Note Title 11/11/2017

One More Example

$$y'' + 9y = \cos 2t \qquad y(0)=0 \quad y'(0)=0$$

Laplace Transform

$$s^2 Y + 9Y = \frac{s}{s^2+4}$$

$$(s^2+1)\,Y = \frac{s}{s^2+4}$$

$$Y = \frac{s}{(s^2+9)(s^2+4)}$$

Inverse Laplace Transform

Partial Fractions

$$\frac{s}{(s^2+9)(s^2+4)} = \frac{As+B}{(s^2+9)} + \frac{Cs+D}{(s^2+4)}$$

$$s = (As+B)(s^2+4) + (Cs+D)(s^2+9)$$

$$= As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2 + 9Cs + 9D$$

$$s = (A+C)s^3 + (B+D)s^2 + (4A+9C)s + (4B+9D)$$

$$\underline{0}\,s^3 + \underline{0}\,s^2 + \underline{1}\,s + 0 =$$

$$s = (A+C)s^3 + (B+D)s^2 + (4A+9C)s + (4B+9D)$$

$$A+C=0 \Rightarrow A=-C$$

$$B+D=0 \Rightarrow B=-D$$

$$4B+9D=0 \Rightarrow -4D+9D=0 \Rightarrow D=0 \Rightarrow B=0$$

$$4A+9C=1 \Rightarrow -4C+9C=1 \Rightarrow C=\frac{1}{5} \Rightarrow A=-\frac{1}{5}$$

$$Y(s) = \frac{-1}{5}\,\frac{s}{s^2+9} + \frac{1}{5}\,\frac{s}{s^2+4}$$

$$y(t) = \frac{-1}{5}\cos 3t + \frac{1}{5}\cos 2t$$

Alternatively

$$\frac{s}{(s^2+9)(s^2+4)} = s\left(\frac{1}{(s^2+9)(s^2+4)}\right) =$$

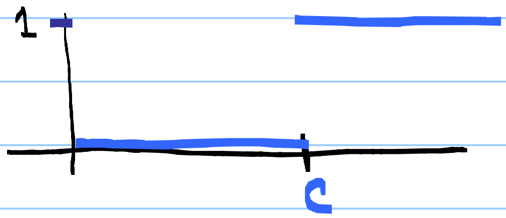
$$= s\left(\frac{A}{(s^2+9)} + \frac{C}{(s^2+4)}\right)$$

why

$$= s\left(\frac{-1/5}{s^2+9} + \frac{1/5}{s^2+4}\right)$$

The Heaviside Function and Time Delay

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$



Multiplying a function by $u_c(t)$ turns it on at time $t=c$. Multiplying by $(1-u_c(t))$ turns it off at $t=c$.

Example

$$f(t) = \begin{cases} t & 0 \leq t \leq 1 \\ e^{(t-1)} & 1 < t \end{cases}$$

f equals t until time 1, then it becomes e^{t-1}

$$f(t) = \underbrace{t\cdot(1-u_1(t))}_{\text{turn off } t} + \underbrace{e^{(t-1)}u_1(t)}_{\text{turn on } e^{(t-1)}}$$

Example

$$f(t) = \begin{cases} 6 & 0 \le t \le 1 \\ e^t & 1 < t \le 2 \\ \frac{t-1}{2} & 2 < t \le 3 \\ 4 & 3 < t \end{cases}$$

$$f(t) = 6\left(1-u_1(t)\right) + e^t\left(u_1(t) - u_2(t)\right) + \left(\tfrac{t-1}{2}\right)\left(u_2(t) - u_3(t)\right) + 4\,u_3(t)$$

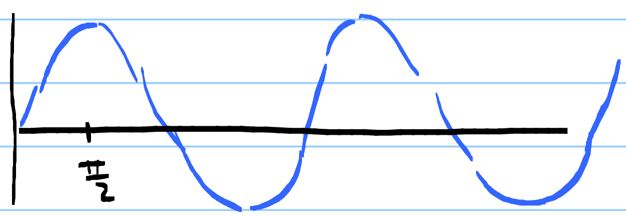
Turn on the 6 term; Turn it off; Turn on the e^t term; Turn off the e^t term; Turn on; Turn off; Turn on 1

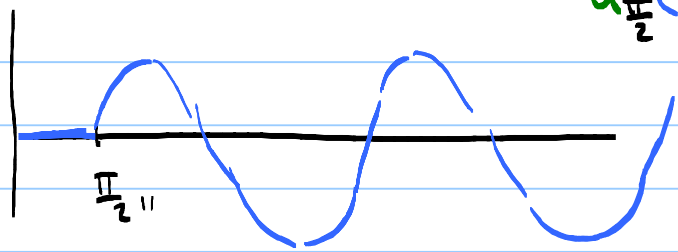
$$1-u_1(t) = \begin{cases} 1 & t<1 & \text{on} \\ 0 & t>1 & \text{off} \end{cases}$$

$$\left(u_1(t) - u_2(t)\right) = \begin{cases} 0 & t<1 & \text{off} \\ 1 & 1<t<2 & \text{on} \\ 0 & 2<t & \text{off} \end{cases}$$

Interpretation of u_c as a delay

sin t





$u_{\frac{\pi}{2}}(t)\sin\left(t-\frac{\pi}{2}\right)$ = "sin t delayed by $\frac{\pi}{2}$"

$$= \begin{cases} 0 & t<\frac{\pi}{2} \\ \sin\left(t-\frac{\pi}{2}\right) & \frac{\pi}{2}<t \end{cases}$$

Laplace Transform of u_c

$$\mathcal{L}\{u_c(t)\,y(t-c)\} = e^{-cs}\,Y(s)$$ ⟸ use this one for calculating inverse transform

Alternatively,

$$\mathcal{L}\{u_c(t)\,y(t)\} = e^{-cs}\,\mathcal{L}\{y(t+c)\}$$ ⟸ use this one for calculating transform

Calculating $\mathcal{L}\{u_c(t)\}$

$$\mathcal{L}\{u_c(t)\,y(t)\} = \int_0^\infty e^{-st}\,u_c(t)\,y(t)\,dt$$

$$= \int_c^\infty e^{-st}\,y(t)\,dt$$

Change variables $t = \tau + c$

$$= \int_0^\infty e^{-s(\tau+c)}\,y(\tau+c)\,d\tau$$

$$= e^{-sc}\int_0^\infty e^{-s\tau}\,y(\tau+c)\,d\tau$$

(1) $$\mathcal{L}\{u_c(t)\,y(t)\} = e^{-sc}\,\mathcal{L}\{y(t+c)\}$$

To see the other formulation $\left(\mathcal{L}\{u_c(t)\,y(t-c)\} = e^{-cs}\,Y(s)\right)$

Define a new function $h(t) = y(t-c)$

then $y(t) = h(t+c)$ and (1) becomes

$$\mathcal{L}\{u_c(t)\,h(t+c)\} = e^{-sc}\,\mathcal{L}\{h(t)\} = e^{-cs}\,H(s)$$

Example 1

$$f(t) = \begin{cases} 0 & 0 \le t \le 1 \\ e^{(t-1)} & 1 < t \end{cases} \qquad f(t) = u_1(t)\, e^{(t-1)}$$

$$F(s) = \mathcal{L}\{u_1(t)\, e^{(t-1)}\} = e^{-1s}\, \mathcal{L}\{e^{t}\} = e^{-s} \cdot \frac{1}{s-1}$$

Example 2

$$f(t) = \begin{cases} 0 & 0 \le t \le 2 \\ e^{t} & 2 < t \end{cases} \qquad f(t) = u_2(t)\, e^{t}$$

$$F(s) = \mathcal{L}\{u_2(t)\, e^{t}\} = e^{-2s}\, \mathcal{L}\{e^{t+2}\}$$

$$= e^{-2s}\, \mathcal{L}\{e^{2} \cdot e^{t}\}$$

$$= e^{-2s} \cdot e^{2}\, \mathcal{L}\{e^{t}\}$$

$$= e^{-2s+2} \cdot \frac{1}{s-1}$$

Example 3

$$f(t) = \begin{cases} \sin t & 0 < t < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < t \end{cases}$$

$$f(t) = \sin t - u_{\pi/2}(t)\sin t$$

$$\mathcal{L}\{f\} = \mathcal{L}\{\sin t\} - \mathcal{L}\{u_{\pi/2}(t)\sin t\}$$

$$= \frac{1}{s^2+1} - e^{-\frac{\pi}{2}s}\mathcal{L}\{\sin(t+\tfrac{\pi}{2})\}$$

$$= \frac{1}{s^2+1} - e^{-\frac{\pi}{2}s}\mathcal{L}\{\cos t\}$$

$$= \frac{1}{s^2+1} - e^{-\frac{\pi}{2}s}\left(\frac{s}{s^2+1}\right)$$