Table of Laplace Transforms

Note Title 11/11/2017

General	$\frac{dy}{dt}$	$sY(s) - y(0)$
	$\frac{d^2y}{dt^2}$	$s^2Y(s) - s\,y(0) - y'(0)$
	$e^{at}\,y(t)$	$Y(s-a)$
	$t\,y(t)$	$-\frac{d}{ds}Y(s)$
Specific	e^{at}	$1/(s-a)$
	$\sin \omega t$	$\omega/(s^2+\omega^2)$
	$\cos \omega t$	$s/(s^2+\omega^2)$
	t^n	$\frac{n!}{s^{n+1}}$

Inverse Laplace Transform

There is a formula ($Y(s) = \int_0^\infty e^{-st} y(t)\,dt$) to compute the Laplace Transform, but there is no comparable formula for the Inverse Laplace Transform. We compute the inverse Laplace transform using the table above.

https://en.wikipedia.org/wiki/Post's_inversion_formula

https://www.rose-hulman.edu/~bryan/invlap.pdf

Table of Laplace Transforms

General

$\frac{dy}{dt}$	$sY(s) - y(0)$
$\frac{d^2y}{dt^2}$	$s^2Y(s) - sy(0) - y'(0)$
$e^{at}y(t)$	$Y(s-a)$
$t\,y(t)$	$-\frac{d}{ds}Y(s)$

$$\boxed{\begin{aligned} e^{at}y(t) &\quad Y(s-a) \\ e^{at}y(t) &= \mathcal{L}^{-1}\{Y(s-a)\} \\ y(t) &= \mathcal{L}^{-1}\{Y(s)\} \end{aligned}}$$

Specific

e^{at}	$1/(s-a)$
$\sin\omega t$	$\omega/(s^2+\omega^2)$
$\cos\omega t$	$s/(s^2+\omega^2)$
t^n	$\frac{n!}{s^{n+1}}$
1	$\frac{1}{s}$

Inverse Laplace Transform

Examples

Compute $\mathcal{L}^{-1}\{y(t)\}$ for each of the following

$$Y(s) = \frac{1}{s-4}$$

$$Y(s) = \frac{1}{(s-4)^2}$$

$$Y(s) = \frac{3s+4}{s^2+9}$$

$$Y(s) = \frac{1}{s^2+2s+5}$$

$$Y(s) = \frac{3s+4}{s^2+2s+5}$$

$$Y(s) = \frac{1}{s^2+2s-8}$$

$$Y(s) = \frac{1}{s-4}$$

This is $\frac{1}{s-a}$ with $a=4$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = e^{4t}$$

Alternatively

Table Entry

$$\left.\begin{aligned} \mathcal{L}^{-1}\{Y(s)\} &= y(t) \\ \mathcal{L}^{-1}\{Y(s-a)\} &= e^{at}y(t) \end{aligned}\right\} \iff e^{at}y(t) \qquad Y(s-a)$$

$$Y(s) = \frac{1}{s-4}$$

$$Y(s+4) = \frac{1}{s}$$

$$\mathcal{L}^{-1}\{Y(s+4)\} = 1$$

$$e^{-4t}y(t) = 1$$

$$y(t) = e^{4t}$$

$$Y(s) = \frac{1}{(s-4)^2}$$

$$\frac{1}{(s-4)^2} \text{ " }= \left.\frac{1}{s^2}\right|_{s \mapsto s-4} \text{ "}$$

" $= \frac{1}{s^2}$ with s replaced by $(s-4)$ "

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\} = e^{4t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^{4t}\cdot t$$

Alternatively,

$$Y(s+4) = \frac{1}{s^2}$$

This one is in table

$$\mathcal{L}^{-1}\{Y(s+4)\} = t$$

Now use $e^{at}y(t) = \mathcal{L}^{-1}\{Y(s-a)\}$

$$e^{-4t}y(t) = t$$

$$\boxed{y(t) = t\,e^{4t}}$$

$$Y(s) = \frac{1}{(s-4)^2}$$

This is minus the derivative of $\frac{1}{s-4}$

$$\frac{d}{ds}\frac{1}{s-4} = \frac{-1}{(s-4)^2}$$

so

$$\frac{1}{(s-4)^2} = -\frac{d}{ds}\frac{1}{(s-4)}$$

so

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^2}\right\} = t\,\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = te^{4t}$$

$$Y(s) = \frac{5s+4}{s^2+9}$$

Two tabe entries

$$\cos 3t \qquad \frac{s}{s^2+9}$$

$$\sin 3t \qquad \frac{3}{s^2+9}$$

$$Y(s) = 5 \cdot \frac{s}{s^2+9} + \frac{4}{3} \cdot \frac{3}{s^2+9}$$

$$\mathcal{L}^{-1}\{Y\} = 5\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{4}{3}\,\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$\boxed{y(t) = 5\cos 3t + \frac{4}{3}\sin 3t}$$

$$Y(s) = \frac{1}{s^2+2s+5}$$

What are the roots?

$$s^2+2s+5 = 0$$

$$s^2+2s+1 = -4$$

$$(s+1)^2 = -4$$

$$s = -1 \pm 4i$$

Complex Roots - Complete the square

$$Y(s) = \frac{1}{(s+1)^2+4} = \text{“}\left.\frac{1}{s^2+4}\right|_{s \to s+1}\text{”}$$

$$\mathcal{L}^{-1}\{Y\} = e^{-t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{e^{-t}}{2}\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{e^{-t}}{2}\sin 2t$$

$$\boxed{y(t) = \frac{e^{-t}}{2}\sin 2t}$$

Alternatively, use

$\mathcal{L}^{-1}\{Y(s)\} = y(t)$ $\mathcal{L}^{-1}\{Y(s-a)\} = e^{at}y(t)$ $\Bigg\} \Longleftrightarrow$	Table Entry $e^{at}y(t) \qquad Y(s-a)$

$$Y(s) = \frac{1}{(s+1)^2+4}$$

$$Y(s-1) = \frac{1}{(s-1+1)^2+4} = \frac{1}{s^2+4}$$

This one is in the table.

$$e^{1t}y(t) = \mathcal{L}^{-1}\{Y(s-1)\} = \frac{1}{2}\sin 2t$$

$$y(t) = \frac{1}{2}e^{-t}\sin 2t$$

$$Y(s) = \frac{3s+4}{s^2+2s+5}$$

$$= \frac{3s+4}{(s+1)^2+4}$$

I want to recognize $Y(s)$ as something with s replaced by $s+1$

$$= \frac{3(s+1)-3+4}{(s+1)^2+4}$$

$$= \frac{3(s+1)+1}{(s+1)^2+4} = \left.\frac{3s+1}{s^2+4}\right|_{s \mapsto s+1}$$

$$\mathcal{L}^{-1}\{Y\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{3s+1}{s^2+4}\right\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{3\frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}\right\}$$

$$= e^{-t}\left[3\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\right]$$

$$\boxed{y(t) = 3e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t}$$

Alternatively, use

$\mathcal{L}^{-1}\{Y(s)\} = y(t)$ $\mathcal{L}^{-1}\{Y(s-a)\} = e^{at}y(t)$	$\iff$ $e^{at}y(t)$	Table Entry $Y(s-a)$

$$Y(s) = \frac{3s+4}{(s+1)^2+4}$$

I want the denominator to be in the table. Not worried about numerator

$$Y(s-1) = \frac{3(s-1)+4}{s^2+4}$$

$$= \frac{3s}{s^2+4} + \frac{1}{s^2+4}$$

$$e^{t}y(t) = \mathcal{L}^{-1}\{Y(s-1)\} = 3\cos 2t + \frac{1}{2}\sin 2t$$

$$y(t) = e^{-t}\left(3\cos 2t + \frac{1}{2}\sin 2t\right)$$

$$Y(s) = \frac{1}{s^2+2s-8}$$

Roots of Denominator

$$s^2+2s+1 = 9 \implies (s+1)^2 = 9$$

$$s = -1 \pm 3 = 2, -4$$

$$Y(s) = \frac{1}{(s-2)(s+4)}$$

$\frac{1}{s-2}$ and $\frac{1}{s+4}$ are in the table

Partial Fractions

$$\frac{1}{(s-2)(s+4)} = \frac{A}{s-2} + \frac{B}{s-4}$$

Cover Up - multiply by $(s-2)$

$$\frac{1}{s+4} = A + \frac{B(s-2)}{s+4}$$

Set $s = 2$ $\quad \frac{1}{2+4} = A = \frac{1}{6}$

Now multiply by $s+4$ and set $s = -4$

to find $B = \frac{-1}{6}$

$$Y(s) = \frac{1}{(s-2)(s+4)} = \frac{1/6}{s-2} - \frac{1/6}{s+4}$$

$$\mathcal{L}^{-1}\left\{\frac{1/6}{s-2} - \frac{1/6}{s-4}\right\} = 1/6\, \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 1/6\, \mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= \frac{1}{6} e^{2t} - \frac{1}{6} e^{-4t}$$

Solve the IVP using Laplace Transforms

$$y'' + 2y' + 10y = 0$$

$$y(0) = 1 \qquad y'(0) = 2$$

$$\mathcal{L}\{y'' + 2y' + 10y\} = \mathcal{L}\{0\} = 0$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 10\mathcal{L}\{y\} = 0$$

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 10[Y(s)] = 0$$

$$[s^2Y(s) - s\cdot 1 - 2] + 2[sY(s) - 1] + 10[Y(s)] = 0$$

$$[s^2 + 2s + 10]Y(s) - s - 4 = 0$$

$$Y(s) = \frac{s+4}{s^2+2s+10}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+4}{s^2+2s+10}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+4}{(s+1)^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+9}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+4}{(s+1)^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+9}\right\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + e^{-t}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$\boxed{y(t) = e^{-t}\cos 3t + e^{-t}\sin 3t}$$

Alternatively, use

$$\left.\begin{aligned} \mathcal{L}^{-1}\{Y(s)\} &= y(t) \\ \mathcal{L}^{-1}\{Y(s-a)\} &= e^{at}y(t) \end{aligned}\right\} \iff e^{at}y(t) \qquad Y(s-a)$$

$$Y(s) = \frac{s+4}{(s+1)^2+9}$$

$$Y(s-1) = \frac{s-1+4}{s^2+9} = \frac{s}{s^2+9} + \frac{3}{s^2+9}$$

$$\mathcal{L}^{-1}\{Y(s-1)\} = \cos 3t + \sin 3t$$

$$e^{t}y(t) =$$

$$y(t) = e^{-t}(\cos 3t + \sin 3t)$$