

Laplace Transform - a method for solving constant coefficient DE's / IVP's

Definition:

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

Notation: The Laplace transform of $f(t)$ is $F(s)$. Also written as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

These two statements mean the same thing:

$$\mathcal{L}\{e^{-t}\} = \frac{1}{s+1} \quad f(t) = e^{-t} \quad F(s) = \frac{1}{s+1}$$

The Laplace transform changes a function of t into a function of s .

Problem - Calculate $\mathcal{L}\{e^{-t}\}$

$$\mathcal{L}\{e^{-t}\} = \int_0^{\infty} e^{-st} e^{-t} dt = \int_0^{\infty} e^{-(s+1)t} dt$$

$$= \left. \frac{e^{-(s+1)t}}{-(s+1)} \right|_0^{\infty} = \frac{0}{-(s+1)} - \frac{1}{-(s+1)} = \frac{1}{s+1}$$

Problem - Calculate $\mathcal{L}\{e^{-t}\}$

$$\mathcal{L}\{e^{-t}\} = \int_0^{\infty} e^{-st} e^{-t} dt = \int_0^{\infty} e^{-(s+1)t} dt$$

Some technical details

$$\int_0^{\infty} e^{-(s+1)t} dt = \lim_{M \rightarrow \infty} \int_0^M e^{-(s+1)t} dt$$

$$= \lim_{M \rightarrow \infty} \left. \frac{e^{-(s+1)t}}{-(s+1)} \right|_0^M$$

$$= \lim_{M \rightarrow \infty} \left(\frac{1}{s+1} - \frac{e^{-M(s+1)}}{s+1} \right)$$

$$= \frac{1}{s+1} - 0 \quad [\text{for } s > -1]$$

We always assume s is big enough so that this limit = 0

Calculate $\mathcal{L}\{e^{rt}\}$ where r is a constant.

$$\mathcal{L}\{e^{rt}\} = \int_0^{\infty} e^{-st} e^{rt} dt = \int_0^{\infty} e^{-(s-r)t} dt$$

$$= \left. \frac{e^{-(s-r)t}}{-(s-r)} \right|_0^{\infty}$$

$$= 0 + \frac{1}{(s-r)}$$

$$\boxed{\mathcal{L}\{e^{rt}\} = \frac{1}{(s-r)}}$$

Why we use Laplace Transform

The Laplace Transform of the derivative

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = s \mathcal{L}\{y\} - y(0)$$

Proof

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = \int_0^{\infty} e^{-ts} \frac{dy}{dt} dt$$

$$= e^{-ts} y(t) \Big|_0^{\infty} - \int_0^{\infty} \frac{d}{dt} e^{-ts} y(t) dt$$

$$= 0 - y(0) - \int_0^{\infty} (-s) e^{-ts} y(t) dt$$

$$= -y(0) + s \int_0^{\infty} e^{-ts} y(t) dt$$

$$= s \mathcal{L}\{y\} - y(0)$$



Laplace Transform Table

$$y(t) \qquad Y(s) = \mathcal{L}\{y(t)\}$$

- ① $e^{rt} \qquad \frac{1}{s-r}$
- ② $\frac{dy}{dt} \qquad sY(s) - y(0)$
- ③ $f(t) + g(t) \qquad F(s) + G(s)$
-

$Y(s)$ is notation for $\mathcal{L}\{y(t)\}$

Table summarizes these facts

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$

Laplace transform is linear

$$\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \qquad \mathcal{L}\{0\} = 0$$

One Additional Fact

Laplace transform is one to one.

IF $\mathcal{L}\{y(t)\} = 0$ then $y(t) = 0$.

Solve IVP $\begin{cases} \frac{dy}{dt} + 4y = 0 \text{ (DE)} \\ y(0) = 1 \text{ (IC)} \end{cases}$ using Laplace transforms

$$\frac{dy}{dt} + 4y = 0$$

$$\mathcal{L}\left\{\frac{dy}{dt} + 4y\right\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + 4\mathcal{L}\{y\} = 0 \quad \text{uses 3 from table}$$

$$sY(s) - y(0) + 4Y(s) = 0 \quad \text{uses 2 from table}$$

$$(s+4)Y(s) - 1 = 0$$

$$Y(s) = \frac{1}{s+4}$$

$$Y(s) = \mathcal{L}\{e^{-4t}\} \quad \text{uses 1 from table}$$

$$\boxed{y(t) = e^{-4t}}$$

We have solved (IVP) without integrating or differentiating. We only used algebra.

Laplace Transform Approach to IVP

$$y'' + 2y' + y = \cos \omega t$$
$$y(0) = 0$$
$$y'(0) = 0$$

$$\frac{2\omega \sin \omega t - (\omega - 1) \cos \omega t - e^{-t} - (2\omega + 1) t e^{-t}}{(\omega^2 + 1)^2}$$

\mathcal{L} Laplace Transform

I haven't explained this step yet

\mathcal{L}^{-1} "Inverse" Laplace Transform

$$s^2 Y + 2sY + Y = \frac{s}{s^2 + \omega^2}$$

where

$$Y(s) = \mathcal{L}\{y\}$$

$$\frac{s}{s^2 + \omega^2} = \mathcal{L}\{\cos \omega t\}$$

$$Y(s) = \frac{s}{(s^2 + 2s + 1)(s^2 + \omega^2)}$$

Laplace Transforms $\mathcal{L}\{y(t)\} := \int_0^{\infty} e^{-st} y(t) dt$

Specific Transforms

$$e^{at}$$

$$\frac{1}{s-a}$$

Notation:

$$\sin \omega t$$

$$\frac{\omega}{s^2 + \omega^2}$$

$$Y(s) := \mathcal{L}\{y(t)\}$$

$$\cos \omega t$$

$$\frac{s}{s^2 + \omega^2}$$

$$1$$

$$\frac{1}{s}$$

$$t$$

$$\frac{1}{s^2}$$

$$\frac{t^n}{n!}$$

$$\frac{1}{s^{n+1}}$$

$$e^{at} \cos \omega t$$

$$\frac{s-a}{(s-a)^2 + \omega^2}$$

$$e^{at} \sin \omega t$$

$$\frac{\omega}{(s-a)^2 + \omega^2}$$

$$e^{at} \frac{t^n}{n!}$$

$$\frac{1}{(s-a)^{n+1}}$$

$$\dot{y}(t)$$

$$sY(s) - y(0)$$

$$\ddot{y}(t)$$

$$s^2 Y(s) - s y(0) - \dot{y}(0)$$