lecture

Note Title 3/4/2020

Inhomogeneous DE's via Laplace Transform

Solve using Laplace Transform

$$\ddot{y} + y = e^{-2t}$$

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

Solution

$$\mathcal{L}\{\ddot{y} + y\} = \mathcal{L}\{e^{-2t}\}$$

$$\mathcal{L}\{\ddot{y}\} + \mathcal{L}\{y\} = \frac{1}{s+2}$$

$$s^2 Y - s\,y(0) - \dot{y}(0) + Y = 1/(s+2)$$

$$s^2 Y - 0 - 0 + Y = \frac{1}{s+2}$$

$$(s^2+1)\,Y = \frac{1}{s+2}$$

$$Y = \frac{1}{(s+2)(s^2+1)}$$

Find $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s^2+1)}\right\}$ - Partial Fractions

General Partial Fractions

$$\frac{Q(s)}{P(s)}$$ degree Q < degree P

P(s) factors $P(s) = P_1(s)\,P_2(s)$

In General

$$\frac{Q(s)}{P_1(s)\,P_2(s)} = \frac{Q_1}{P_1(s)} + \frac{Q_2}{P_2(s)}$$

$\deg Q_1 = \deg P_1 - 1$ $\deg Q_2 = \deg P_2 - 1$

IF P_1 and P_2 have no common factor.

Example:

degree < 3 degree 0 degree 1

$$\frac{1}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+2}$$

degree 3 degree 1 polynomial degree 2 polynomial

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degree < 3 Example

$$\frac{6s^2+3s+4}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+2}$$

degree 3

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degree < 3 Bad Example

$$\frac{6s^2+3s+4}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs}{s^2+2}$$

degree 3

Not good enough!!
Need most general degree 1 polynomial
Bs + C

Examples

$$\frac{1}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4} \quad \checkmark$$

$$\frac{6s^3+4s^2+3s+6}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4} \quad \checkmark$$

$$\frac{1}{(s+3)^2(s+2)^2} = \frac{As+B}{(s+3)^2} \quad \frac{Cs+D}{(s+2)^2}$$

Correct, but not as useful as the one below

$$\frac{1}{(s+3)^2(s+2)^2} = \frac{A}{s+3} + \frac{\tilde{B}}{(s+3)^2} + \frac{C}{s+2} + \frac{\tilde{D}}{(s+2)^2}$$

If you check $\tilde{B} = B-3A$
$\tilde{D} = D-2C$

We use this one because $\frac{1}{s+3}$, $\frac{1}{(s+3)^2}$, $\frac{1}{s+2}$, $\frac{1}{(s+2)^2}$ are easier to find in the transform table.

Example

$$\frac{1}{(s+1)(s+2)(s+3)(s+4)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} + \frac{D}{s+4}$$

Back to Initial Value Problem on page 1

Find $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s^2+1)}\right\}$ - Partial Fractions

$$\frac{1}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+1}$$

Clear denominators and write as polynomial equality

$$1 = A(s^2+1) + (Bs+C)(s+2)$$

$$1 = As^2 + A + Bs^2 + Cs + 2Bs + 2C$$

$$1 = (A+B)s^2 + (C+2B)s + (A+2C)$$

$$0s^2 + 0s + 1 = (A+B)s^2 + (C+2B)s + (A+2C)$$

$A + 2C = 1 \qquad A + B = 0 \qquad C + 2B = 0$

Eliminate A

$$\begin{array}{rl} & A + 2C = 1 \\ - & A + B = 0 \\ \hline & 2C - B = 1 \end{array}$$

Combine with $C + 2B = 0 \Rightarrow C = -2B$

$$2C - B = 1 \Rightarrow -4B - B = 1 \Rightarrow B = -\frac{1}{5}$$

$$A = -B = \frac{1}{5} \text{ and } C = -2B = \frac{2}{5}$$

$$\frac{1}{(s+2)(s^2+1)} = \frac{1/5}{s+2} + \frac{-1/5\, s + 2/5}{s^2+1}$$

$$= 1/5 \frac{1}{s+2} - 1/5 \frac{s}{s^2+1} + \frac{2}{5} \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s^2+1)}\right\} = 1/5\, \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - 1/5\, \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{2}{5} \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\boxed{y(t) = 1/5\, e^{-2t} - 1/5 \cos t + \frac{2}{5} \sin t}$$

How to find the constants?

Example

$$\frac{1}{(s-1)(s-2)(s-3)} = \frac{A_1}{(s-1)} + \frac{A_2}{s-2} + \frac{A_3}{(s-3)}$$

$$1 = A_1(s-2)(s-3) + A_2(s-1)(s-3) + A_3(s-1)(s-2)$$

Always works, but lots of calculating

$$1 = (A_1 + A_2 + A_3)s^2 + (-5A_1 - 4A_2 - 3A_3)s + (6A_1 + 4A_2 + 2A_3)$$

$$A_1 + A_2 + A_3 = 0$$
$$-5A_1 - 4A_2 - 3A_3 = 0$$
$$6A_1 + 4A_2 + 7A_3 = 1$$

Three equations in three unknowns
Solve them

Faster, especially for simple roots

$$1 = A_1(s-2)(s-3) + A_2(s-1)(s-3) + A_3(s-1)(s-2)$$

Set $s = 1$

$$1 = A_1(1-2)(1-3) + 0 + 0$$

$$\boxed{\frac{1}{2} = A_1}$$

Set $s = 2$

$$1 = 0 + A_2(2-1)(2-3)$$

$$\boxed{-1 = A_2}$$

Set $s = 3$

$$1 = 0 + 0 + A_3(3-1)(3-2)$$

$$\boxed{\frac{1}{2} = A_3}$$

Example

$$\frac{s+2}{(s-1)(s-2)^2} = \frac{A_1}{s-1} + \frac{A_2}{s-2} + \frac{B_2}{(s-2)^2}$$

$$s+2 = A_1(s-2)^2 + A_2(s-2)(s-1) + B_2(s-1)$$

Set $s=1$

$$1+2 = A_1(1-2)^2 + 0 + 0$$

$$\boxed{3 = A_1}$$

Set $s=2$

$$2+2 = 0 + 0 + B_2(2-1)$$

$$\boxed{4 = B_2}$$

set $s=0$ (or anything else

$$0+2 = 3(0-2)^2 + A_2(0-2)(0-1) + 4(0-1)$$

$$2-12+4 = 2A_2$$

$$\boxed{7 = A_2}$$

Example

$$\frac{1}{(s-1)(s^2+4)} = \frac{A_1}{s-1} + \frac{A_2 s + B_2}{s^2+4}$$

$$1 = A_1(s^2+4) + (A_2 s + B_2)(s-1)$$

set $s = 1$ $\boxed{A_1 = \frac{1}{5}}$

Now multiply it out

$$1 = (A_1 + A_2)s^2 + (B_2 - A_2)s + (4A_1 - B_2)$$

$A_1 + A_2 = 0$ so $\boxed{A_2 = -\frac{1}{5}}$

$B_2 - A_2 = 0$ so $\boxed{B_2 = -1/5}$

$4\left(\frac{1}{5}\right) - \left(\frac{-1}{5}\right) = 1$ ✓