Lecture 17 Damped Forced

Note Title 11/2/2017

Damped Force Oscillations (Filters)

Example $\ddot{y} + \rho\dot{y} + y = \cos(\omega t)$

① Find the steady state solution

② Find the amplitude and phase as functions of ρ and ω

③ At what frequency ω is the amplitude largest?

Answer

Seek $y = A\cos\omega t + B\sin\omega t$

$$\rho\dot{y} = -\rho A\omega\sin\omega t + \rho B\omega\cos\omega t$$

$$+ \quad \ddot{y} = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

$$\ddot{y} + \rho\dot{y} + y = [(1-\omega^2)A + \rho\omega B]\cos\omega t + [(1-\omega^2)B - \rho\omega A]\sin\omega t$$

$\cos\omega t =$

so

$$[(1-\omega^2)A + \rho\omega B] = 1$$

$$[(1-\omega^2)B - \rho\omega A] = 0$$

$$(1-\omega^2)A + \xi\omega B = 1$$

$$(1-\omega^2)B - \xi\omega A = 0 \implies A = \frac{(1-\omega^2)}{\xi\omega} B$$

$$\frac{(1-\omega^2)^2}{\xi\omega} B + \xi\omega B = 1 \implies \left[(1-\omega^2)^2 + (\xi\omega)^2\right] B = \xi\omega$$

$$B = \frac{\xi\omega}{(1-\omega^2)^2 + (\xi\omega)^2} \qquad A = \frac{(1-\omega^2)}{(1-\omega^2)^2 + (\xi\omega)^2}$$

$$y_{ss} = \frac{(1-\omega^2)}{(1-\omega^2)^2 + (\xi\omega)^2} \cos\omega t + \frac{\xi\omega}{(1-\omega^2)^2 + (\xi\omega)^2} \sin\omega t$$

We want to know how big the steady state solution will become, so we write y_{ss} in amplitude phase form

$$y_{ss} = A\cos(\omega t - \varphi)$$

Amplitude Phase Form

$$A\cos(\omega t-\varphi) = A\cos\varphi\cos\omega t + A\sin\varphi\sin\omega t$$

$$y_{ss} = \frac{(1-\omega^2)}{(1-\omega^2)^2+(\xi\omega)^2}\cos\omega t + \frac{\xi\omega}{(1-\omega^2)^2+(\xi\omega)^2}\sin\omega t$$

$$A\cos\varphi = \frac{(1-\omega^2)}{(1-\omega^2)^2+(\xi\omega)^2} \qquad A\sin\varphi = \frac{\xi\omega}{(1-\omega^2)^2+(\xi\omega)^2}$$

$$A^2 = \frac{(1-\omega^2)^2+(\xi\omega)^2}{\left[(1-\omega^2)^2+(\xi\omega)^2\right]^2}$$

$$A = \frac{1}{\left[(1-\omega^2)^2+(\xi\omega)^2\right]^{1/2}} \qquad \tan\varphi = \frac{\xi\omega}{(1-\omega^2)}$$

$$\boxed{y_{ss} = \frac{\cos\left(\omega t - \operatorname{atan}\left(\frac{\xi\omega}{1-\omega^2}\right)\right)}{\left[(1-\omega^2)^2+(\xi\omega)^2\right]^{1/2}}}$$

$$\boxed{y_{ss} = \frac{\cos\left(\omega t - \operatorname{atan}\left(\frac{\zeta\omega}{1-\omega^2}\right)\right)}{\left[(1-\omega^2)^2 + (\zeta\omega)^2\right]^{1/2}}}$$

This formula answers many important questions.

$$A(\omega,\zeta) = \frac{1}{\left[(1-\omega^2)^2 + (\zeta\omega)^2\right]^{1/2}}$$

The amplitude tells us how much the forcing amplitude is magnified or damped.

<u>Example</u> $\zeta = 0.1$

$$\ddot{y} + 0.1\dot{y} + y = \cos(1.1t) + \cos(3t)$$

Because the equation is linear, y_{ss} can be written as a sum of two solutions:

$$\ddot{y}_1 + 0.1\,\dot{y}_1 + y_1 = \cos(1.1t)$$

$$\ddot{y}_3 + 0.1\,\dot{y}_3 + y_3 = \cos(3t)$$

$$y_1 = \frac{1}{\left[(1-1.1^2)^2 + (0.11)^2\right]^{1/2}} \cos\left(1.1t + \operatorname{atan}\left(\frac{0.1}{1-1.1^2}\right)\right)$$

$$y_2 = \frac{1}{\left[(1-3^2)^2 + (0.33)^2\right]^{1/2}} \cos\left(3t + \operatorname{atan}\left(\frac{0.1}{1-3^2}\right)\right)$$

$$y_1 = 4.2\cos(1.1t + 2.7)$$

$$y_2 = 0.125\cos(3t + 3.10)$$

$$\frac{4.2}{0.125} = 33.7$$

The response to $\cos(1.1t)$ is 34 times bigger than the response to $\cos(3t)$.

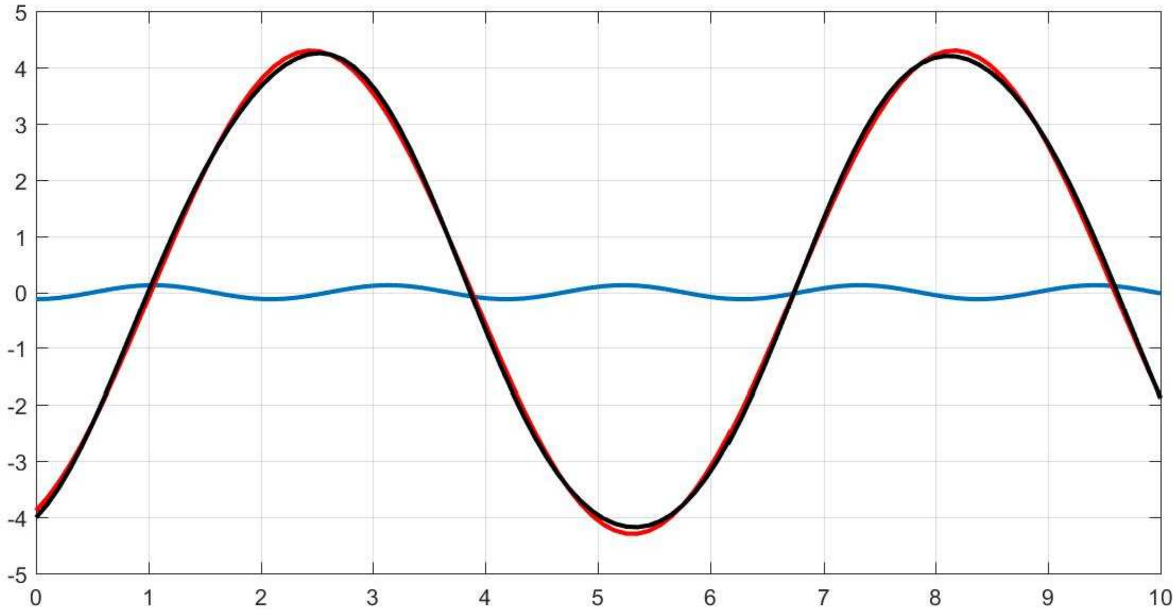
Summary

$$\ddot{y} + 0.1\dot{y} + y = \cos(1.1t) + \cos(3t)$$

$$y_{ss} = 4.2\cos(1.1t + 2.7) + 0.125\cos(3t + 3.10)$$

$4.2\cos(1.1t + 2.7)$ $\quad$ $4.2\cos(1.1t+2.7) + 0.125\cos(3t+3.10)$

$0.125\cos(3t + 3.10)$



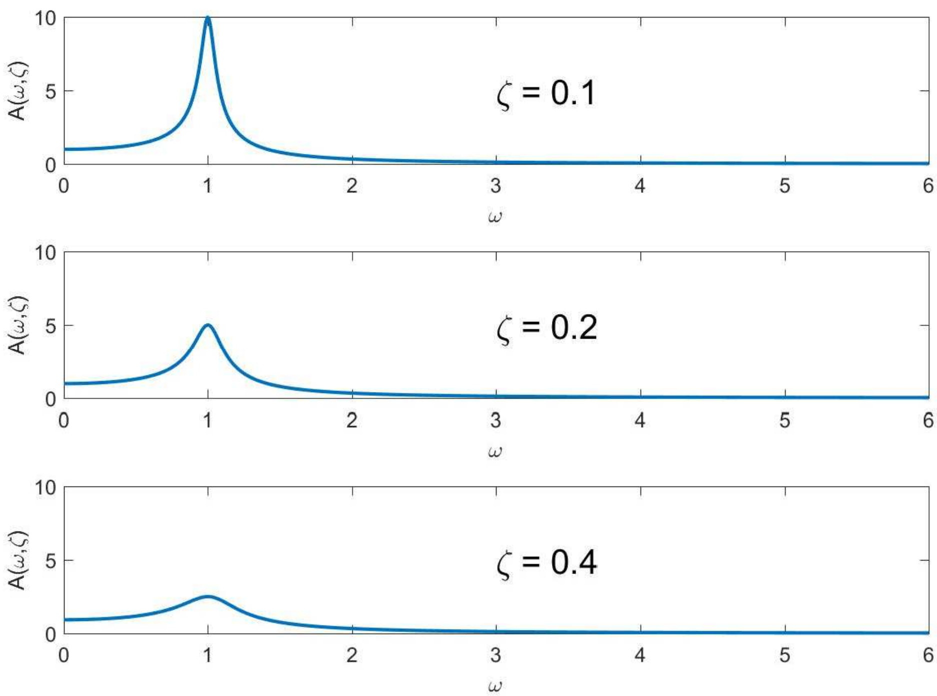
The steady state solution magnifies frequencies near $\omega = 1$ and suppresses frequencies far from $\omega = 1$. It "filters out" frequencies away from $\omega = 1$.

$\ddot{y} + \zeta \dot{y} + y = \cos \omega t$

$y_{ss} = A(\omega, \zeta) \cos(\omega t - \varphi(\omega, \zeta))$

$$A(\omega, \zeta) = \frac{1}{\left[(1-\omega^2)^2 + (\zeta \omega)^2\right]^{1/2}}$$

The smaller the value of ζ, the higher and narrower the peak.



If we replace the DE with

$$\ddot{y} + \omega_0 \zeta \dot{y} + \omega_0^2 y = \cos \omega t$$

then the frequency peak moves to ω_0

③ At what frequency ω is the amplitude largest?

$$A(\omega,\varrho) = \frac{1}{\left[(1-\omega^2)^2+(\varrho\omega)^2\right]^{1/2}}$$

A is largest when $(1-\omega^2)^2+(\varrho\omega)^2$ is smallest

$$0 = \frac{d}{d\omega}(1-\omega^2)^2+(\varrho\omega)^2 = -\cancel{2}\cancel{\omega}2(1-\omega^2) + \cancel{2}\varrho^2\cancel{\omega}$$

$$0 = 2 - 2\omega^2 + \varrho^2$$

$$\omega_{max} = \sqrt{1-\frac{\varrho^2}{2}}$$

④ What is the maximum amplitude?

$$A(\omega_{max},\varrho) = \frac{1}{\sqrt{\left(1-\left(1-\frac{\varrho^2}{2}\right)\right)^2 + \varrho^2\left(1-\frac{\varrho^2}{2}\right)}}$$

$$= \frac{1}{\sqrt{\frac{\varrho^4}{4}+\varrho^2-\frac{\varrho^4}{2}}}$$

$$= \frac{1}{\varrho\sqrt{1-\frac{\varrho^2}{4}}}$$