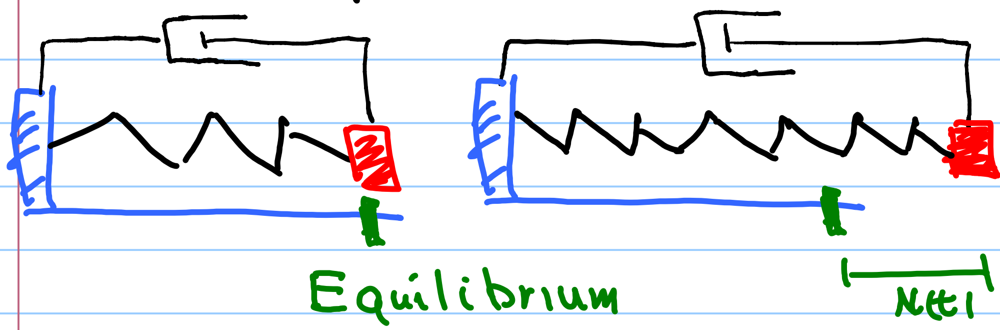
Note Title 2/1/2020

Damped Harmonic Oscillator



Suppose the spring constant is 145 $kg\,sec^{-2}$

Damping coeff. = 2 $kg\,sec^{-1}$. $m = 1$ kg

Initial displacement = 1 meters

Initial velocity = 2 meters/sec

① Write the Initial Value Problem

② Find a formula for $x(t)$

Answer

$$m\ddot{x} = -\gamma\dot{x} - kx$$

$$1\ddot{x} = -2\dot{x} - 145x$$

$$\ddot{x} + 2\dot{x} + 145x = 0 \qquad \text{(IVP)}$$

$$x(0) = 1 \qquad \dot{x}(0) = 2$$

$$\ddot{x} + 2\dot{x} + 145x = 0$$

(IVP)

$$x(0) = 1 \qquad \dot{x}(0) = 2$$

Seek $\quad x(t) = e^{rt}$

$$r^2 e^{rt} + 2r e^{rt} + 145 e^{rt} = 0$$

$$r^2 + 2r + 145 = 0$$

$$r^2 + 2r + 1 = -144$$

$$(r+1)^2 = -144$$

$$r + 1 = \pm 12 i$$

$$r = -1 \pm 12 i$$

We could write

$$x(t) = c_1 e^{(-1+12i)t} + c_2 e^{(-1-12i)t}$$

But, instead we write

$$x(t) = D_1 e^{-t} \cos 12t + D_2 e^{-t} \sin 12t$$

$$x(t) = D_1 e^{-t} \cos 12t + D_2 e^{-t} \sin 12t$$

and require $\quad x(0) = 1 \quad \dot{x}(0) = 2$

$$1 = x(0) = D_1 + \cancel{D_2 \cdot 0}$$

$$2 = \dot{x}(0) = D_1 \left(-e^{-t}\cos 12t - 12 e^{-t} \sin 12t\right)\Big|_{t=0} + D_2\left(-e^{-t}\sin 12t + 12 e^{-t}\cos 12t\right)\Big|_{t=0}$$

$$2 = D_1(-1 - 0) + D_2(0 + 12)$$

so $\quad 1 = D_1$

and $\quad 2 = -D_1 + 12 D_2$

so $\quad D_2 = \frac{3}{12} = \frac{1}{4}$

$$\boxed{x(t) = 1\, e^{-t}\cos 12t + \frac{1}{4} e^{-t} \sin 12t}$$

Critically Damped $\qquad \ddot{x} + 2\dot{x} + x = 0$ (DE)

$x(0) = 0 \quad \dot{x}(0) = 10$ (IC)

① Solve the IVP ② Is the displacement ever $= 0$?

③ What is the maximum displacement?

Seek $x(t) = e^{rt}$ $\qquad r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$ so $r = -1$

When there is only one value of r that solves the indicial equation, the general solution is $c_1 e^{rt} + c_2 t e^{rt}$.

explanation after we finish the problem.

General Solution $\quad c_1 e^{-t} + c_2 t e^{-t}$

Impose IC's

$0 = x(0) = c_1 + 0 c_2$ so $c_1 = 0$

$10 = \dot{x}(0) = -c_1 + c_2$ so $c_2 = 10$

① $x(t) = 0 e^{-t} + 10 t e^{-t} = 10 t e^{-t}$

$x(t) = 10t\,e^{-t}$

② where is displacement $= 0$

$0 \overset{?}{=} 10t\,e^{-t}$

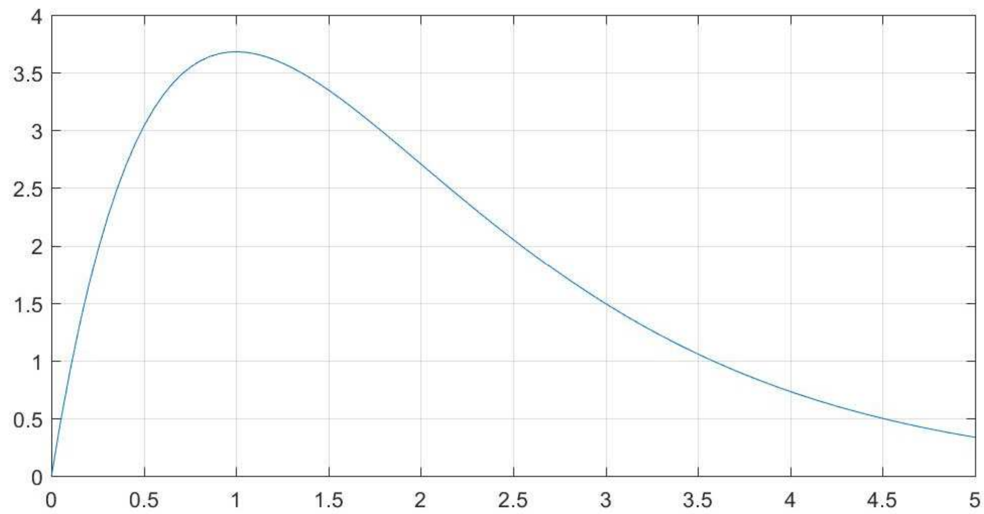
Only zero at $t = 0$

③ What is the max displacement?

Max displacement occurs when $\dot{x} = 0$

$0 = \dot{x}(t) = 10(e^{-t} - t e^{-t}) = 10e^{-t}(1-t)$

$\dot{x}$ vanishes at $t=1$. Max displacement $= 10e^{-1}$



Looks just like overdamped

Find the ge eral Solution to

$$\ddot{x} + 2\dot{x} + (1-\alpha)x = 0 \qquad \text{(DE)}$$

The form of the solution will look different for different values of α.

Seek $x = e^{rt}$ $\quad \dot{x} = re^{rt}$ $\quad \ddot{x} = r^2 e^{rt}$

(DE) $\quad r^2 e^{rt} + 2re^{rt} + (1-\alpha)e^{rt} = 0$

$$r^2 + 2r + 1 - \alpha = 0$$

$$r^2 + 2r + 1 = \alpha$$

$$(r+1)^2 = \alpha$$

$$r = -1 \pm \sqrt{\alpha}$$

$$x(t) = c_1 e^{(-1+\sqrt{\alpha})t} + c_2 e^{(-1-\sqrt{\alpha})t}$$

Case 1 $\underline{\alpha > 0}$ $\sqrt{\alpha}$ is a real number

$$x(t) = C_1 e^{(-1+\sqrt{\alpha})t} + C_2 e^{(-1-\sqrt{\alpha})t}$$

General Solution is a linear combination of decaying exponentials. If $\alpha = \frac{1}{2}$

$$x(t) = C_1 e^{\left(-1+\frac{1}{\sqrt{2}}\right)t} + C_2 e^{\left(-1-\frac{1}{\sqrt{2}}\right)t}$$

Overdamped – No oscillations

Case 2 $\alpha < 0$ $\sqrt{\alpha} = i\sqrt{-\alpha}$

$$x(t) = C_1 e^{(-1+i\sqrt{-\alpha})t} + C_2 e^{(-1+i\sqrt{-\alpha})t}$$

e.g. $\underline{\alpha = -1}$

$$x(t) = C_1 e^{(-1+i)t} + C_2 e^{(-1-i)t}$$

$$x(t) = D_1 e^{-t} \cos t + D_2 e^{-t} \sin t$$

Underdamped – Decaying oscillations

Case 3 $\alpha = 0$ Critically Damped

$$x(t) = c_1 e^{(-1+0)t} + c_2 e^{(-1-0)t}$$

$$x(t) = c_1 e^{-t} + c_2 e^{-t} \leftarrow$$ Both terms are the same

what is the other solution?

Fact

When the indicial equation only has one root r, both e^{rt} and te^{rt} are solutions to the DE1.

How do you see this?

Boyce-DiPrima problems 20-21-22 §3.5

I'll follow 21

Philosophy

General Solutions - without initial conditions aren't necessarily physical, so they can behave strangely. Solutions to the (IVP) are physical, so they can't do strange things.

Example

$$\ddot{x} + 2\dot{x} + (1-\alpha)x = 0$$

$$x(0) = 0$$
$$\dot{x}(0) = 1$$

We will solve this IVP explicitly for $\alpha > 0$ and then take the limit as $\alpha \to 0$

$$x(t) = c_1 e^{(-1+\sqrt{\alpha})t} + c_2 e^{(-1-\sqrt{\alpha})t}$$

Impose initial conditions

$$0 = x(0) = c_1 + c_2$$

$$1 = \dot{x}(0) = (-1+\sqrt{\alpha})c_1 + (-1-\sqrt{\alpha})c_2$$

$$0 = x(0) = C_1 + C_2$$

$$1 = \dot{x}(0) = (-1+\sqrt{\alpha})C_1 + (-1-\sqrt{\alpha})C_2$$

so $\quad C_1 = -C_2$ and $\quad 1 = 2\sqrt{\alpha}\, C_1$

$$x(t) = \frac{1}{2\sqrt{\alpha}} e^{(-1+\sqrt{\alpha})t} - \frac{1}{2\sqrt{\alpha}} e^{(-1-\sqrt{\alpha})t}$$

$$x(t) = e^{-t}\left(\frac{e^{\sqrt{\alpha}\,t} - e^{-\sqrt{\alpha}\,t}}{2\sqrt{\alpha}} \right)$$

Now, let $\alpha \to 0$ $\quad x(t) \to e^{-t}\left(\frac{0}{0}\right)$

The substitution $\alpha = b^2$ makes it easier to apply L'Hopital's rule

$$\lim_{\alpha \to 0} \left(\frac{e^{\sqrt{\alpha}\,t} - e^{-\sqrt{\alpha}\,t}}{2\sqrt{\alpha}} \right) = \lim_{b \to 0} \left(\frac{e^{bt} - e^{-bt}}{2b} \right)$$

$$\lim_{\to 0} \left(\frac{e^{bt} - e^{-bt}}{2b} \right) = \lim_{\to 0} \frac{\frac{d}{db}\left(e^{bt} - e^{-bt}\right)}{\frac{d}{d}(2b)} = \frac{t}{2}$$

$$\boxed{x(t) = t e^{-t}}$$

This is the second solution.

I won't ask you to reproduce this, but I will ask you to solve an IVP with a parameter.

e.g.

Find the solution to

$$\ddot{x} + \omega^2 x = 0$$
$$x(0) = 0$$
$$\dot{x}(0) = 1$$

<u>Solution</u>

$$x(t) = \frac{\sin \omega t}{\omega}$$