 - \cos 3(t-2) \right)$$

Problem 2 Solve the IVP

$$y'' + y = g(t)$$

$$y(0) = 0 \quad y'(0) = 0$$

$$g(t) = \begin{cases} e^{-t} & t < 2 \\ 0 & 2 < t \end{cases}$$

Laplace Transforms

$$s^2 Y + Y = \mathcal{L}\{g\}$$

$$Y = \frac{1}{s^2+1} \mathcal{L}\{g\}$$

Compute $\mathcal{L}\{g\}$

$$g(t) = e^{-t} (1 - u_2(t))$$

$$\mathcal{L}\{g\} = \mathcal{L}\{e^{-t}\} - \mathcal{L}\{u_2(t) e^{-t}\}$$

Version of Table
Entry for Fwd
Transform

$$\mathcal{L}\{u_c(t) y(t)\} = e^{-cs} \mathcal{L}\{y(t+c)\}$$

$$= \frac{1}{s+1} - e^{-2s} \mathcal{L}\{e^{-(t+2)}\}$$

$$= \frac{1}{s+1} - e^{-2s} \mathcal{L}\{e^{-2} e^{-t}\}$$

$$= \frac{1}{s+1} - e^{-2s} e^{-2} \mathcal{L}\{e^{-t}\}$$

$$= \frac{1}{s+1} - e^{-2s-2} \frac{1}{s+1}$$

$$s^2 Y + Y = \mathcal{L}\{g\}$$

$$Y = \frac{1}{s^2+1} \mathcal{L}\{g\}$$

$$\mathcal{L}\{g\} = \frac{1}{s+1} - e^{-2s-2} \frac{1}{s+1}$$

so

$$Y = \frac{1}{(s^2+1)(s+1)} - e^{-2} e^{-2s} \frac{1}{(s^2+1)(s+1)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} - e^{-2} \mathcal{L}^{-1}\left\{e^{-2s} \frac{1}{(s^2+1)(s+1)}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} - e^{-2} u_2(t) \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} \Big|_{t=t-2}$$

Calculate $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\}$

$$\frac{1}{(s^2+1)(s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$1 = (s^2+1)A + (Bs+C)(s+1)$$

$$= (A+B)s^2 + (B+C)s + (A+C)$$

$$A+B=0 \text{ and } B+C=0 \text{ and } A+C=1$$

$$\text{so } A=-B=C \text{ and } 2A=1$$

$$\frac{1}{(s^2+1)(s+1)} = \frac{1/2}{s+1} + \frac{-1/2s+1/2}{s^2+1}$$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ &= \frac{1}{2}(e^{-t} - \cos t + \sin t)\end{aligned}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} - e^{-2}u_2(t) \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s+1)}\right\} \Big|_{t=t-2}$$

$$y(t) = \frac{1}{2}(e^{-t} - \cos t + \sin t) - e^{-2}u_2(t) \frac{1}{2}(e^{-(t-2)} - \cos(t-2) + \sin(t-2))$$

1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$u_a(t)$	$\frac{e^{-as}}{s}$
$\delta_a(t)$	e^{-as}
$y'(t)$	$sY(s) - y(0)$
$y''(t)$	$s^2Y(s) - y(0)s - y'(0)$
$e^{at}y(t)$	$Y(s - a)$
$ty(t)$	$-\frac{d}{ds}Y(s)$
$u_a(t)y(t - a)$	$e^{-as}Y(s)$
$u_a(t)y(t)$	$e^{-as}\mathcal{L}\{y(t + a)\}$
$y(at)$	$\frac{1}{a}Y\left(\frac{s}{a}\right)$