Note Title 11/10/2017

Laplace Transforms $\mathcal{L}\{y(t)\} := \int_0^\infty e^{-st}\, y(t)\, dt$

Specific Transforms

e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	?
$\cos \omega t$	?
1	?
t	?
$\frac{t^n}{n!}$	?

We will fill in the rest of the table using "general rules" rather than by computing integrals.

Notation:

$$Y(s) := \mathcal{L}\{y(t)\}$$

Rule 1 $\mathcal{L}\{\dot{y}\} = sY(s) - y(0)$ $\left[= s\mathcal{L}\{y\} - y(0)\right]$

Rule 2 $\mathcal{L}\{e^{at}y(t)\} = Y(s-a)$

Rule 3 $\mathcal{L}\{t\, y(t)\} = -\frac{d}{ds}Y(s)$

Rule 4 $\mathcal{L}\{y(at)\} = \frac{1}{a}Y\left(\frac{s}{a}\right)$

General Rules

$$\mathcal{L}\{\dot{y}\} = sY(s) - y(0)$$

Proof

$$\int_0^\infty \dot{y}\, e^{-st}\, dt = y\, e^{-st}\Big|_0^\infty + s\int_0^\infty y(t)\, e^{-st}\, dt$$

$$= -y(0) + sY(s)$$

Combine $\mathcal{L}\{0\} = 0$ and $\mathcal{L}\{\dot{y}\} = s\mathcal{L}\{y\} - y(0)$

to calculate $\mathcal{L}\{1\}$ using the fact that $\frac{d}{dt}1 = 0$

$$0 = \mathcal{L}\{0\} = \mathcal{L}\{\tfrac{d}{dt}1\} = s\mathcal{L}\{1\} - 1$$

$$0 = s\mathcal{L}\{1\} - 1$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}}$$

Compute $\mathcal{L}\{t\}$ using $\dot{t} = 1$

General Rule $\mathcal{L}\{\dot{y}\} = s\mathcal{L}\{y\} - y(0)$

$$\mathcal{L}\{\dot{t}\} = s\mathcal{L}\{t\} - 0$$

$$\frac{1}{s} = \mathcal{L}\{1\} = s\mathcal{L}\{t\}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

Compute $\mathcal{L}\{t^n\}$

using $(t^n)^{\boldsymbol{\cdot}} = n t^{n-1}$

$$\mathcal{L}\{\dot{y}\} = s\mathcal{L}\{y\} - y(0)$$

$$\mathcal{L}\{n t^{n-1}\} = s\mathcal{L}\{t^n\} - 0 \quad \text{for } n \geqslant 1$$

$$n\mathcal{L}\{t^{n-1}\} = s\mathcal{L}\{t^n\}$$

$$\boxed{\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}}$$

$$\mathcal{L}\{t^2\} = \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}$$

$$\mathcal{L}\{t^3\} = \frac{3}{s}\mathcal{L}\{t^2\} = \frac{3}{s}\cdot\frac{2}{s^3} = \frac{3!}{s^4}$$

$$\boxed{\mathcal{L}\left\{\frac{t^n}{n!}\right\} = \frac{1}{s^{n+1}}}$$

Compute Laplace Transform of Sine and cosine

$(\sin t)' = \cos t$ so we may apply Rule 1

$$\mathcal{L}\{(\cos t)\} = \mathcal{L}\{(\sin t)'\} = s\,\mathcal{L}\{\sin t\} - 0 \qquad (1)$$

$(-\cos t)' = \sin t$ so we may apply Rule 1

$$\mathcal{L}\{\sin t\} = \mathcal{L}\{-(\cos t)'\} = -(s\,\mathcal{L}\{\cos t\} - 1)$$

$$= -s\,\mathcal{L}\{\cos t\} + 1 \qquad (2)$$

$$\mathcal{L}\{(\cos t)\} \overset{(1)}{=} s\,\mathcal{L}\{\sin t\} \overset{(2)}{=} s\left(-s\,\mathcal{L}\{\cos t\} + 1\right)$$

So $\mathcal{L}\{(\cos t)\} = -s^2\,\mathcal{L}\{\cos t\} + s$

$$(1+s^2)\,\mathcal{L}\{\cos t\} = s$$

$$\boxed{\mathcal{L}\{\cos t\} = \frac{s}{(1+s^2)}}$$

$$\mathcal{L}\{\sin t\} = -s\,\mathcal{L}\{\cos t\} + 1 \quad \text{from } (2)$$

$$= \frac{-s^2}{1+s^2} + 1 = \frac{\cancel{-s^2} + 1 + \cancel{s^2}}{1+s^2}$$

$$\boxed{\mathcal{L}\{\sin t\} = \frac{1}{1+s^2}}$$

More general Rules

Compute $\mathcal{L}\{e^{at}y(t)\}$

$$\mathcal{L}\{e^{at}y(t)\} = \int_0^\infty e^{-st}e^{at}y(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t}y(t)\,dt$$

$$= Y(s-a) = \mathcal{L}\{y(t)\}\Big|_{s-a}$$

$$\boxed{\mathcal{L}\{e^{at}y(t)\} = Y(s-a)} \qquad \underline{\text{Rule 2}}$$

Compute $\mathcal{L}\{e^{at}\cos\omega t\}$

$$\mathcal{L}\{e^{at}\cos\omega t\} = \mathcal{L}\{\cos\omega t\}\Big|_{s\to s-a}$$

$$= \frac{(s-a)}{(s-a)^2+\omega^2}$$

$$\mathcal{L}\{e^{at}\sin\omega t\} = \mathcal{L}\{\sin\omega t\}\Big|_{s\to s-a}$$

$$= \frac{\omega}{(s-a)^2+\omega^2}$$

One more rule $\mathcal{L}\{t\,y(t)\} = \,?$

Calculate $\frac{d}{ds} Y(s)$

$$\frac{d}{ds} Y(s) = \frac{d}{ds}\int_0^\infty e^{-st}\, y(t)\, dt$$

$$= \int_0^\infty \left(\frac{d}{ds} e^{-st}\right) y(t)\, dt$$

$$= \int_0^\infty \left(-t e^{-st}\right) y(t)\, dt$$

$$= -\int_0^\infty e^{-st}\, t\, y(t)\, dt$$

$$= -\mathcal{L}\{t\,y(t)\}$$

$$\mathcal{L}\{t\,y(t)\} = -\frac{d}{ds} Y(s) \qquad \underline{\text{Rule 3}}$$

Compute $\mathcal{L}\{t e^{at}\}$

$$\mathcal{L}\{t e^{at}\} = -\frac{d}{ds}\mathcal{L}\{e^{at}\} = -\frac{d}{ds}\left(\frac{1}{s-a}\right)$$

$$\boxed{\mathcal{L}\{t e^{at}\} = \frac{1}{(s-a)^2}}$$

Rule 4 $\mathcal{L}\{y(at)\} = \frac{1}{a} Y\left(\frac{s}{a}\right)$

Use Rule 4 to calculate $\mathcal{L}\{\sin \omega t\}$

$$\mathcal{L}\{\sin \omega t\} = \frac{1}{\omega} \mathcal{L}\{\sin t\}\Big|_{s \to \frac{s}{\omega}}$$

$$= \frac{1}{\omega} \frac{1}{1+\left(\frac{s}{\omega}\right)^2}$$

$$= \frac{1}{\omega} \frac{\omega^2}{s^2+\omega^2} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\{\cos \omega t\} = \frac{1}{\omega} \mathcal{L}\{\cos t\}\Big|_{s \to \frac{s}{\omega}}$$

$$= \frac{1}{\omega} \frac{s/\omega}{1+\left(\frac{s}{\omega}\right)^2}$$

$$= \frac{1}{\omega} \frac{\omega s}{s^2+\omega^2} = \frac{s}{s^2+\omega^2}$$

Alternative Calculation of $\mathcal{L}\{te^{at}\}$

$$\mathcal{L}\{te^{at}\} = -\frac{d}{ds}\mathcal{L}\{e^{at}\}$$

$$= -\frac{d}{ds}\left(\frac{1}{s-a}\right)$$

$$= \frac{1}{(s-a)^2}$$

Proof of Rule 4

$$\mathcal{L}\{y(at)\} = \int_0^\infty e^{-st}\, y(at)\, dt$$

$$= \int_0^\infty e^{-\frac{s}{a}(at)}\, y(at)\, dt$$

Let $\tau = at$

$$= \int_0^\infty e^{-\left(\frac{s}{a}\right)\tau}\, y(\tau)\, \frac{d\tau}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)\tau}\, y(\tau)\, d\tau$$

$$= \frac{1}{a}\, \mathcal{L}\{y(t)\}\Big|_{s \mapsto \frac{s}{a}} = \frac{1}{a} Y\left(\frac{s}{a}\right)$$

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1	$\frac{1}{s}$
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$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$

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