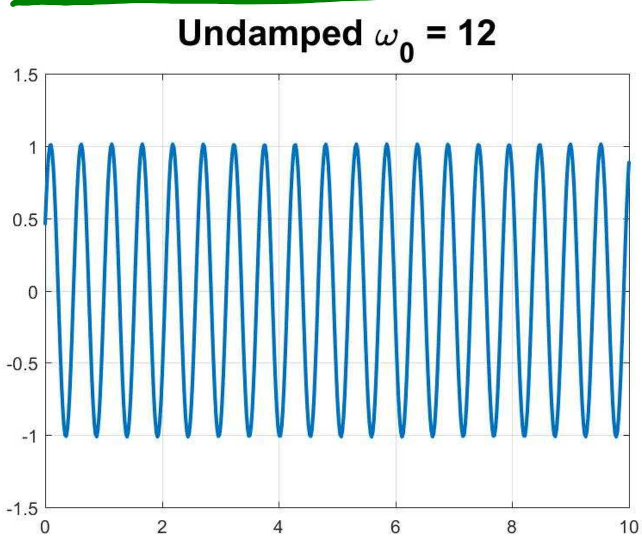
Note Title 11/8/2017

Summary -

Second Order Constant Coefficient ODE
Harmonic Motion

Unforced Motion



Undamped

$\ddot{y} + \frac{k}{m} y = 0$ mass-spring

$\ddot{y} + \omega_0^2 y = 0$ harmonic oscillator

$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$

$= A \cos(\omega_0 t - \varphi)$

Damped

$\ddot{y} + 2\zeta\omega_0 \dot{y} + \omega_0^2 y = 0$ harmonic oscillator

$\ddot{y} + \frac{\gamma}{m}\dot{y} + \frac{k}{m} y = 0$ mass-spring

Overdamped

$y = c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$

$r^2 + \frac{\gamma}{m} r + \frac{k}{m} = 0$

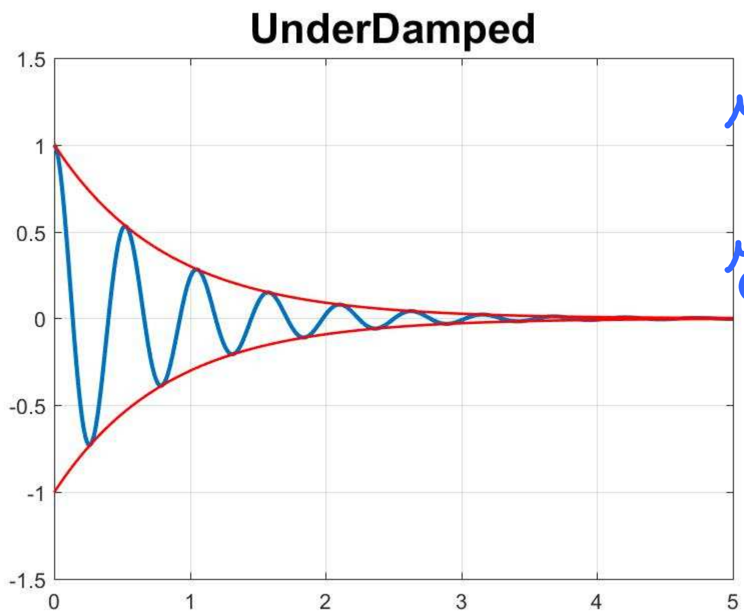
has 2 real roots

Critically Damped

$r^2 + \frac{\gamma}{m} r + \frac{k}{m} = 0$

has 1 (real) root

$y = c_1 e^{-rt} + c_2 t e^{-rt}$



$$y = c_1 e^{-\frac{r}{2m}t} \cos \omega_d t + C_2 e^{-\frac{r}{2m}t} \sin \omega_d t$$

$$y = A e^{-\frac{r}{2m}t} \cos(\omega_d t - \alpha)$$

$$r^2 + \frac{r}{m} r + \frac{k}{m} = 0$$

has complex roots

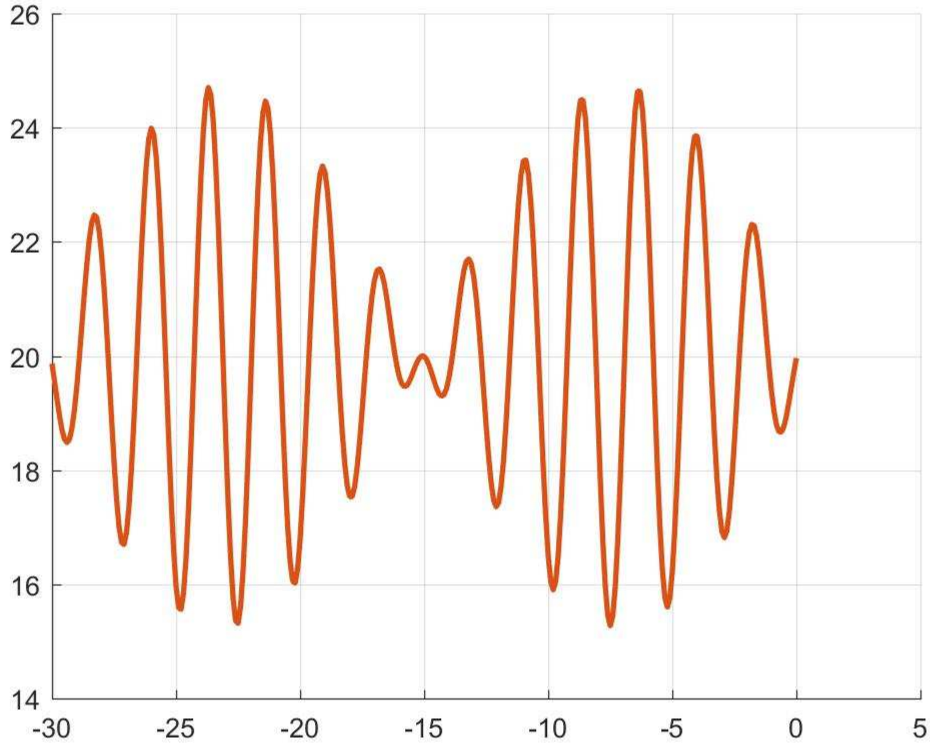
Forced Motion

Undamped

Beats $\ddot{y} + \omega_0^2 y = \cos \omega t$

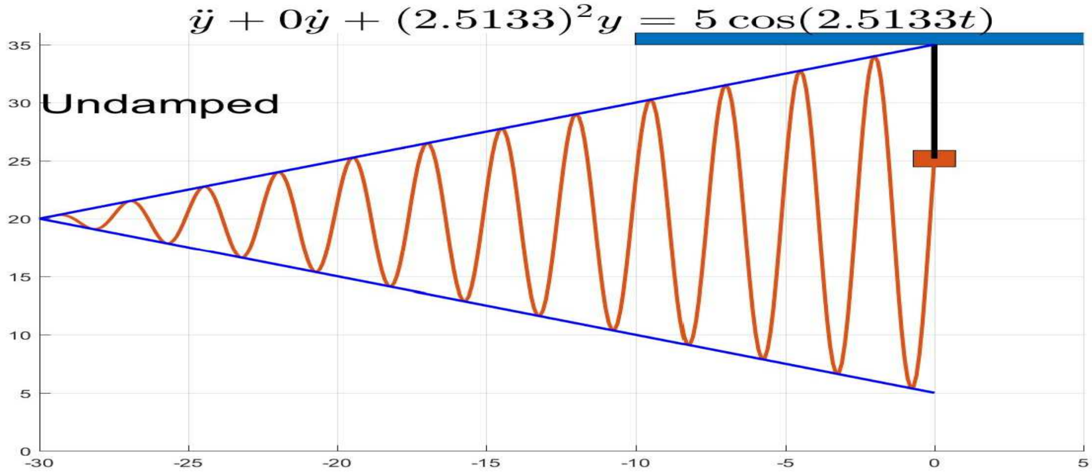
$$y = \frac{2 \sin\left(\frac{\omega - \omega_0}{2} t\right) \sin\left(\frac{\omega + \omega_0}{2} t\right)}{\omega^2 - \omega_0^2}$$

average freq.
half difference



Resonance $\ddot{y} + \omega_0^2 y = \cos\omega_0 t \quad y(0) = 0 \quad \dot{y}(0) = 1$

$$y = t\cos\omega_0 t$$



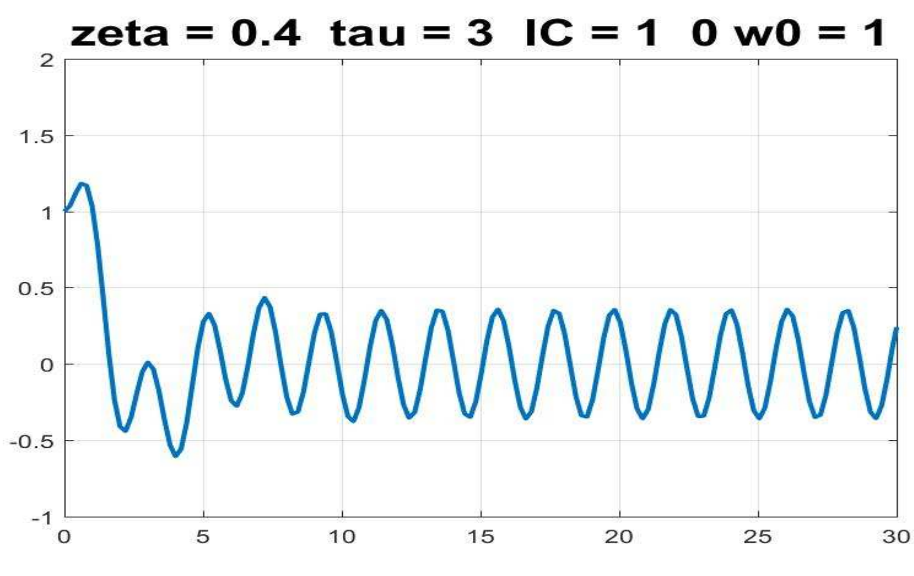
Damped Forced $\ddot{y} + \frac{\gamma}{m}\dot{y} + \frac{k}{m}y = \cos\omega t$

$$\left[\text{or } \ddot{y} + 2\zeta\omega_0\dot{y} + \omega_0^2 y = \cos\omega t\right]$$

$$y_{ss} = A\cos(\omega t - \varphi)$$

we can calculate A and φ in terms of ζ, ω_0, ω

$$A = \frac{1}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\zeta^2\omega_0^2\omega^2}}$$



I don't expect you to memorize any of these formulas. I expect you to be able to work them out in specific cases.

Mathematical Methods for
Second Order Constant Coefficient ODE

① Find homogeneous solutions as linear combinations of functions e^{rt} or te^{rt}.

② Particular Solutions are linear combinations of forcing functions and their derivatives.

If forcing terms satisfy homogeneous equation, particular solution must be multiplied by t. Repeat until no term satisfies homogeneous equation.