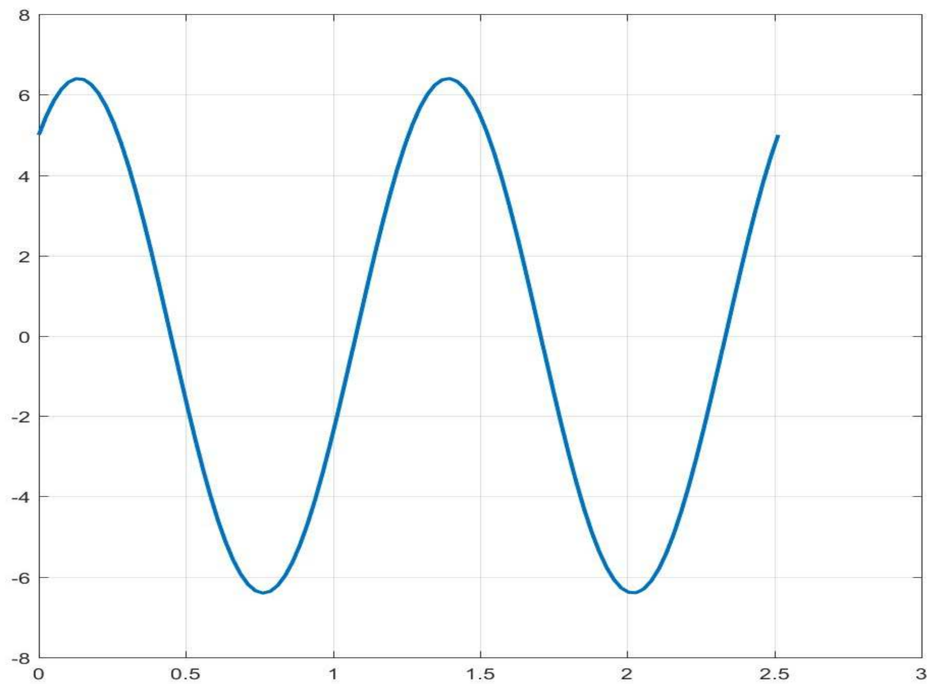
Find Frequency Amplitude Phase from the graph

$$y(t) = A\cos(\omega t - \varphi)$$



A = amplitude $\quad \omega$ = angular frequency (omega)

φ = phase (lag)

slightly different form

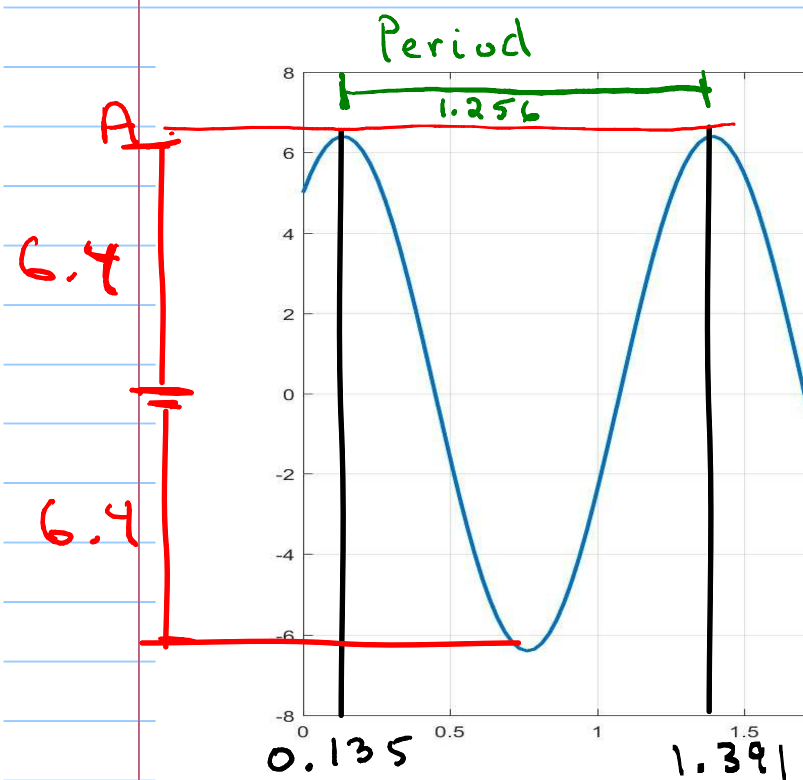
$$y(t) = A\cos(\omega(t - t_0))$$

Because its easier to find t_0 first

t_0 = time lag

$\varphi = \omega t_0$

$$y(t) = A\cos(\omega(t - t_0))$$



$A = 6.4$ $\qquad \omega = 2\pi/\text{Period} = 2\pi/1.256 = 5$

$t_0 = 0.135$ Because $A\cos(\omega(t-t_0)) = A$
when $\omega(t-t_0) = 0$

$$y(t) = 6.4\cos(5(t - 0.135))$$
$$= 6.4\cos(5t - 0.675)$$

$\varphi = 0.675$

Linear DE's $\ddot{y} + a\dot{y} + by = f(t)$

Two Principles

① If $y_1(t)$ solves and $y_2(t)$ solves

then $c_1 y_1(t) + c_2 y_2(t)$ solves

② If $y_1(t)$ and $y_2(t)$ are independent $(y_1(t) \neq C\, y_2(t))$ solutions

then every solution $y(t)$ can be written as $y(t) = C_1\, y_1(t) + C_2\, y_2(t)$

① is easy to understand

② is harder

I will expect you to use these two principles. You don't need to remember the proofs.

Example of ①

If
$$\ddot{y}_1 + 4y_1 = f_1$$
$$\ddot{y}_2 + 4y_2 = f_2$$

then
$$w = 3y_1 + 8y_2$$

solves
$$\ddot{w} + 4w = 3f_1 + 8f_2$$

Proof
$$3\ddot{y}_1 + 3\cdot 4\, y_1 = 3f_1$$
$$+ \quad 8\ddot{y}_2 + 8\cdot 4\, y_2 = 8f_2$$

$$3\ddot{y}_1 + 8\ddot{y}_2 + 4\cdot(3y_1 + 8y_2) = 3f_1 + 8f_2$$

derivative is linear

$$(3y_1 + 8y_2)^{\cdot\cdot} + 4\cdot(3y_1 + 8y_2) = 3f_1 + 8f_2$$

$$\ddot{w} + 4w = 3f_1 + 8f_2$$

So far, we have only used this in the case that $f_1 = 0$ and $f_2 = 0$

Thm There is exactly one solution to a second order initial value problem.

Proof by example

$$\ddot{y} = 5y \qquad y(0) = a \qquad \dot{y}(0) = b$$

$$\ddot{y}(0) = 5y(0) = 5a$$

$$\dddot{y}(0) = 5\dot{y}(0) = 5b$$

$$y^{(iv)}(0) = 5\ddot{y}(0) = 25a$$

$$y^{(v)}(0) = 5\dddot{y}(0) = 25b$$

etc.

$$y(t) = \sum_{t=0}^{\infty} \frac{y^{(n)}(0)}{n!} t^n$$

We have produced $y(t)$ as a formula involving only a and b

Corollary

$$y_1(t) = y_2(t) \iff \begin{array}{c} y_1(0) = y_2(0) \\ \text{and} \\ \dot{y}_1(0) = \dot{y}_2(0) \end{array}$$

Corollary

$$y_1(t) = c\, y_2(t) \iff \begin{array}{c} y_1(0) = c\, y_2(0) \\ \text{and} \\ \dot{y}_1(0) = c\, \dot{y}_2(0) \end{array} \iff y_1(0)\,\dot{y}_2(0) - \dot{y}_1(0)\, y_2(0) = 0$$

Corollary

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \iff \begin{array}{c} y(0) = c_1 y_1(0) + c_2 y_2(0) \\ \text{and} \\ \dot{y}(0) = c_1 \dot{y}_1(0) + c_2 \dot{y}_2(0) \end{array}$$

Every solution $y(t) = c_1 y_1(t) + c_2 y_2(t)$

$\iff$ I can solve for c_1 and c_2 from

$$e = c_1 y_1(0) + c_2 y_2(0)$$

and

$$f = c_1 \dot{y}_1(0) + c_2 \dot{y}_2(0)$$

for every e and f

Two equations in two unknowns

$$a c_1 + b c_2 = e$$

$$c c_1 + d c_2 = f$$

Formula for solution

$$c_1 = \frac{be - cf}{ad - bc}$$

$$c_2 = \frac{ae - df}{ad - bc}$$

as long as $ad - bc \neq 0$

Corollary

I can solve for c_1 and c_2 from

$e = c_1 y_1(0) + c_2 y_2(0)$

and

$f = c_1 \dot{y}_1(0) + c_2 \dot{y}_2(0)$

for every e and f

$$\Longleftrightarrow$$

$y_1(0)\dot{y}_2(0) - \dot{y}_1(0)\, y_2(0) \neq 0 \iff y_1(t) \neq C\, y_2(t)$

$ad - bc \neq 0$

Back to stuff you're responsible for

Example

$$\ddot{y} + 4y = 0$$

Some solutions

$y_1 = \cos 2t$

$y_2 = \sin 2t$

$y_3 = \cos(2t-3)$

$y_4 = 6\cos(2t-3) + P\sin 2t$

I can always find a solution to the

(IVP) $\quad \ddot{y} + 4y = 0$

$y(0) = a$

$\dot{y}(0) = b$

of the form

$$c_1 \cos 2t + c_2 \cos(2t-3)$$

or

$$c_1 \sin 2t + c_2 \cos(2t-3)$$

or

$\vdots$

choose any 2

$c_1 \sin(2t) + c_2 \, 2\sin(2t)$ doesn't work

$c_1 \sin(2t) + c_2 \cos(2t - \frac{\pi}{2})$ doesn't work

Forced Harmonic Oscillator

$m\ddot{y} + \gamma\dot{y} + ky = F(t)$ = external force

Problem Solve the (IVP)

$$\ddot{y} + 3\dot{y} + 2y = e^{-4t}$$

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

Technique

Seek $y = y_p + y_H$

where

① Find general solution to

$\ddot{y}_H + 3\dot{y}_H + 2y_H = 0$ called "homogeneous solution"

② Find any solution to

$\ddot{y}_p + 3\dot{y}_p + 2y_p = e^{-4t}$ called "particular solution"
or "steady state solution"

③ Choose the constants from ①
to satisfy IC's.

① Find general solution to homogeneous equation.

$$\ddot{y}_H + 3\dot{y}_H + 2y_H = 0$$

Seek $y_H = e^{rt}$

$$r^2 + 3r + 2 = 0$$
$$(r+2)(r+1) = 0$$

$$y_H = C_1 e^{-2t} + C_2 e^{-t}$$

② Find any solution to

$$\ddot{y}_p + 3\dot{y}_p + 2y_p = e^{-4t}$$

Seek $y_p = Ae^{-4t}$

Substitute y_p into (DE) and solve for A

$$(-4)^2 A e^{-4t} + 3(-4) A e^{-4t} + 2Ae^{-4t} = e^{-4t}$$

$$(16 - 12 + 2) A = 1$$

$$6A = 1$$
$$A = \frac{1}{6}$$

$$\boxed{y_p = \frac{1}{6} e^{-4t}}$$

Recall $y(t) = y_H + y_p$

so $y = \frac{1}{6} e^{-4t} + C_1 e^{-2t} + C_2 e^{-t}$

Now use IC's to find C_1 and C_2.

(IC)'s $\quad y(0) = 0 \qquad \dot{y}(0) = 0$

$$\left.\begin{aligned} 0 &= y(0) = \tfrac{1}{6} + C_1 + C_2 \\ 0 &= \dot{y}(0) = -\tfrac{4}{6} - 2C_1 - C_2 \end{aligned}\right\} \quad \begin{aligned} C_1 &= -1/2 \\ C_2 &= 1/3 \end{aligned}$$

$$\boxed{y = \tfrac{1}{6} e^{-4t} - \tfrac{1}{2} e^{-2t} + \tfrac{1}{3} e^{-t}}$$