Note Title 2/12/2020

Two Formulas from Last Time

$$\ddot{y} + y = \cos \omega t \qquad y(0) = 0 \qquad \dot{y}(0) = 0$$

$\omega \neq 1$ $\quad y(t) = \dfrac{\cos \omega t - \cos t}{1 - \omega^2}$

$\omega = 1$ $\quad y(t) = \dfrac{t \sin t}{2}$

Slightly more general

$$\ddot{y} + \omega_0^2 y = \cos \omega t \qquad y(0) = 0 \qquad \dot{y}(0) = 0$$

$\omega \neq \omega_0$ $\quad y(t) = \dfrac{\cos \omega t - \cos \omega_0 t}{\omega_0^2 - \omega^2}$

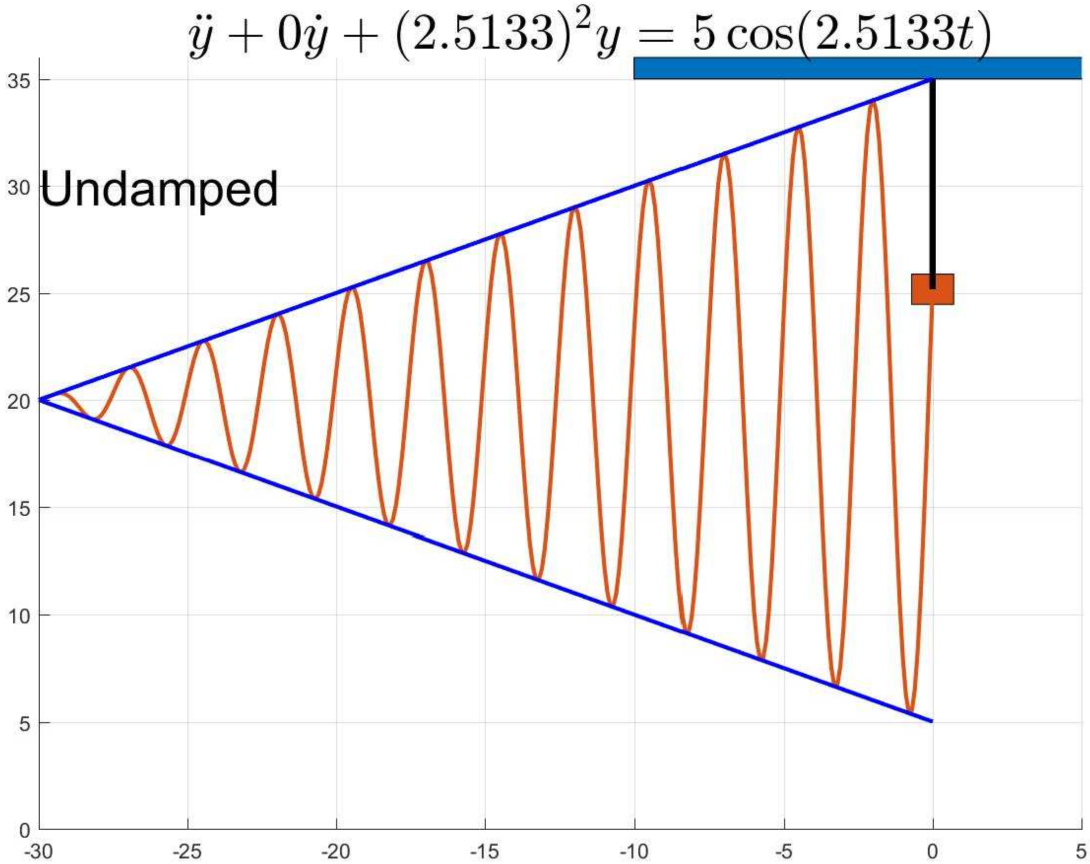
$\omega = \omega_0$ $\quad y(t) = \dfrac{t \sin \omega_0 t}{2 \omega_0}$

ω_0 = Natural Frequency

ω = Forcing Frequency

What do they look like?

$\omega_0 = \omega$ Resonance

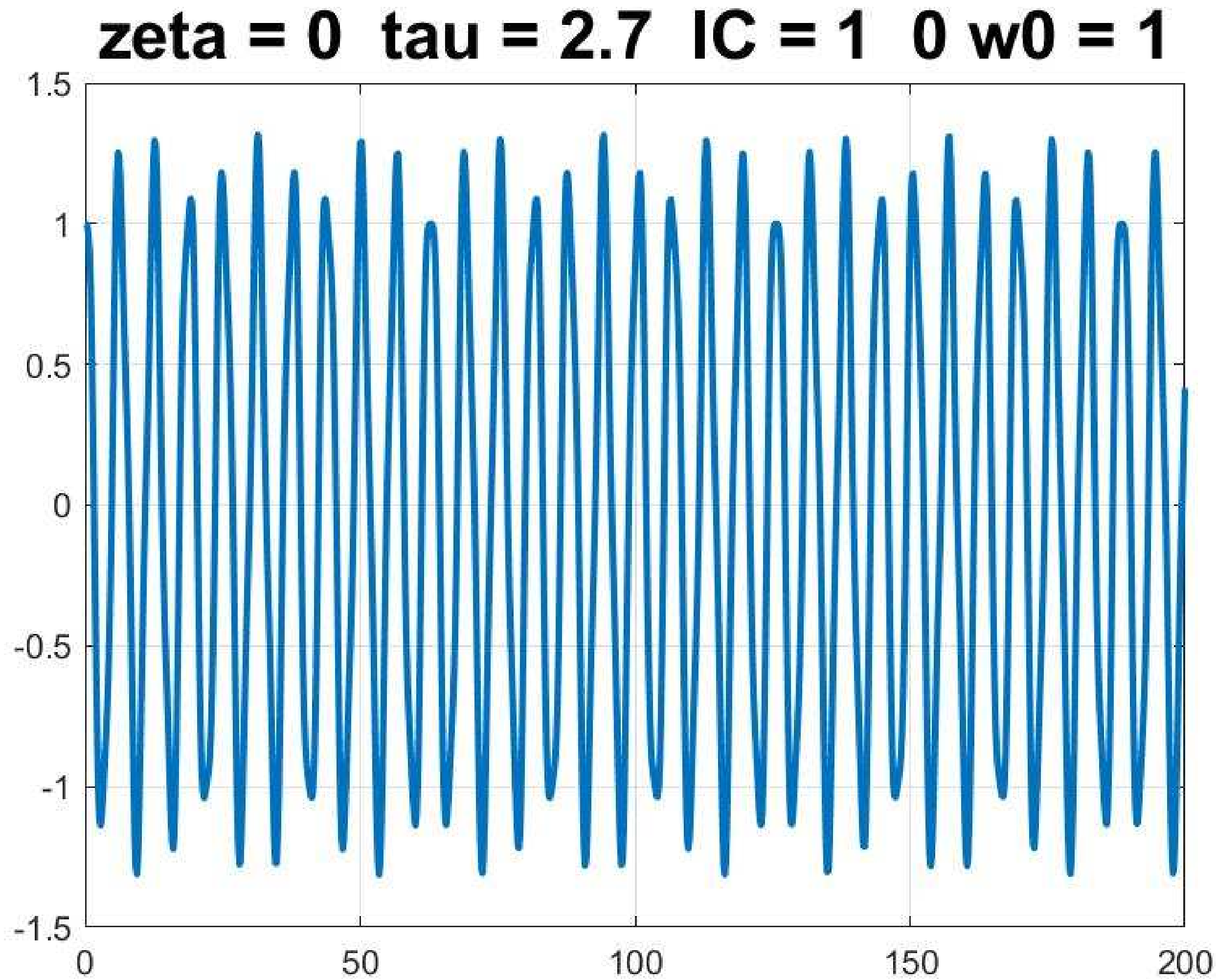


$$y(t) = \frac{t \sin(2.5133t)}{2 \cdot 2.5133}$$

Resonance - displacement $y(t)$ grows larger and larger - this can lead to disaster

ω_0 far from ω - Nothing Special

$$y(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}$$



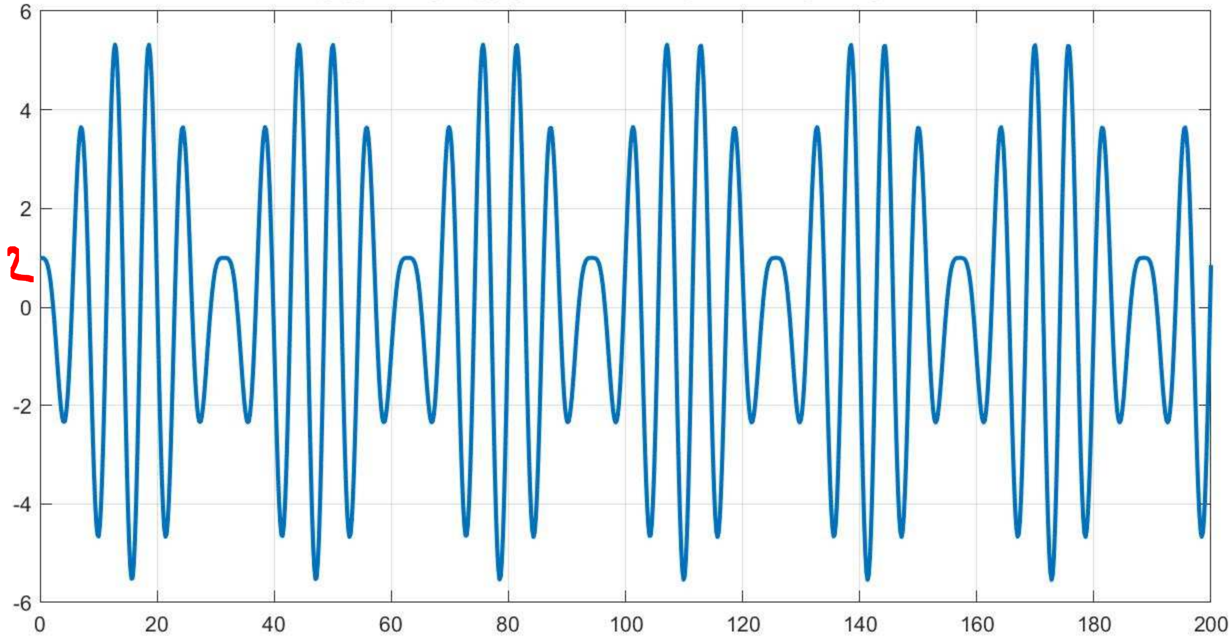
ω_0 close to ω - Beats

$$y(t) = \frac{\cos \omega t - \cos \omega_0 t}{\omega_0^2 - \omega^2}$$

zeta = 0 tau = 1.2 IC = 1 0 w0 = 1

$\omega_0 = 1$

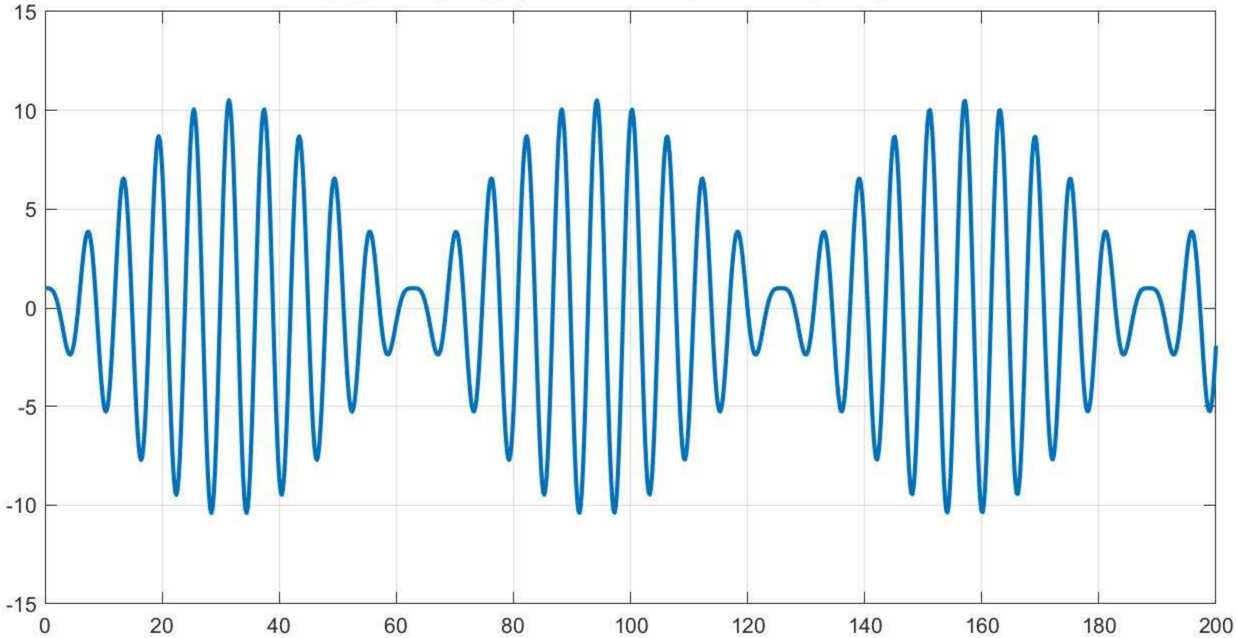
$\omega = 1.2$



zeta = 0 tau = 1.1 IC = 1 0 w0 = 1

$\omega_0 = 1$

$\omega = 1.1$

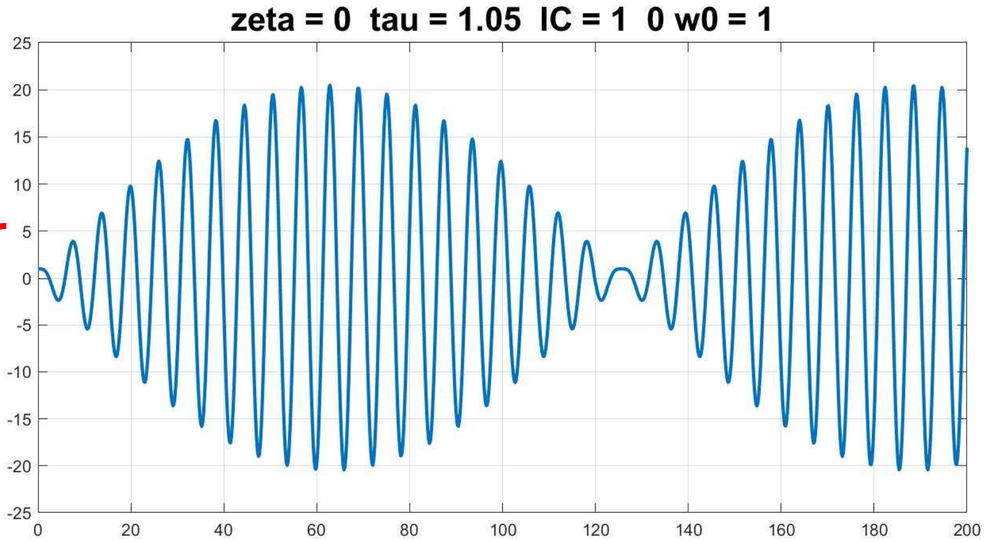


ω_0 close to ω - Beats

$$y(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}$$

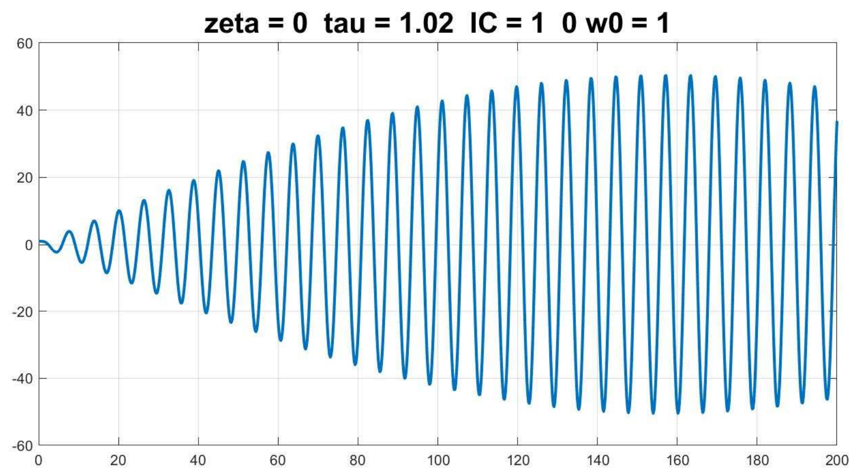
$\omega_0 = 1$

$\omega = 1.05$



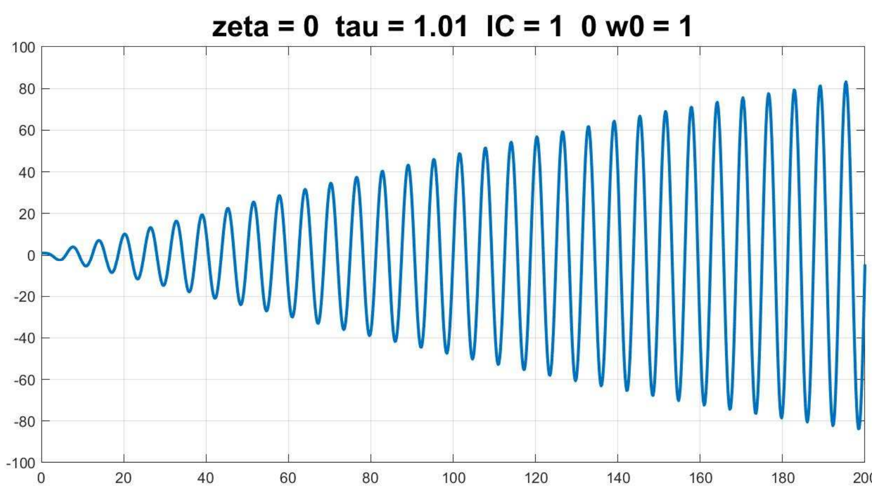
$\omega_0 = 1$

$\omega = 1.02$



$\omega_0 = 1$

$\omega = 1.01$

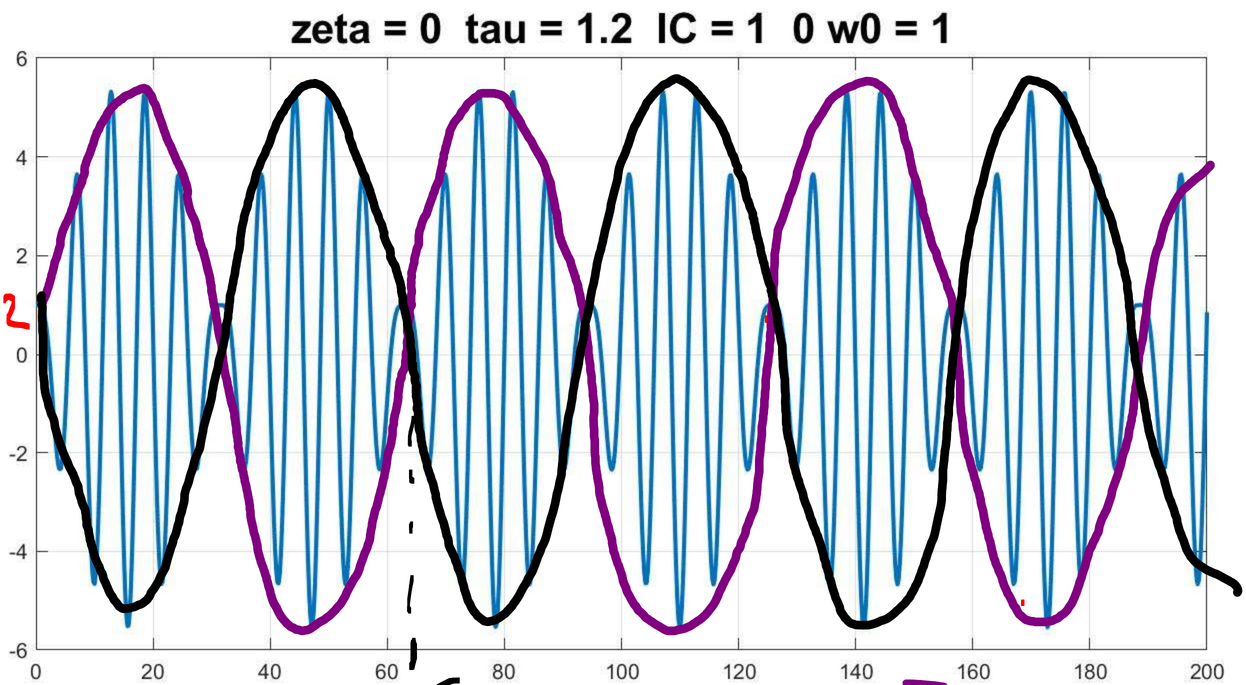


ω_0 close to ω - Beats

$$y(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}$$

$\omega_0 = 1$

$\omega = 1.2$

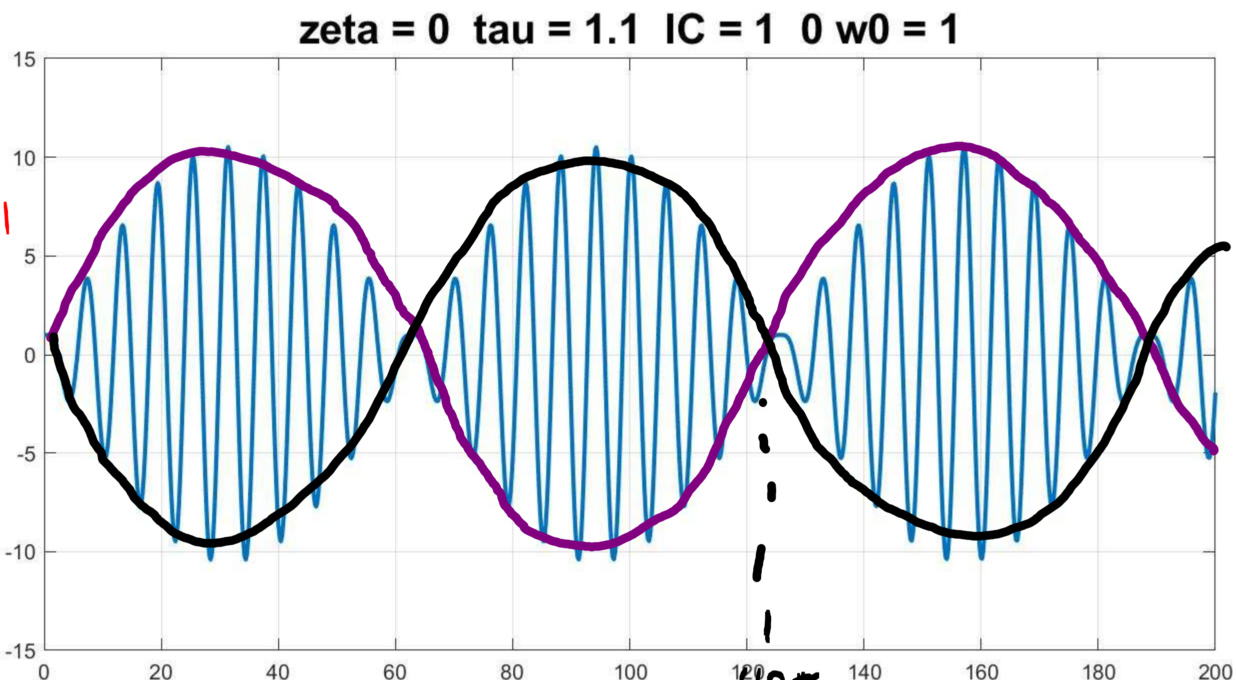


20π

Beat period $= 20\pi$ $\quad \omega_{Beat} = \frac{2\pi}{20\pi} = 0.1$

$\omega_0 = 1$

$\omega = 1.1$



40π

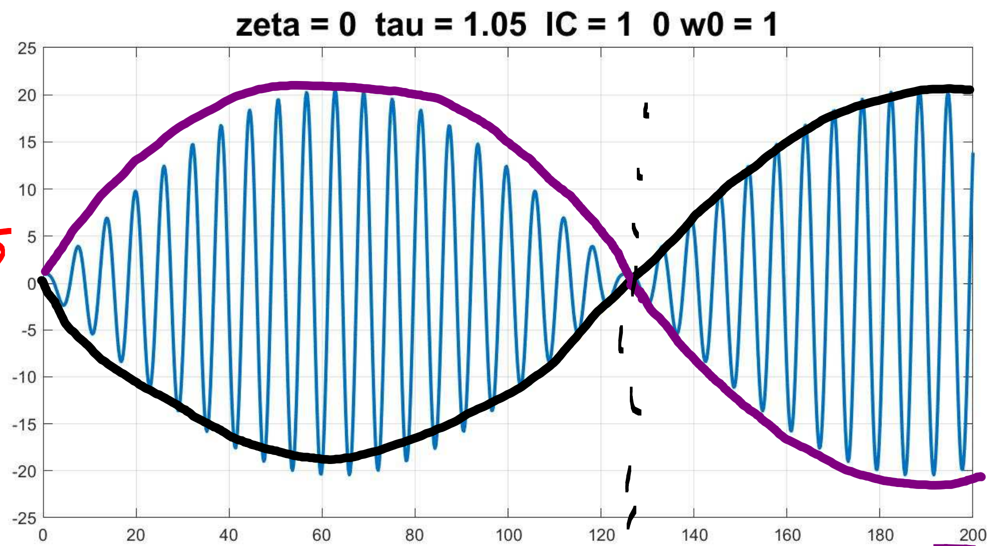
Beat period $= 40\pi$ $\quad \omega_{Beat} = \frac{2\pi}{40\pi} = 0.05$

ω_0 close to ω - Beats

$$y(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}$$

$\omega_0 = 1$

$\omega = 1.05$

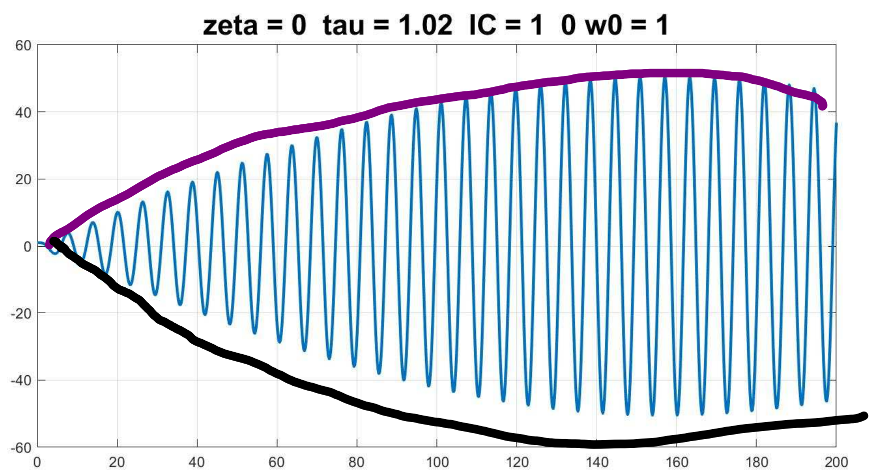


40π

Beat period $= 80\pi$ $\quad \omega_{Beat} = \frac{2\pi}{80\pi} = 0.025$

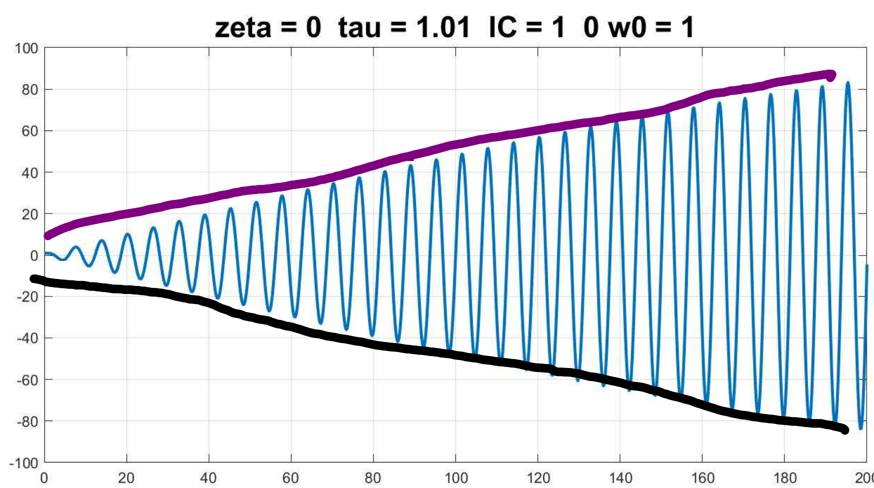
$\omega_0 = 1$

$\omega = 1.02$



$\omega_0 = 1$

$\omega = 1.01$



Observation

$$\omega_{Beat} = \frac{\omega_0 - \omega}{2}$$

We can't easily see this from the formula:

$$y(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}$$

but this formula can be rewritten in a way that makes the beats phenemonon visible

$$y(t) = \frac{-2\sin\left(\frac{\omega_0 - \omega}{2} t\right)}{\omega_0 - \omega} \cdot \frac{\sin\left(\frac{\omega_0 + \omega}{2} t\right)}{\omega_0 + \omega}$$

Product of Sines with

$\frac{\omega_0 + \omega}{2}$ average frequency $\frac{\omega_0 - \omega}{2}$ half difference

$$\omega = \frac{\omega+\omega_0}{2} + \frac{\omega-\omega_0}{2}$$

$$\omega_0 = \frac{\omega+\omega_0}{2} - \frac{\omega-\omega_0}{2}$$

$\frac{\omega+\omega_0}{2}$ Average

$\frac{\omega-\omega_0}{2}$ Half Difference

Sum of angles formula for cosine

$$\cos\omega t = \cos\left(\frac{\omega+\omega_0}{2}t + \frac{\omega-\omega_0}{2}t\right) =$$

$$\cos\left(\frac{\omega+\omega_0}{2}t\right)\cos\left(\frac{\omega-\omega_0}{2}t\right) - \sin\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right) \quad (1)$$

$$\cos\omega_0 t = \cos\left(\frac{\omega+\omega_0}{2}t - \frac{\omega-\omega_0}{2}t\right) =$$

$$\cos\left(\frac{\omega+\omega_0}{2}t\right)\cos\left(\frac{\omega-\omega_0}{2}t\right) + \sin\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right) \quad (2)$$

subtract (2) from (1)

$$\cos\omega t - \cos\omega_0 t = -2\sin\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right)$$

$$y(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}$$

$$= \frac{-2\sin\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right)}{(\omega_0-\omega)(\omega_0+\omega)}$$

$$y(t) = \frac{-2\sin\left(\frac{\omega-\omega_0}{2}t\right)}{\omega-\omega_0} \cdot \frac{\sin\left(\frac{\omega+\omega_0}{2}t\right)}{\omega+\omega_0}$$

Example

$$\ddot{y} + 4^2 y = \cos 5t$$

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

Solution (after some calculations)

$$y = \frac{\cos 4t - \cos 5t}{9}$$

We write y as a product to visualize the "beats".

$$\omega_{AV} = \frac{4+5}{2} \qquad \omega_{HD} = \frac{5-4}{2}$$

[use positive difference for convenience]

$$\cos 4t = \cos\left(\tfrac{9}{2}t - \tfrac{1}{2}t\right) = \cos\left(\tfrac{9}{2}t\right)\cos\left(\tfrac{t}{2}\right) + \sin\left(\tfrac{9}{2}t\right)\sin\tfrac{t}{2}$$

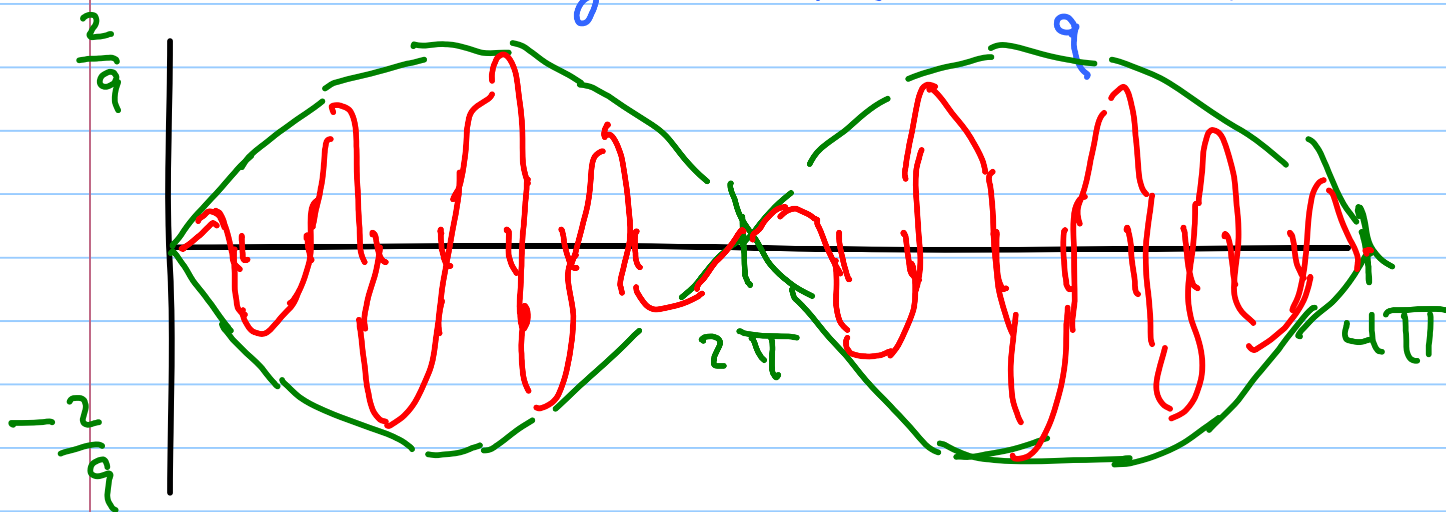
$$\cos 5t = \cos\left(\tfrac{9}{2}t + \tfrac{1}{2}t\right) = \cos\left(\tfrac{9}{2}t\right)\cos\left(\tfrac{t}{2}\right) - \sin\left(\tfrac{9}{2}t\right)\sin\tfrac{t}{2}$$

$$\cos 4t - \cos 5t = -2\sin\tfrac{9}{2}t \, \sin\tfrac{t}{2}$$

$$y(t) = \frac{-2\sin\frac{9}{2}t \, \sin\frac{t}{2}}{9}$$

Sketch

$y(t) = \dfrac{2\sin\frac{9}{2}t\,\sin\frac{t}{2}}{9}$



$\sin\frac{t}{2}$ $\quad P = \dfrac{2\pi}{\frac{1}{2}} = 4\pi \approx 12$

$\sin\frac{9}{2}t$ $\quad$ period $= \dfrac{2\pi}{9/2} = \dfrac{4\pi}{9} \approx \dfrac{12}{9}$

oscillates 9 times faster than $\sin\frac{t}{2}$

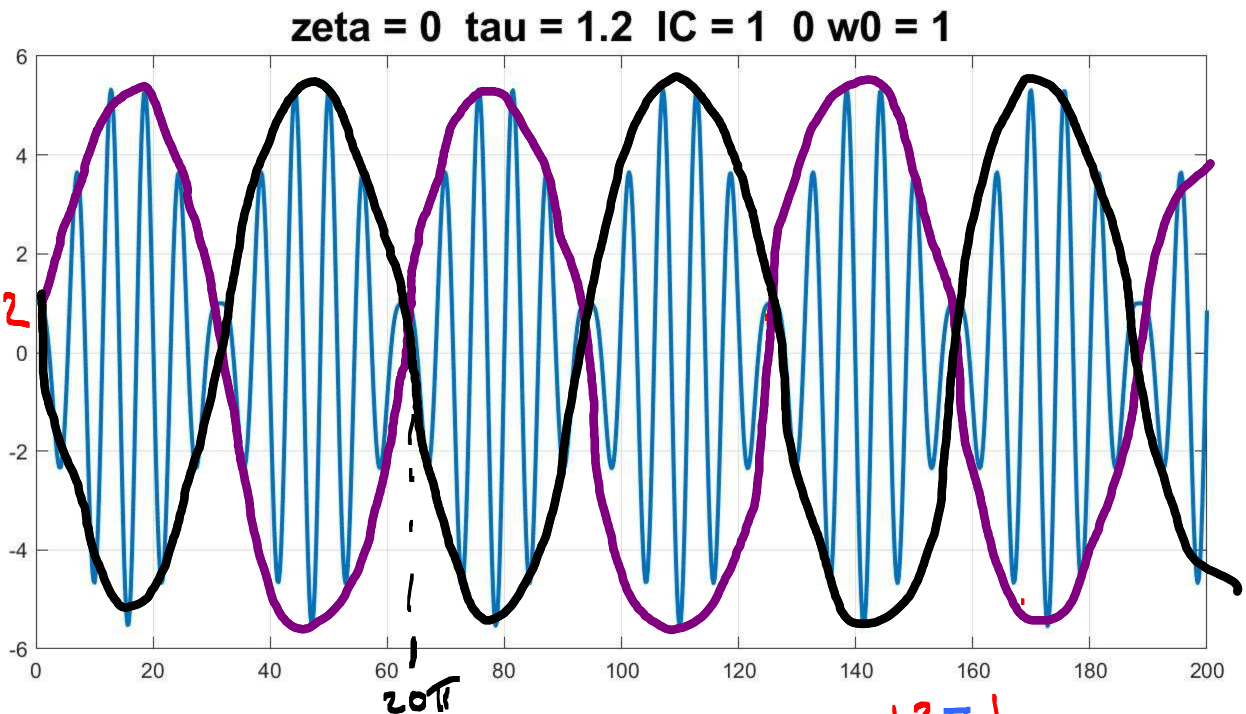
$\sin\frac{9}{2}t$ has 9 zeroes for every zero of $\sin\frac{t}{2}$

ω_0 close to ω - Beats

$$y(t) = \frac{-2\sin\left(\frac{\omega_0-\omega}{2}t\right)}{\omega_0-\omega}\cdot\frac{\sin\left(\frac{\omega_0+\omega}{2}t\right)}{\omega_0+\omega}$$

$\omega_0 = 1$

$\omega = 1.2$

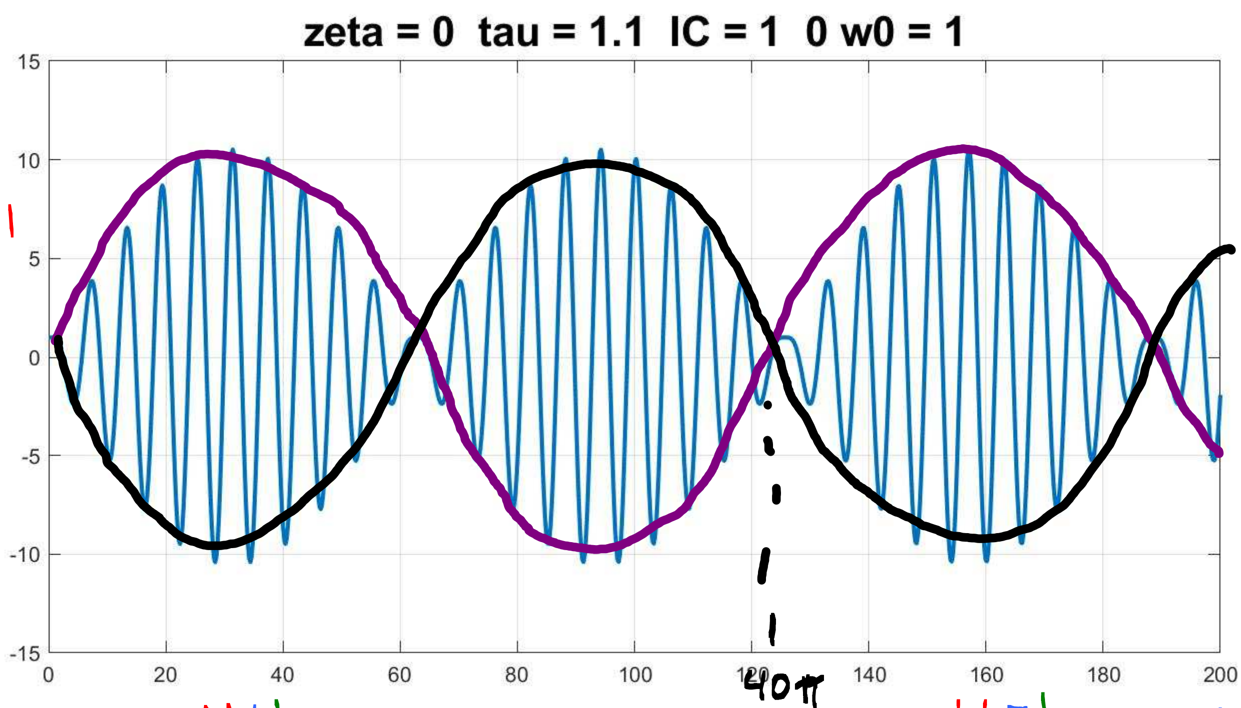


$\omega_{AV} = \frac{1+1.2}{2} = 1.1$

$\omega_{Beat} = \frac{1.2-1}{2} = 0.1$

$\omega_0 = 1$

$\omega = 1.1$



$\omega_{AV} = \frac{1.1+1}{2} = 1.05$

$\omega_{Beat} = \frac{1.1-1}{2} = .05$