Note Title 2/18/2020

$$\ddot{x} + \frac{\gamma}{m}\dot{x} + \frac{k}{m}x = 0$$ Mass-Spring

Often rewritten as

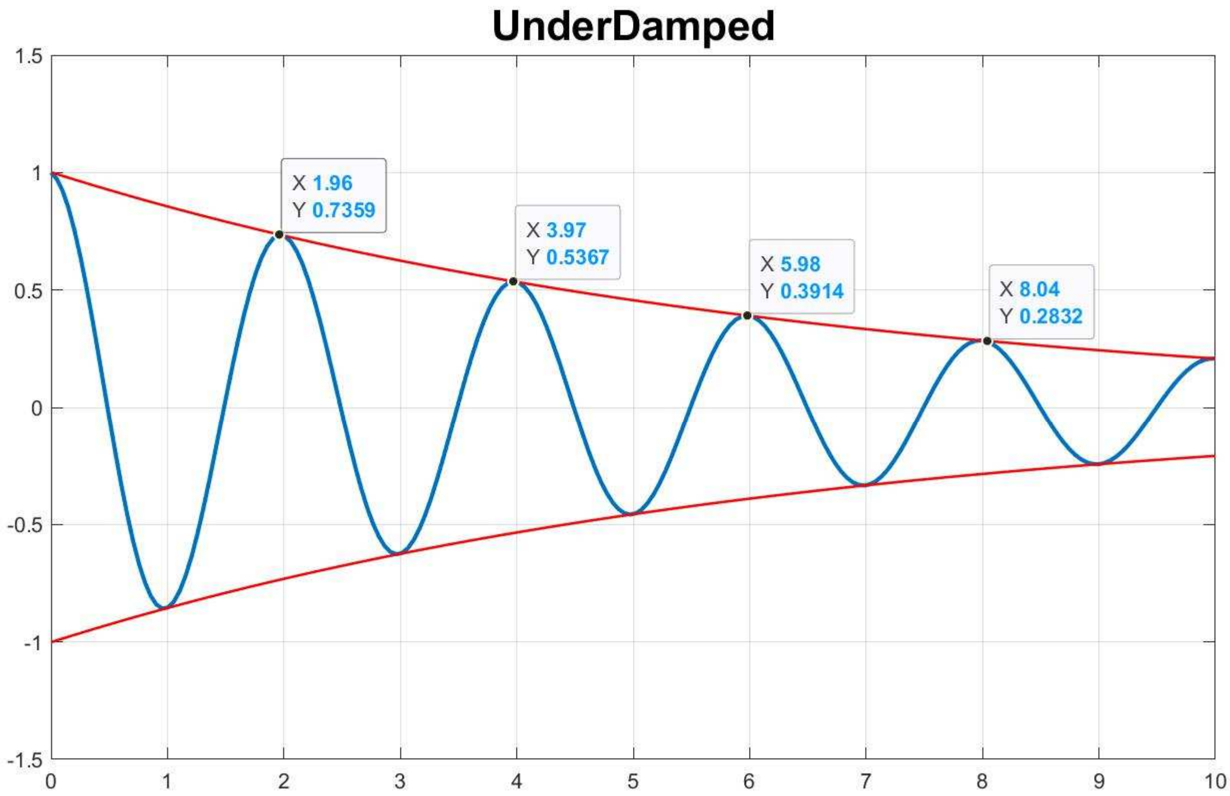
$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0 \qquad \text{(DE)}$$

Solution in polar form

$$x(t) = A e^{-\zeta\omega_0 t}\cos(\omega_d t - \phi)$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_0$$

(DE) is written this way because its easy to find ω_d, ζ, ω_0 from the graph.



$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0$$

Solution in polar form

$$x(t) = A e^{-\zeta\omega_0 t}\cos(\omega_d t - \phi)$$

Derivation of Solution

Seek $x(t) = e^{rt}$

$$r^2 + 2\zeta\omega_0 r + \omega_0^2 = 0$$

$$(r + \zeta\omega_0)^2 = -\omega_0^2(1-\zeta^2)$$

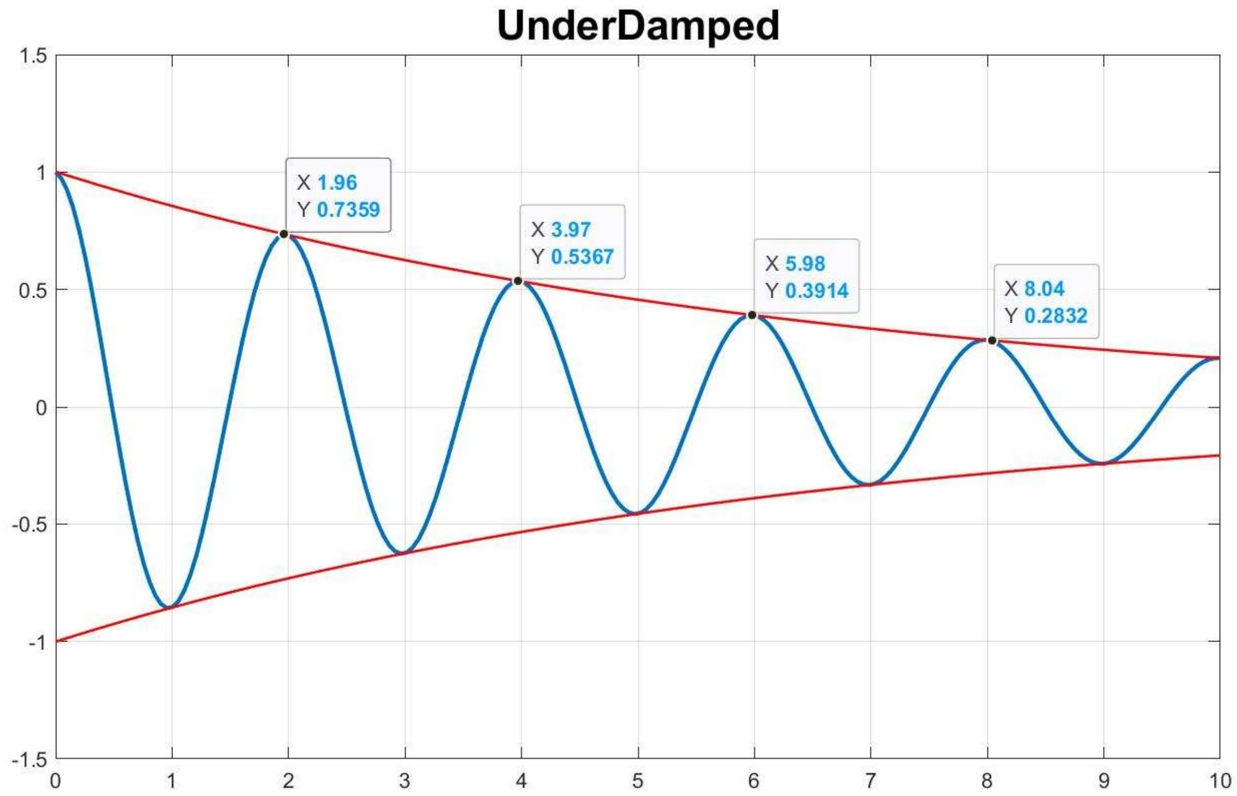
$$r = -\zeta\omega_0 \pm i\omega_0\sqrt{1-\zeta^2}$$

Definition $\omega_d = \sqrt{1-\zeta^2}\,\omega_0$

$$x(t) = e^{-\zeta\omega_0 t}\left(c_1\cos\omega_d t + c_1\sin\omega_d t\right)$$

$$= e^{-\zeta_0\omega t} A\cos(\omega_d t - \phi)$$

Problem: Find ω_d and ζ $\left(\omega_0 = \frac{\omega_d}{\sqrt{1-\zeta^2}}\right)$



Relevant Facts

① The time between peaks (local maxima) is $\frac{2\pi}{\omega_d}$

② The cosine term $\cos(\omega_d t - \phi)$ has the same value at all peaks

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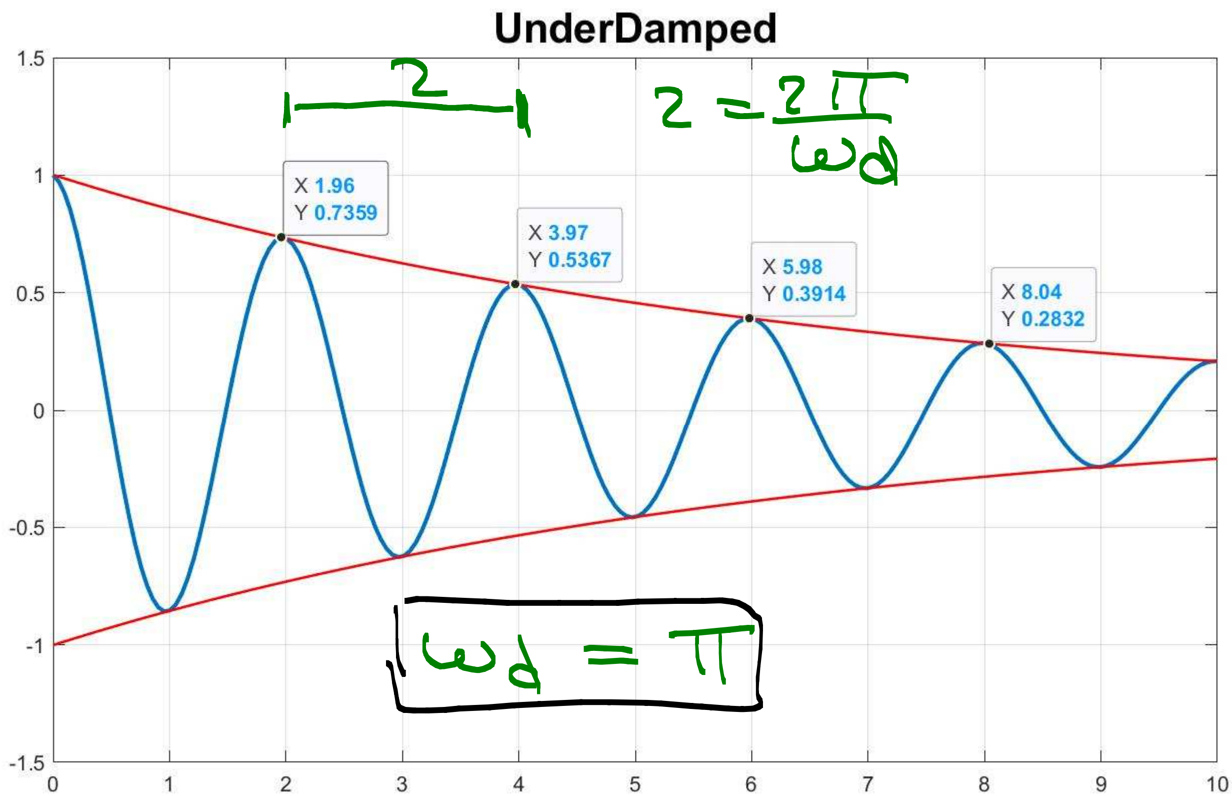

$$e^{-\zeta\omega_0(1.96)} \cos(\omega_d \cdot 1.96 - \phi) = 0.7359$$

$$e^{-\zeta\omega_0(3.97)} \cos(\omega_d \cdot 3.97 - \phi) = 0.5367$$

② The cosine term $\cos(\omega_d t - \phi)$ has the same value at all peaks

$$\frac{e^{-\zeta\omega_0(1.96)}}{e^{-\zeta\omega_0(3.97)}} = \frac{0.7359}{0.5367}$$

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$$e^{\zeta\omega_0 \cdot 2} = 1.3712 \quad \omega_d$$

$$\zeta\omega_0 = \frac{\ln(1.3712)}{2} = 0.1578$$

<u>Usual Approximation</u> $\quad \omega_0 \approx \omega_d$

$$\omega_0 = \omega_d = \pi$$

$$\zeta = \frac{0.1578}{\pi} = 0.0502$$

<u>Exact Calculation</u>

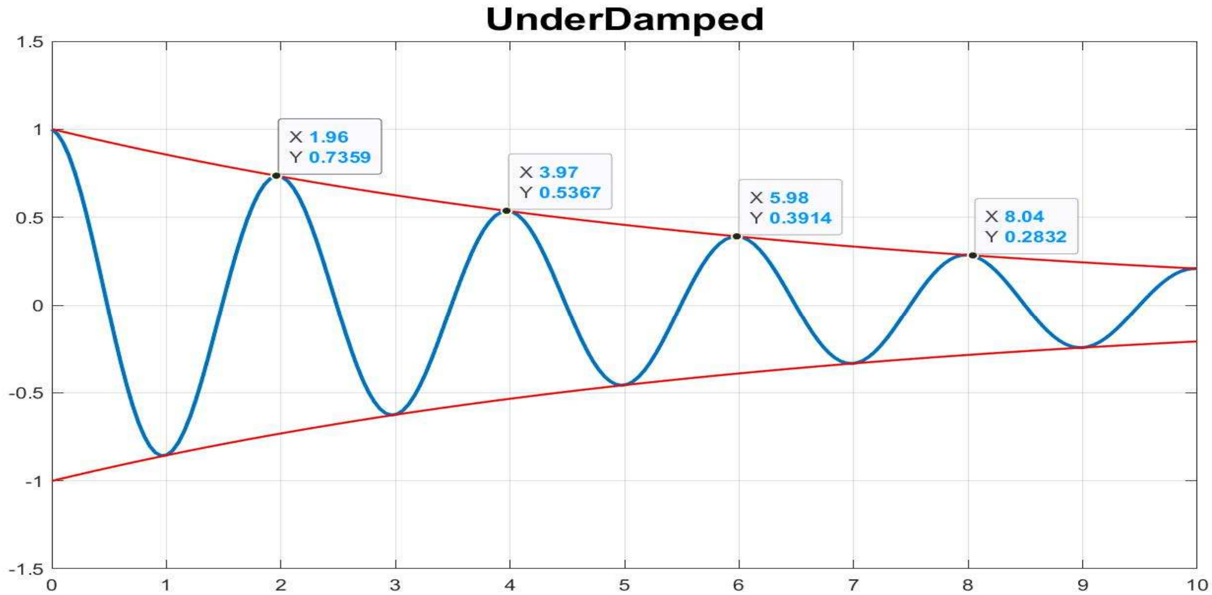
$$(\zeta\omega_0)^2 + \omega_d^2 = (\zeta\omega_0)^2 + (1-\zeta^2)\omega_0^2$$

$$= \omega_0^2$$

$$(0.1578)^2 + \pi^2 = \omega_0^2$$

$$3.1455 = \omega_0$$

$$0.0502 = \frac{0.1578}{3.1455} = \zeta$$



$$x(t) = A\, e^{-\zeta \omega_0 t} \cos(\omega_d t - \phi)$$

① The time between peaks is $\frac{2\pi}{\omega_d}$

② The cosine term $\cos(\omega_d t - \phi)$ has the same value at all peaks

"Approximate Proof" of ① and ②

$\cos(\omega_D t - \phi) \approx 1$ at the peaks *

Both cosines are 1 at peaks proves ②

so $\omega_D t_0 - \phi = 0$ at first peak

$\omega_D t_1 - \phi = 2\pi$ at next peack

subtract

$\boxed{\omega_d (t_1 - t_0) = 2\pi}$ proves ①

* This is only approximately true

"Exact Proof"

Max occurs at t_0 where $\frac{d\varphi}{dt} = 0$

$$\frac{d}{dt}\left(e^{-\zeta\omega_0 t}\cos(\omega_d t - \varphi)\right)$$

$$-\zeta\omega_0 \cancel{e^{-\zeta\omega_0 t}} \cos(\omega_d t - \varphi) = \cancel{e^{-\zeta\omega_0 t}}\, \omega_d \sin(\omega_d t - \varphi)$$

$$-\frac{\zeta\omega_0}{\omega_d} = \tan(\omega_d t - \varphi)$$

so at 2 consecutive peaks t_0 & t_1

$$\tan(\omega_d t_0 - \varphi) = \tan(\omega_d t_1 - \varphi)$$

$$\omega_d t_0 - \cancel{\varphi} = \omega_d t_1 - \cancel{\varphi} + N\pi$$

At consecutive maxima

$$\omega_d t_0 - \omega_d t_1 = 2\pi$$

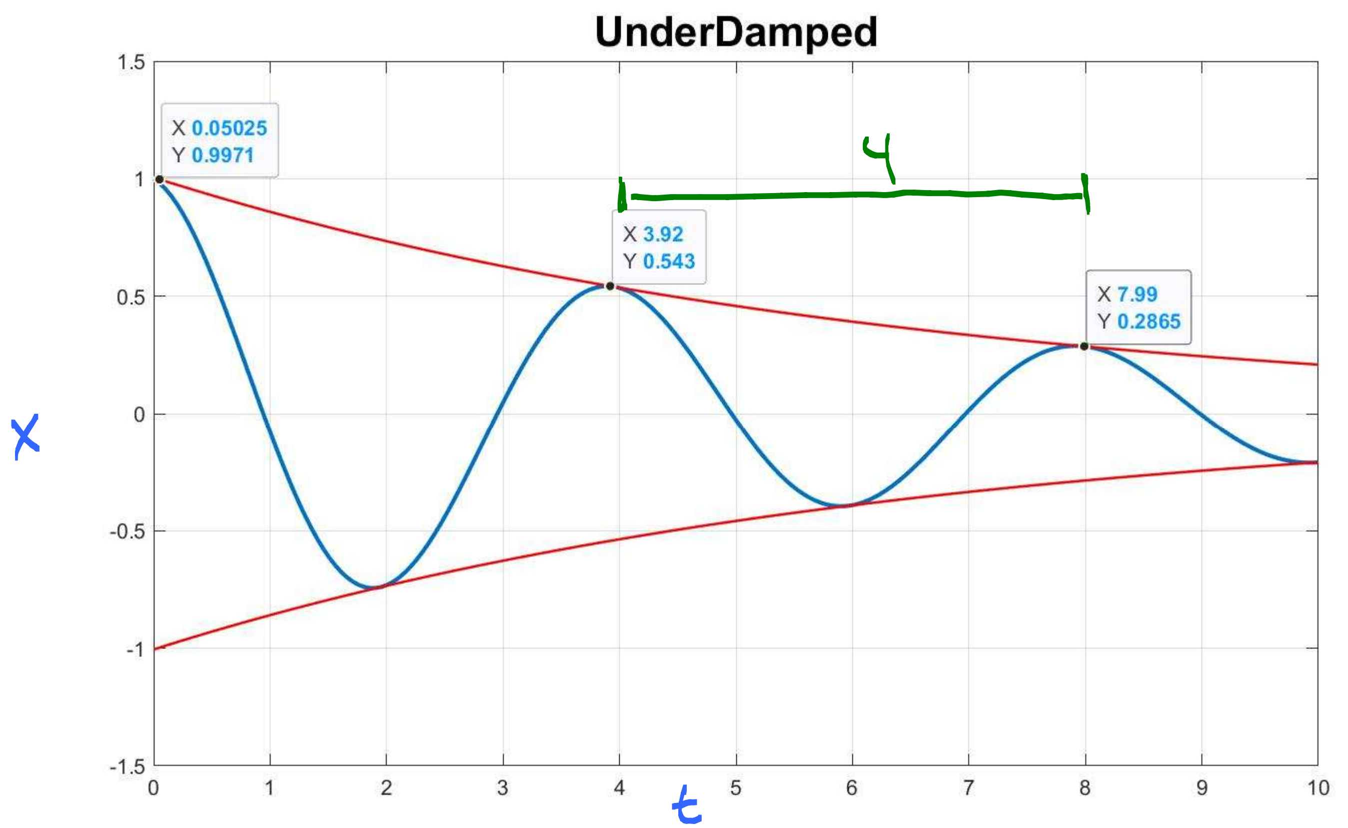
② IF $\tan(\omega_d t_0 - \varphi) = \tan(\omega_d t_1 - \varphi)$

then $\cos(\omega_d t_0 - \varphi) = \pm \cos(\omega_d t_1 - \varphi)$

at maxima $\cos(\omega_d t_0 - \varphi) = \cos(\omega_d t_1 - \varphi)$

Problem Find ζ and ω_0

$$x(t) = A\, e^{-\zeta\omega_0 t}\cos(\omega_d t - \phi)$$



$$\frac{2\pi}{4} = \omega_d \approx \omega_0$$

$$4\zeta\omega_0 = \ln\left(\frac{0.543}{0.286}\right) = 0.64$$

$$\zeta = \frac{0.64}{2\pi} = 0.102$$