Method of Undetermined Coefficients

Problem

Find a particular solution to

$$\ddot{y}_p + y_p = \cos t$$

General Procedure

① Find general homogeneous solution to $\ddot{y}_H + y_H = 0$

Answer $\quad y_H = C_1 \cos t + C_2 \sin t$

② Seek particular solution as a linear combination of the forcing term and its derivatives

Answer $\quad y_p = A \underbrace{\cos t}_{\text{Forcing term}} + B \underbrace{\sin t}_{\text{derivative of forcing term}}$

③ Ask homogeneous question: Are any terms of y_p solutions to the homogeneous equation?

No: Insert y_p in (1)(2) and solve for coefficients

Yes: Multiply those terms by t and repeat step ③

$$\ddot{y}_p + y_p = \cos t$$

$$y_H = C_1 \cos t + C_2 \sin t \qquad y_p = A\cos t + B\sin t$$

(3) Ask homogeneous question: Are any terms of y_p solutions to the homogeneous equation?

No → Proceed to step (4)

Yes → Multiply those terms by t and repeat step (3)

(4) Insert y_p in (1)(2) and solve for coefficients

<u>Answer</u>

Yes, every term in y_p is a term in y_H

Make new $y_p = t \cdot (\text{old } y_p)$

$$y_p = At\cos t + Bt\sin t$$

<u>Repeat step (3)</u>

Are any terms of y_p solutions to the homogeneous equation?

<u>Answer</u> No - proceed to step (4)

Step (4)

$$\ddot{y}_p + y_p = \cos t$$

$$y_p = At\cos t + Bt\sin t$$

$$\dot{y}_p = A\cos t - At\sin t + B\sin t + Bt\cos t$$

$$\ddot{y}_p = -A\sin t - A\sin t - At\cos t + B\cos t + B\cos t - Bt\sin t$$

Organize terms

$$\ddot{y}_p = -At\cos t - Bt\sin t - 2A\sin t + 2B\cos t$$

$$\ddot{y}_p + y_p = -At\cos t - Bt\sin t - 2A\sin t + 2B\cos t + At\cos t + Bt\sin t$$

Simplify

$$= -2A\sin t + 2B\cos t$$

So we must find A and B so that

$$-2A\sin t + 2B\cos t = \cos t$$

$$A = 0 \quad \text{and} \quad B = \tfrac{1}{2}$$

$$\boxed{y_p = \frac{1}{2} t\sin(t)}$$

Where does the "t" come from?

$$\ddot{y} + y = \cos(1.0001t)$$

$$y_p = A\cos(1.0001t) + B\sin(1.0001t)$$

But

$$\ddot{y} + y = \cos t$$

$$y_p = At\cos t + Bt\sin t$$

To see why we solve

$$\ddot{y} + y = \cos\omega t$$

with initial conditions

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

(These initial conditions make the calculation simpler, but any initial conditions will work)

We will write down a formula for the solution for $\omega \neq 1$, and then take the limit as $\omega \to 1$.

$$\ddot{y} + y = \cos \omega t \qquad (DE)$$

$$y(0) = 0 \qquad \dot{y}(0) = 0 \qquad (IC)$$

Homogeneous Solution

$$\ddot{y}_H + y_H = 0 \qquad y_H = c_1 \cos t + c_2 \sin t$$

Seek particular solution

$$y_p = A \cos \omega t + B \sin \omega t$$

$$\ddot{y}_p = -\omega^2 A \cos \omega t - \omega^2 B \sin \omega t$$

Insert y_p in (DE)

$$(1-\omega^2) A \cos \omega t + (1-\omega^2) B \sin \omega t = \ddot{y}_p + y_p = \cos \omega t$$

Conclude: $\quad A = \dfrac{1}{1-\omega^2} \qquad B = 0$

so $\quad y(t) = \dfrac{\cos \omega t}{1-\omega^2} + c_1 \cos t + c_2 \sin t$

Now use (IC)

$$0 = y(0) = \frac{1}{1-\omega^2} + c_1 \Longrightarrow c_1 = \frac{-1}{1-\omega^2}$$

$$0 = \dot{y}(0) = c_2 \Longrightarrow c_2 = 0$$

$$y(t) = \frac{\cos \omega t - \cos t}{\omega^2 - 1}$$

The solution to

$$\ddot{y} + y = \cos \omega t \qquad \text{(DE)}$$

$$y(0) = 0 \qquad \dot{y}(0) = 0 \qquad \text{(IC)}$$

is $$y(t) = \frac{\cos\omega t - \cos t}{\omega^2 - 1}$$

Now we can let $\omega \to 1$

$$\lim_{\omega\to 1} \frac{\cos\omega t - \cos t}{\omega^2 - 1} = \frac{\cos t - \cos t}{1 - 1} = \text{“}\frac{0}{0}\text{”}$$

so we use L'Hôpital's rule

$$\lim_{\omega\to 1} \frac{\cos\omega t - \cos t}{\omega^2 - 1} = \lim_{\omega\to 1} \frac{\frac{d}{d\omega}(\cos\omega t - \cos t)}{\frac{d}{d\omega}(\omega^2 - 1)}$$

$$= \lim_{\omega\to 1} \frac{t \omega \sin\omega t}{2\omega} = \frac{t \sin t}{2}$$

so we see that the forcing function $\cos t$ results in a solution $y(t)$ of the form $A \underbrace{t} \sin t$. This is one way to see why we multiply the forcing term by t when the forcing term matches a term from the homogeneous solution.

Another Example

Find the correct form for y_p

$$\ddot{y} + 5\dot{y} + 6y = t^2 e^{-2t}$$

① $y_h = C_1 e^{-2t} + C_2 e^{-3t}$

② Seek y_p as sum of forcing term and derivatives

$$y_p = A t^2 e^{-2t} + B t e^{-2t} + C e^{-2t}$$

③ Homogeneous Question – Yes

Multiply y_p by t

$$y_p = A t^3 e^{-2t} + B t^2 e^{-2t} + C t e^{-2t}$$

③ Homogeneous Question – No

Proceed to step ④

Steady State and Transients

$$\ddot{y} + 0.2\dot{y} + y = \cos 2t$$

① $r^2 + 0.2r + 1 = 0$

$$(r+0.1)^2 = -0.99$$

$$r = -0.1 \pm i\sqrt{0.99}$$

$$y_H = C_1 e^{-0.1t}\cos(0.995t) + C_2 e^{-0.1t}\sin(0.995t)$$

② $y_p = A\cos 2t + B\sin 2t$

$$\dot{y}_p = -2A\sin 2t + 2B\cos 2t$$

$$\ddot{y}_p = -4A\cos 2t - 4B\sin 2t$$

$$\ddot{y}_p + 0.2\dot{y}_p + y_p = (-3A + 0.4B)\cos 2t$$

$$+ (-3B - 0.4A)\sin 2t$$

$\cos 2t =$

$$\ddot{y}_p + 0.2\,\dot{y}_p + y_p = (-3A + 0.4\,B)\cos 2t$$
$$+ (-3B - 0.4\,A)\sin 2t$$

$$\cos 2t = (-3A + 0.4\,B)\cos 2t + (-3B - 0.4\,A)\sin 2t$$

$$\left.\begin{aligned} 1 &= -3A + 0.4B \\ 0 &= -0.4A - 3B \end{aligned}\right\} \quad \begin{aligned} A &= -0.375 \\ B &= 0.0437 \end{aligned}$$

$$\ddot{y} + 0.2\dot{y} + y = \cos 2t$$

$$y = -0.375\cos 2t + 0.0437\sin 2t$$

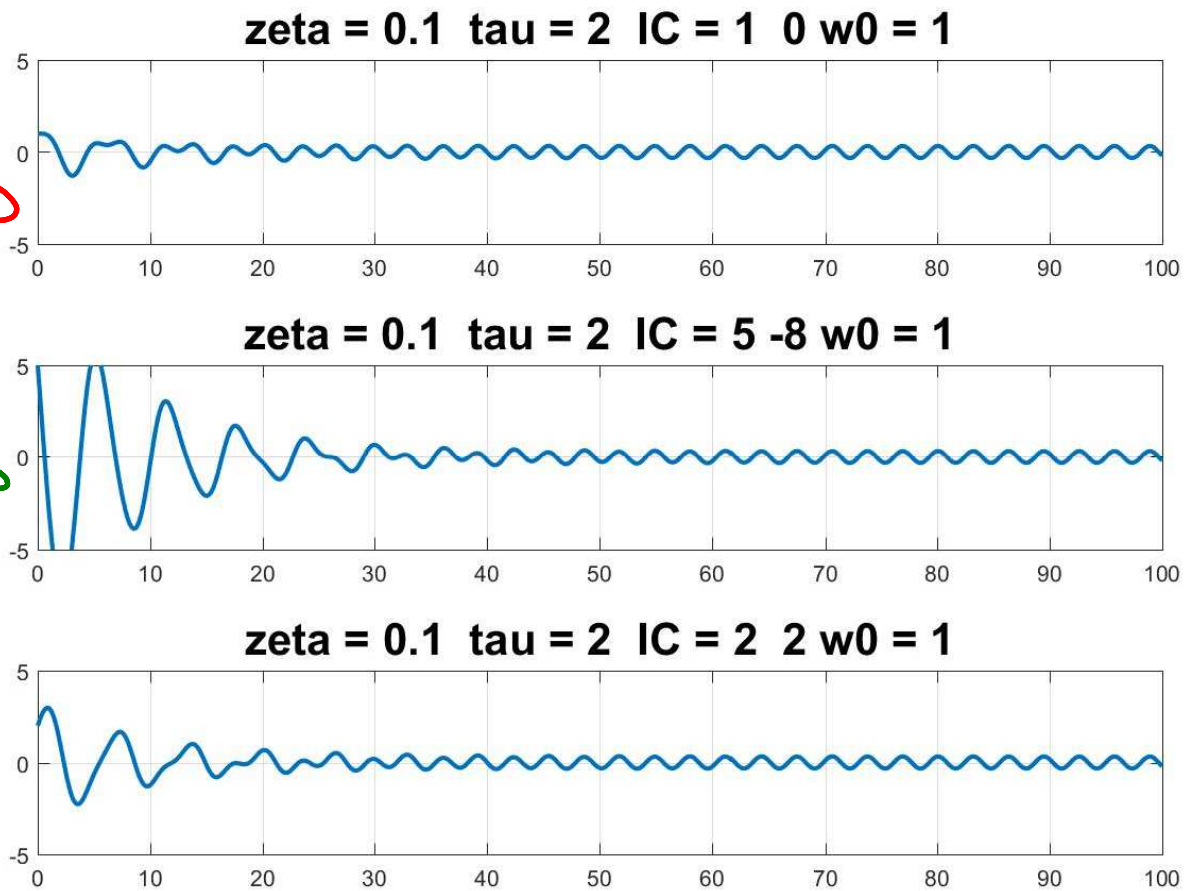
$$+ C_1 e^{-0.1t}\cos(0.995t) + C_2 e^{-0.1t}\sin(0.995t)$$

Initial Conditions

$y(0) = 1$
$\dot{y}(0) = 0$

$y(0) = 5$
$\dot{y}(0) = -8$

$y(0) = 2$
$\dot{y}(0) = 2$



All solutions are essentially the same for $t > 50$. Why?

All solutions are essentially the same for $t > 50$. Why?

We want to compare the sizes of y_p and y_H so we write them in amplitude-phase form.

$$y_p = -0.375 \cos 2t + 0.0437 \sin 2t$$

$$= 0.3775 \cos(2t - 0.116)$$

$$y_H = c_1 e^{-0.1t} \cos(0.995t) + c_2 e^{-0.1t} \sin(0.995t)$$

$$= e^{-0.1t} (c_1^2 + c_2^2)^{1/2} \cos(0.995t + \operatorname{atan2}(c_2, c_1))$$

$$e^{-0.1 \cdot 50} = e^{-5} = 0.067$$

For $t \geq 50$ the amplitude of y_p is 0.3775, while the amplitude of y_H is less than 0.067 times the initial amplitude.

$$y_P = 0.3775\cos(2t - 0.116)$$

constant (steady) amplitude 0.3775

$$y_H = \underbrace{e^{-0.1t}(c_1^2+c_2^2)^{1/2}}_{\text{decaying amplitude}}\cos(0.995t + \operatorname{atan2}(c_2, c_1))$$

Something which decays with time is called transient. IF you wait long enough, its gone.

When the homogeneous solution decays, we call y_P the steady state solution, and y_H the transient.

$$y = 0.3775\cos(2t - 0.116) + e^{-0.1t}(c_1^2+c_2^2)^{1/2}\cos(0.995t + \operatorname{atan2}(c_2, c_1))$$

$$y = y_{ss} + y_{tr}$$

$$y = 0.3775\cos(2t - 0.116) + e^{-0.1t}(c_1^2+c_2^2)^{1/2}\cos(0.995t + \mathrm{atan2}(c_2,c_1))$$

$$y = y_{ss} + y_{tr}$$

The steady state does not depend on the initial conditions. It only depends on the forcing term.

Only the transient part depends on the initial conditions.

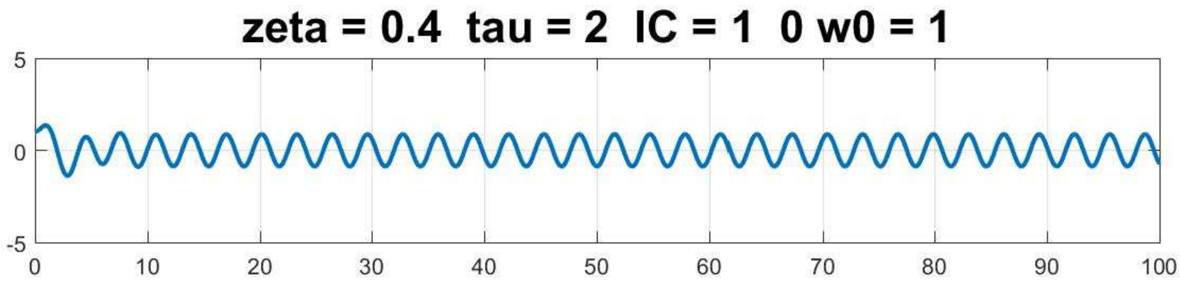
The more damping, the faster the transient part decays

An Example with more damping

$$\ddot{y} + 0.8\dot{y} + y = 3\cos 2t$$

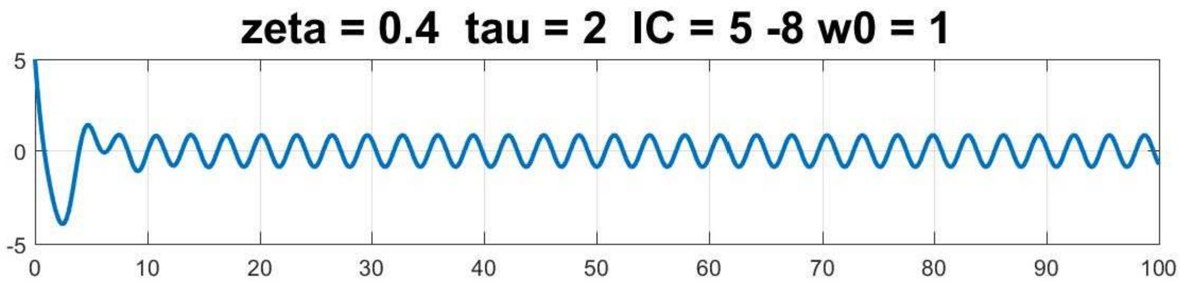
$y(0) = 1$

$\dot{y}(0) = 0$



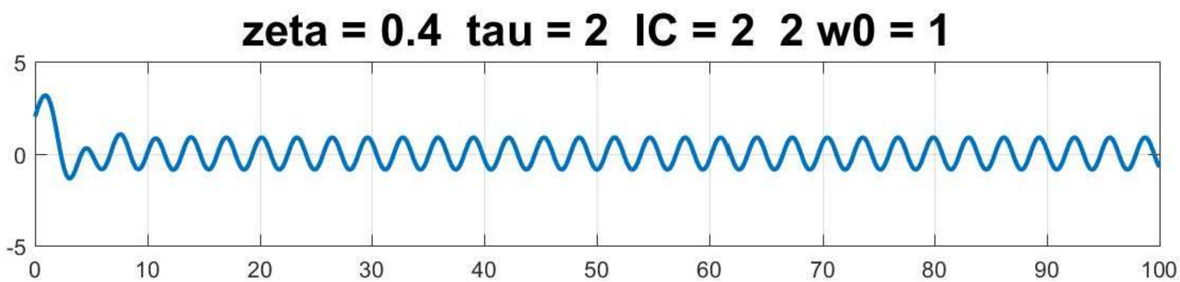
$y(0) = 5$

$\dot{y}(0) = -8$



$y(0) = 2$

$\dot{y}(0) = 2$



$$y_H = y_{tr} = e^{-0.4t} A\cos(0.6t - \phi_1)$$

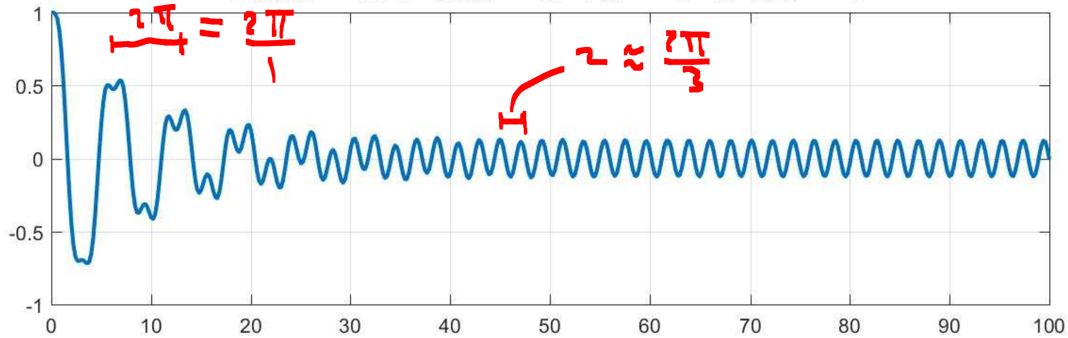
These depend on IC's

$$y_p = y_{ss} = 0.8824\cos(2t - 2.65)$$

$$\ddot{y} + 0.1\dot{y} + y = \cos 3t$$

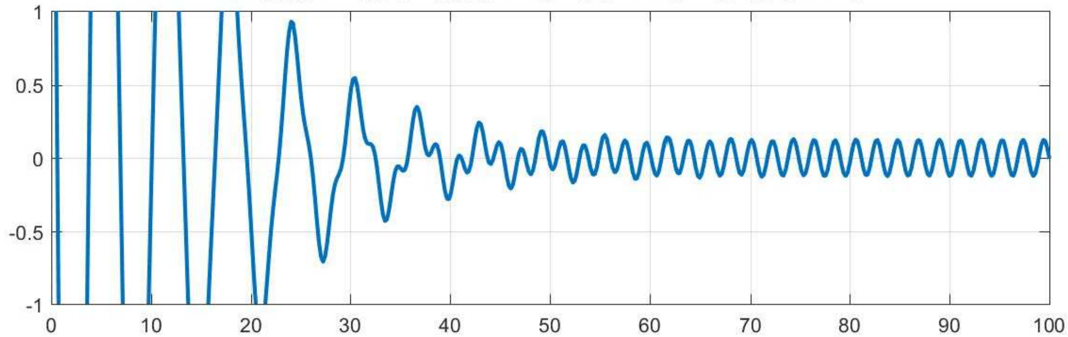
$y(0)=1$ $\dot{y}(0)=0$

zeta = 0.1 tau = 3 IC = 1 0 w0 = 1



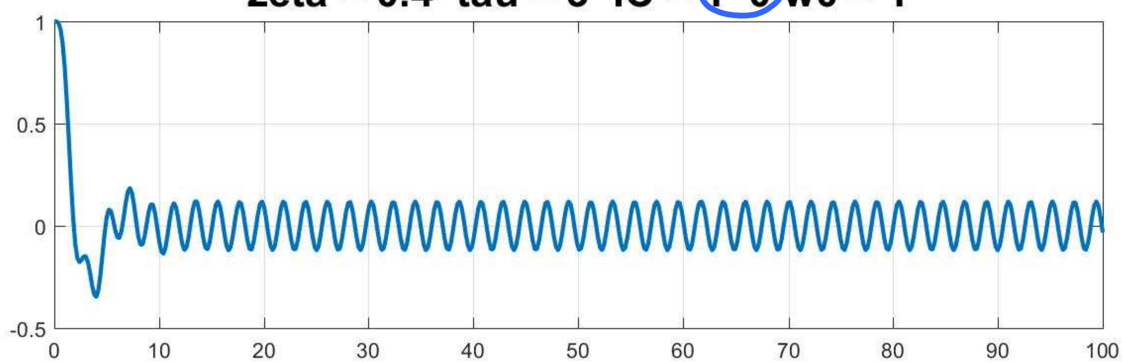
$y(0)=5$ $\dot{y}(0)=-8$

zeta = 0.1 tau = 3 IC = 5 -8 w0 = 1



$$\ddot{y} + 0.4\dot{y} + y = \cos 3t$$

zeta = 0.4 tau = 3 IC = 1 0 w0 = 1 IC's



zeta = 0.4 tau = 3 IC = 5 -8 w0 = 1 IC's