lecture 14

Note Title 10/26/2017

Forced Harmonic Oscillator

$m\ddot{y} + \gamma\dot{y} + ky = F(t)$ = external force

Problem Solve the (IVP)

$$\ddot{y} + 3\dot{y} + 2y = \cos 2t$$

$$y(0) = 0 \qquad \dot{y}(0) = 1$$

Technique

Seek $y = y_p + y_H$

where

(1) Find general solution to

$\ddot{y}_H + 3\dot{y}_H + 2y_H = 0$ called "homogeneous solution"

(2) Find any solution to

$\ddot{y}_p + 3\dot{y}_p + 2y_p = \cos 2t$

called "particular solution" or "steady state solution"

(3) Choose the constants from (1) to satisfy IC's.

$$\ddot{y} + 3\dot{y} + 2y = \cos 2t$$

$$y(0) = 0 \qquad \dot{y}(0) = 1$$

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1. $\ddot{y}_H + 3\dot{y}_H + 2y_H = 0$

Seek $y = e^{rt}$ $\quad r^2 + 3r + 2 = 0$

$$\boxed{y_H(t) = C_1 e^{-t} + C_2 e^{-2t}}$$

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2. $\ddot{y}_p + 3\dot{y}_p + 2y_p = \cos 2t$

Seek $y_p = A\cos 2t$ ← Not good enough

Seek $y_p = A\cos 2t + B\sin 2t$ ← This one works

Rule: The particular solution must contain constants times

1. The forcing term
2. All terms that are derivatives of forcing terms

$$\ddot{y}_p + 3\dot{y}_p + 2y_p = \cos 2t$$

This doesn't work

Try $y_p = A\cos 2t$

$$2y_p = 2A\cos 2t$$

$$3\dot{y}_p = -2A\sin 2t \cdot 3$$

$$+ \quad \ddot{y}_p = -4A\cos 2t$$

$$\cos 2t = \underbrace{-2A}_{=1}\cos 2t \underbrace{-6A}_{=0}\sin 2t$$

Can't solve this

$$2\, y_p = 2A\cos 2t + 2B\sin 2t$$

$$3 \cdot \dot{y}_p = 3\cdot(-2)A\sin 2t + 3\cdot 2B\cos 2t$$

$$+ \quad \ddot{y}_p = -4A\cos 2t - 4B\sin 2t$$

$$\cos 2t = (2A + 6B - 4A)\cos 2t + (2B - 6A - 4B)\sin 2t$$

$$0\sin 2t + 1\cos 2t = (-2B-6A)\sin 2t + (6B - 2A)\cos 2t$$

$$0 = (-2B-6A) \qquad B = -3A$$

$$1 = (6B - 2A) \qquad 1 = -20A$$

$$A = -\frac{1}{20} \qquad B = \frac{3}{20}$$

$$y_p = -\frac{1}{20}\cos 2t + \frac{3}{20}\sin 2t$$

Now go to step 3

$$y(t) = y_p + y_H$$

$$= \frac{-1}{20}\cos 2t + \frac{3}{20}\sin 2t + C_1 e^{-t} + C_2 e^{-2t}$$

$$0 = y(0) = \frac{-1}{20}\cdot 1 + \frac{3}{20}\cdot 0 + C_1\cdot 1 + C_2 \cdot 1$$

$$1 = \dot{y}(0) = \frac{2}{20}\cdot 0 + \frac{6}{20}\cdot 1 - C_1 - 2C_2$$

$$\left.\begin{aligned} C_1 + C_2 &= \frac{1}{20} \\ -C_1 - 2C_2 &= \frac{14}{20} \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 &= \frac{16}{20} \\ C_2 &= \frac{-15}{20} \end{aligned}$$

$$y(t) = \frac{-1}{20}\cos 2t + \frac{3}{20}\sin 2t + \frac{16}{20} e^{-t} - \frac{15}{20} e^{-2t}$$

Notice: As t gets larger, $y_H \to 0$

So, after a long time

$$y(t) \approx \frac{-1}{20}\cos 2t + \frac{3}{20}\sin 2t$$

so we call $y_p(t)$ the steady state solution.

Back to discussion of particular solution

(2) $\ddot{y}_p + 3\dot{y}_p + 2y_p = \cos 2t$

why make this choice?

Seek $y_p = A\cos 2t + B\sin 2t$

I want sums of functions $y_p, \dot{y}_p, \ddot{y}_p$ that will sum up to $\cos 2t$.

Start with $y_p = A\cos 2t$

but y_p will get differentiated, so I will see terms like $B\sin 2t$

and $B\sin 2t$ will get differentiated, so I'll see terms like $C\cos 2t$.

But I already have this term, so I stop here.

Same as previous page with fewer words

$\cos 2t$ 1

$\Big\downarrow \frac{d}{dt}$

$2\sin 2t$ 2

$\Big\downarrow \frac{d}{dt}$

$-4\cos 2t$ old STOP

$y_p(t) =$

$A\cos 2t + B\sin 2t$

Another example

$\ddot{y}_p + 2y_p = t\cos t$

$t\cos t$ 1

$\Big\downarrow \frac{d}{dt}$

2 $\cos t$ $-$ $t\sin t$ 3

$\Big\downarrow \frac{d}{dt}$ $\qquad \Big\downarrow \frac{d}{dt}$

$-\sin t$ $\qquad t\cos t$ old

4 $\qquad + \sin t$ old

$\Big\downarrow$

$-\cos t$ old $\qquad$ stop

$y_p =$
$(At+B)\cos t$

$+ (Ct+D)\sin t$

Slightly More Complicated Example

$$\ddot{y} + 3\dot{y} + 2y = t^2 e^{-4t}$$

Homogeneous Solution is the same

What is the form of y_p ?

Start with $y_p = At^2 e^{-4t} + ??$

Differentiate

$$\dot{y}_p = 2t e^{-4t} - 8t^2 e^{-4t}$$

I could add the whole derivative

$$y_p = At^2 e^{-4t} + B(2t e^{-4t} - 8t^2 e^{-4t})$$
$$= (A - 8B) t^2 e^{-4t} + 2Bte^{-4t}$$
$$= A_1 t^2 e^{-4t} + B_1 te^{-4t}$$

but its simpler to just add the new term

$$\dot{y}_p = \underbrace{2t e^{-4t}}_{\text{new term}} \underbrace{- 8t^2 e^{-4t}}_{\text{old term}}$$

$$y_p = At^2 e^{-4t} + Bte^{-4t} + ??$$

$$y_p = At^2e^{-4t} + Bte^{-4t} + ??$$

Now differentiate again. Its good enough to just differentiate the new term.

$$(te^{-4t})' = \underbrace{e^{-4t}}_{\text{new term}} - 4\underbrace{te^{-4t}}_{\text{old term}}$$

Add the new term

$$y_p = At^2e^{-4t} + Bte^{-4t} + Ce^{-4t} + ??$$

Now differentiate again. Its good enough to just differentiate the new term.

$$(e^{-4t})' = -4\underbrace{e^{-4t}}_{\text{old term}}$$

No new term - I can stop!!

$$y_p = At^2e^{-4t} + Bte^{-4t} + Ce^{-4t}$$

This is exactly the same calculation as the previous pages, with fewer words

Purple means "not important"

$$t^2 e^{-4t} \quad ①$$

$$\Big\downarrow \frac{d}{dt}$$

$$2t e^{-4t} \; ② \quad -4 t^2 e^{-4t} \; \text{old}$$

STOP

$$\Big\downarrow \frac{d}{dt}$$

$$2 e^{-4t} \; ③ \quad -4 t e^{-4t} \; \text{old}$$

STOP

$$\Big\downarrow$$

$$-8 e^{-4t} \; \text{old}$$

STOP

$$y_p(t) = At^2 e^{-4t} + Bte^{-4t} + Ce^{-4t}$$

or

$$(At^2 + Bt + C)e^{-4t}$$

$$y_p = At^2e^{-4t} + Bte^{-4t} + Ce^{-4t}$$

Now substitute y_p into

$$\ddot{y}_p + 3\dot{y}_p + 2y_p = t^2e^{-4t}$$

and solve for A, B, C

$$y_p = At^2e^{-4t} + Bte^{-4t} + Ce^{-4t}$$

$$\dot{y}_p = -4At^2e^{-4t} + (2A-4B)te^{-4t} + (B-4C)e^{-4t}$$

$$\ddot{y}_p = 16At^2e^{-4t} + (-16A+16B)te^{-4t} + (2A-8B+16C)e^{-4t}$$

$$2y_p = 2At^2e^{-4t} + 2Bte^{-4t} + 2Ce^{-4t}$$

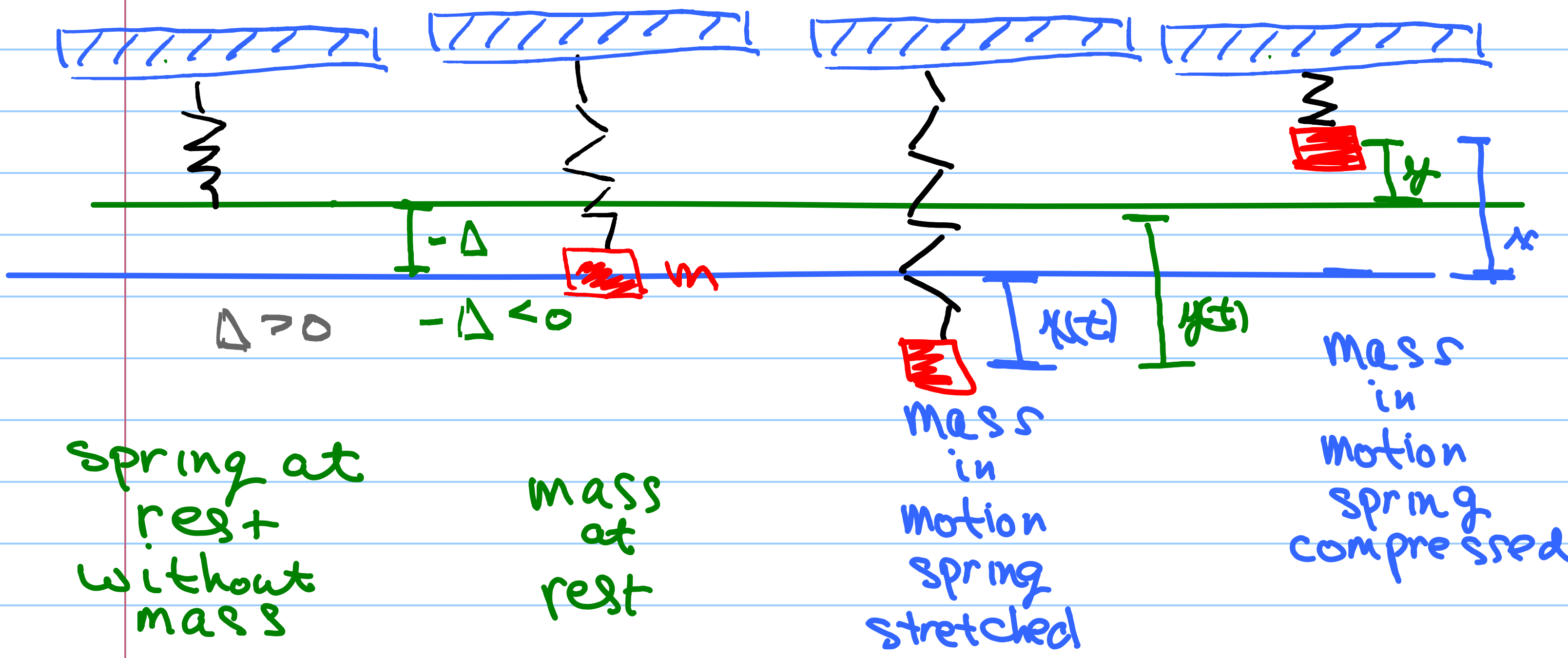
$$+\ 3\dot{y}_p = -12At^2e^{-4t} + (6A-12B)te^{-4t} + (3B-12C)e^{-4t}$$

$$t^2e^{-4t} = 6At^2e^{-4t} + (-10A-6B)te^{-4t} + (2A-5B+6C)e^{-4t}$$

$$\left.\begin{aligned} 6A &= 1 \\ -10A-6B &= 0 \\ 2A-5B+6C &= 0 \end{aligned}\right\} \qquad \begin{aligned} A &= \frac{1}{6} \\ B &= \frac{-10}{36} = \frac{-5}{18} \\ C &= \frac{-31}{18\cdot 6} \end{aligned}$$

I didn't check this carefully.

Where did gravity go?



<u>Mass at Rest</u> – Net Force = 0

$0 = -k(\Delta) - mg$ = Spring Force + Gravitational Force

DE for $y(t)$ $\quad m\ddot{y} = -ky - mg$

But Everyone writes DE for $x(t) = y + \Delta$

$$m\ddot{x} = m\ddot{y} = -ky - mg = -k(x-\Delta) - mg$$

$$= -kx + \underbrace{k\Delta - mg}_{=0}$$

$$\boxed{m\ddot{x} = -kx}$$

Experimental Method to determine k

① Attach m

② measure Δ $\qquad$ ③ $k = \dfrac{mg}{\Delta}$

You are not responsible for this

How do you force a mass spring system?

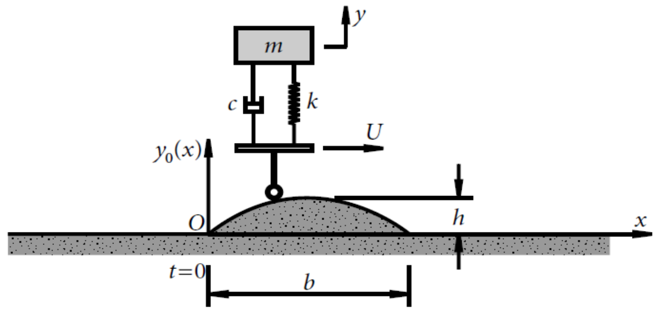
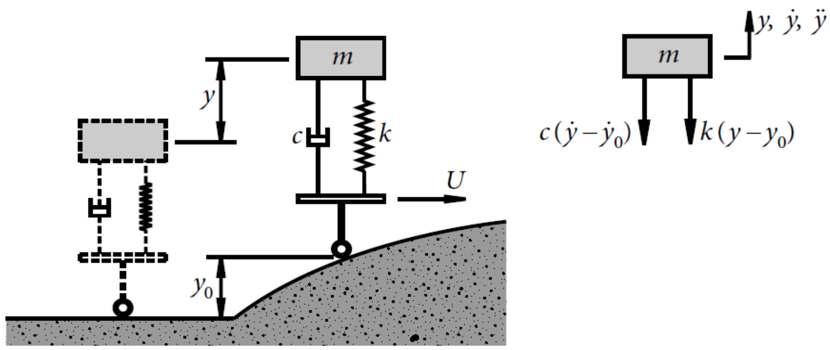


Figure 5.20 A vehicle passing a speed bump.



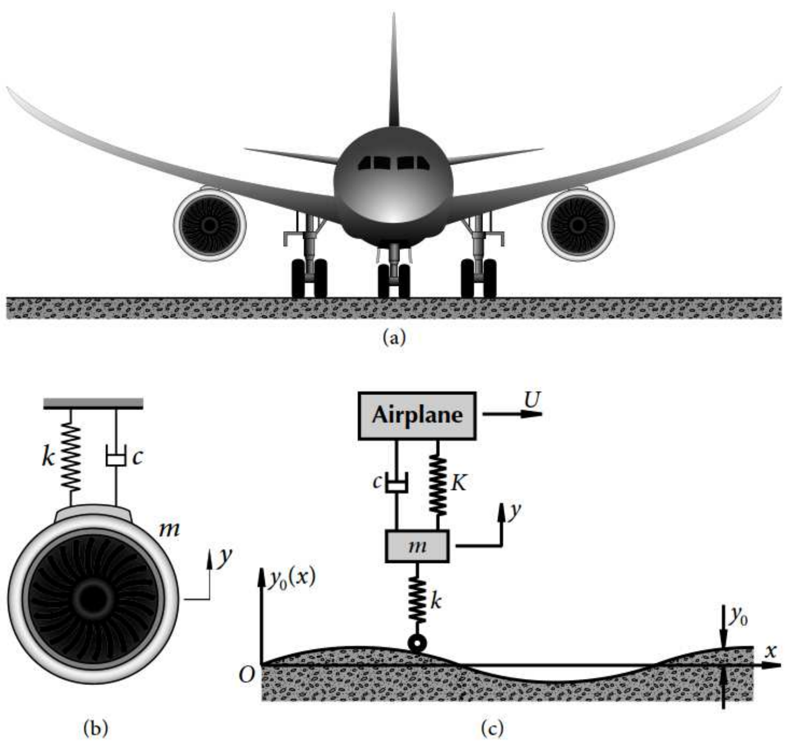


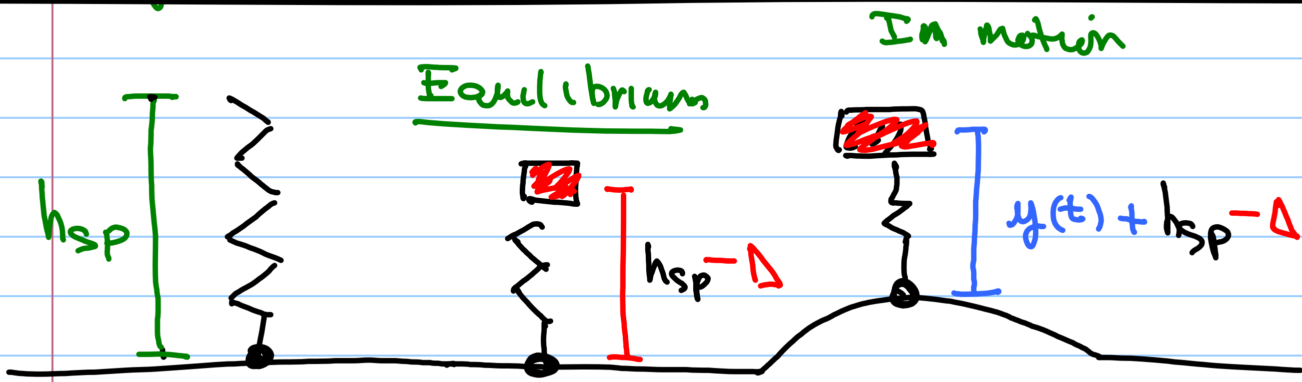
Figure 5.8 Mathematical modeling of jet engine and landing gear.

The forcing comes from driving over an uneven surface, or from a moving surface, as in an earthquake. The force is transmitted through a spring.

Height of the ground

e.g. $H_g(x) = A\cos(\alpha x)$

Mass Spring moving along bumpy ground at speed c



$h_g(t)$ = height of the ground

$\left[= H_g(ct) \text{ where } \right.$

$\left. H_g(x) = \text{height at position } x \right]$

$\boxed{\begin{array}{l}\underline{\text{At equilibrium}} \\ 0 = -mg - k(-\Delta) \\ \text{or } k\Delta = mg\end{array}}$

Newton

$$m\begin{pmatrix}\text{block}\\ \text{acceleration}\end{pmatrix} = \sum \text{Forces}$$

$$m\ddot{(\text{block height})} = -k\begin{pmatrix}\text{spring}\\ \text{compressed}\end{pmatrix} - mg$$

$$m\ddot{\left(y(t)+h_{sp}-\Delta+h_g(t)\right)} = -k(y-\Delta) - mg$$

$$m\ddot{y} + m\ddot{h}_g = -ky + \underbrace{k\Delta - mg}_{=0}$$

$$\boxed{m\ddot{y} + ky = -m\ddot{h}_g(t)}$$

$$m\ddot{y} + ky = -m\ddot{h}_g(t)$$

$$h_g(t) = H_g(ct) =$$

$$\ddot{h}_g = c^2 \ddot{H}_g(ct)$$

e.g. $H_g(x) = A\cos(\alpha x)$

$$\ddot{h}_g = \alpha^2 c^2 A \cos(\alpha c t)$$

$$m\ddot{y} + ky = m\alpha^2 c^2 A\cos(\alpha c t)$$

Notice: Forcing frequency depends on speed c and "wavenumber" α.