Note Title 2/1/2020

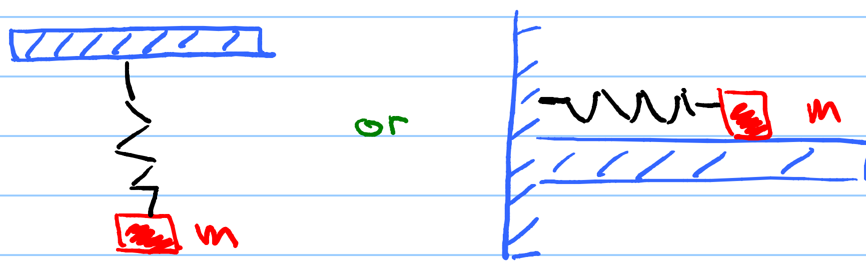
Second Order Differential Equations

Force = Mass · Acceleration

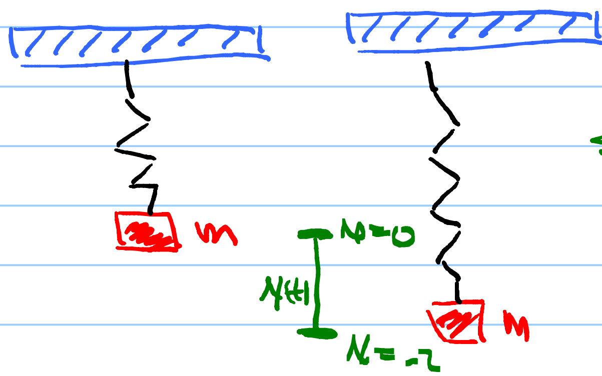
$= m \cdot \ddot{x}$

Primary Example Mass and Spring

a.k.a Simple Harmonic Oscillator



mass at rest



spring is stretched

Hook's Law

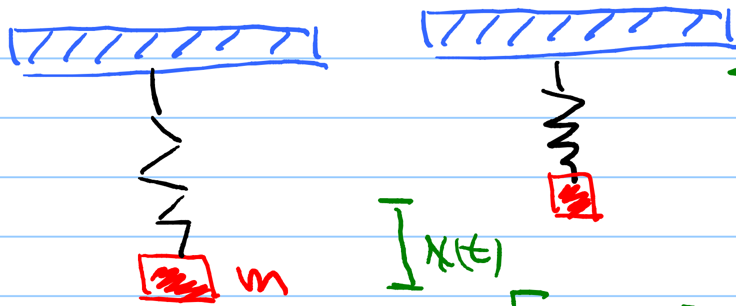
$$F_{spring} = -kx$$

Spring force opposes displacement.

You may choose up [or ~~down~~] to be the positive direction.

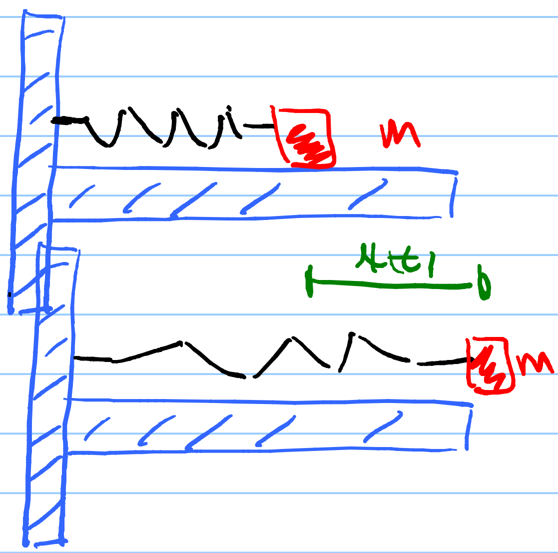
Hook's Law

mass at rest



spring is compressed

$$F_{spring} = -k\,x$$



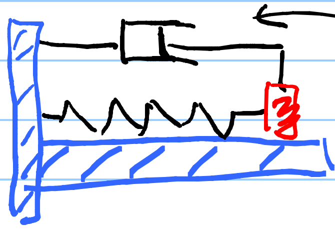
$$F_{spring} = -k\,x$$

stretched

$$\boxed{\begin{array}{c}\underline{\text{Equation of Motion}} \\ m\ddot{x} = -k\,x\end{array}}$$

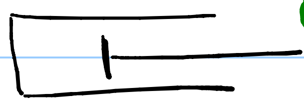
k is called the spring constant

Damped Harmonic Oscillator



Damper

Resists motion with a force proportional to velocity and in the opposite direction to the velocity



γ = damping coefficient

piston moving through a fluid

damping isn't dry friction

examples of damping

air resistance

shock absorber

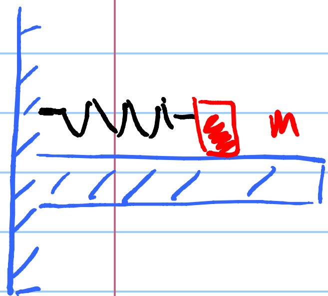
$$F_{damping} = -\gamma \dot{x}$$

Equations of Motion for Damped Harmonic Oscillator

$$m\ddot{x} = -\gamma \dot{x} - kx$$

Usually written

$$m\ddot{x} + \gamma \dot{x} + kx = 0$$



Suppose a 2 kg mass is attached to a spring with spring constant 24 kg/sec^2 and a damping coefficient of 16 kg/sec. The mass is set in motion from its equilibrium position with an initial velocity of 1 m/sec. Formulate and solve the Initial Value Problem.

Solution

y = displacement from equilibrium

$$m\ddot{y} = -\gamma\dot{y} - ky$$

$$2\ddot{y} = -16\dot{y} - 24y$$

starts from equilibrium $y(0)=0$

initial velocity $\dot{y}(0)=1$

$$2\ddot{y} + 16\dot{y} + 24y = 0$$

$$y(0)=0 \quad \dot{y}(0)=1$$

(IVP)

IVP $\ddot{y} + 8\dot{y} + 12y = 0$ (DE)

$y(0) = 0 \quad \dot{y}(0) = 1$ (IC)

Seek $y(t) = e^{rt}$

then $\dot{y}(t) = r e^{rt}$ and $\ddot{y}(t) = r^2 e^{rt}$

Insert into (DE)

$$r^2 e^{rt} + 8 r e^{rt} + 12 e^{rt} = 0$$

$$r^2 + 8r + 12 = 0$$

$$(r+6)(r+2) = 0$$

$$r = -6 \quad \text{or} \quad r = -2$$

General Solution $y(t) = c_1 e^{-6t} + c_2 e^{-2t}$

Recall – This is a linear equation, so we may add solutions and multiply them by constants

Impose Initial Conditions

$$0 = y(0) = c_1 + c_2$$

$$1 = \dot{y}(0) = -6c_1 - 2c_2$$

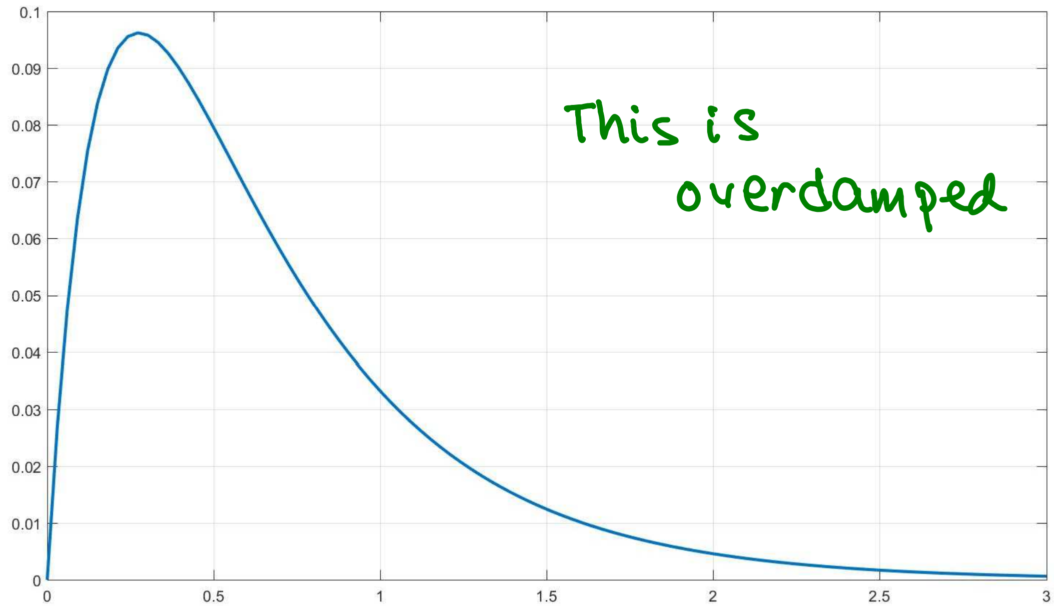
$$0 = y(0) = c_1 + c_2$$

$$1 = \dot{y}(0) = -6c_1 - 2c_2$$

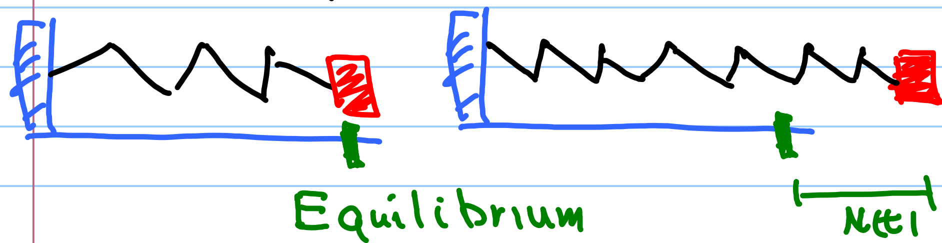
$$
\begin{array}{rl}
0 &= 2c_1 + 2c_2 \\
+\;1 &= -6c_1 - 2c_2 \\
\hline
1 &= -4c_1
\end{array}
$$

so $c_1 = -\frac{1}{4}$ and $c_2 = \frac{1}{4}$

so $y(t) = -\frac{1}{4}e^{-6t} + \frac{1}{4}e^{-2t}$



Undamped Harmonic Oscillator



Suppose the spring constant is 50 kg sec^{-2}

<u>No damping</u> $m = 2$ kg

Initial displacement = 5 meters

Initial velocity = 20 meters/sec

① <u>Write the Initial Value Problem</u>

② <u>Find a formula for $x(t)$</u>

<u>Answer</u> mass · acceleration = Σ Forces

$$m\ddot{x} = -kx$$

$$2\ddot{x} = -50x$$

or $$\ddot{x} + 25x = 0$$

$$x(0) = 5 \qquad \dot{x}(0) = 20$$

IVP $\ddot{x} + 25x = 0$

$x(0) = 5 \qquad \dot{x}(0) = 20$

Seek $x(t) = e^{rt}$

$$\begin{aligned} \ddot{x} &= r^2 e^{rt} \\ + \ 25x &= 25 e^{rt} \\ \hline \ddot{x} + 25x &= (r^2+25) e^{rt} = 0 \end{aligned}$$

so $r^2 + 25 = 0$

or $r = \pm 5i$

Two solutions:

$x_1(t) = e^{5ti}$ and $x_2(t) = e^{-5ti}$

Because this is a Linear DE

$x(t) = c_1 e^{5ti} + c_2 e^{-5ti}$

is a solution for any constants c_1 and c_2

But we seek real solutions, so we use Euler's formula, and write our general solution as:

$$x(t) = D_1 \cos 5t + D_2 \sin 5t$$

and solve for D_1 and D_2.

General Solution

$$y(t) = C_1 e^{5it} + C_2 e^{-5it}$$

This is a real physical problem. It should always have a real solution. The displacement shouldn't be a complex number.

$$y(t) = C_1 e^{5it} + C_2 e^{-5it}$$

Euler's Formula

$\cos(-4t) = \cos(4t)$

$\sin(-4t) = -\sin(4t)$

cosine is even and sine is odd

$$e^{5it} = \cos(5t) + i\sin(5t)$$

why?

$$e^{-5it} = \cos(5t) - i\sin(5t)$$

Add two solutions and divide by 2

$$\frac{e^{5it} + e^{-5it}}{2} = \cos(5t)$$

Subtract and divide by $2i$

$$\frac{e^{5it} - e^{-5it}}{2i} = \sin(5t)$$

So the general solution may be written as

$$y(t) = D_1 \cos(5t) + D_2 \sin(5t)$$

$x(t) = D_1 \cos 5t + D_2 \sin 5t$

$5 = x(0) = D_1 \cos(0) + D_2 \sin(0) = D_1$

$20 = \dot{x}(0) = -5D_1 \sin(0) + 5D_2 \cos(0) = 5D_2$

so $D_1 = 5$ and $D_2 = 4$

$$\boxed{x(t) = 5\cos 5t + 4\sin 5t}$$

Summary - Linear Constant Coefficient Differential Equations, Second Order Homogeneous

$$\ddot{x} + r\dot{x} + kx = 0$$

① If you find 2 different solutions y_1 and y_2, then every solution is a linear combination of y_1 and y_2

i.e. $y(t) = C_1 y_1(t) + C_2 y_2(t)$

② All solutions are sums of exponentials.*

* This is a little bit of a lie. I'll explain more as we go on.

Examples of Damped Harmonic Oscillators

I won't test you on this.

A single story shear building consists of a rigid girder with mass m, which is supported by columns with combined stiffness k. The columns are assumed to be weightless, inextensible in the axial (vertical) direction, and they can only take shear forces but not bending moments. In the horizontal direction, the columns act as a spring of stiffness k. As a result, the girder can only move in the horizontal direction, and its motion can be described by a single variable $x(t)$; hence the system is called a single degree-of-freedom (DOF) system. The number of degrees-of-freedom is the total number of variables required to describe the motion of a system.

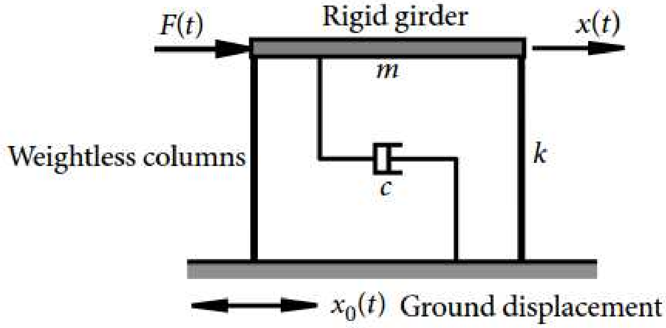


Figure 5.1 A single-story shear building.

The combined stiffness k of the columns can be determined as follows. Apply a horizontal static force P on the girder. If the displacement of the girder is Δ as shown in Figure 5.2, then the combined stiffness of the columns is $k = P/\Delta$.

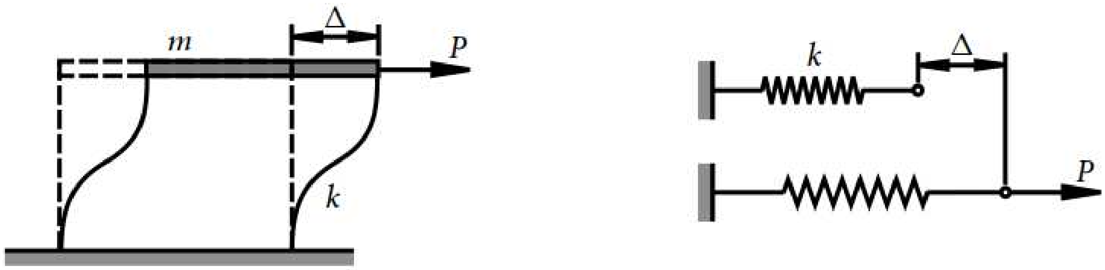
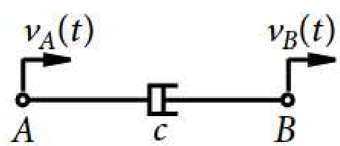


Figure 5.2 Determination of column stiffness.

The internal friction between the girder and the columns is described by a viscous dashpot damper with damping coefficient c. A dashpot damper is shown schematically in Figure 5.3 and provides a damping force $-c(v_B - v_A)$, where v_A and v_B are the velocities of points A and B, respectively, and $(v_B - v_A)$ is the relative velocity between points B and A. The damping force is opposite to the direction of the relative velocity.



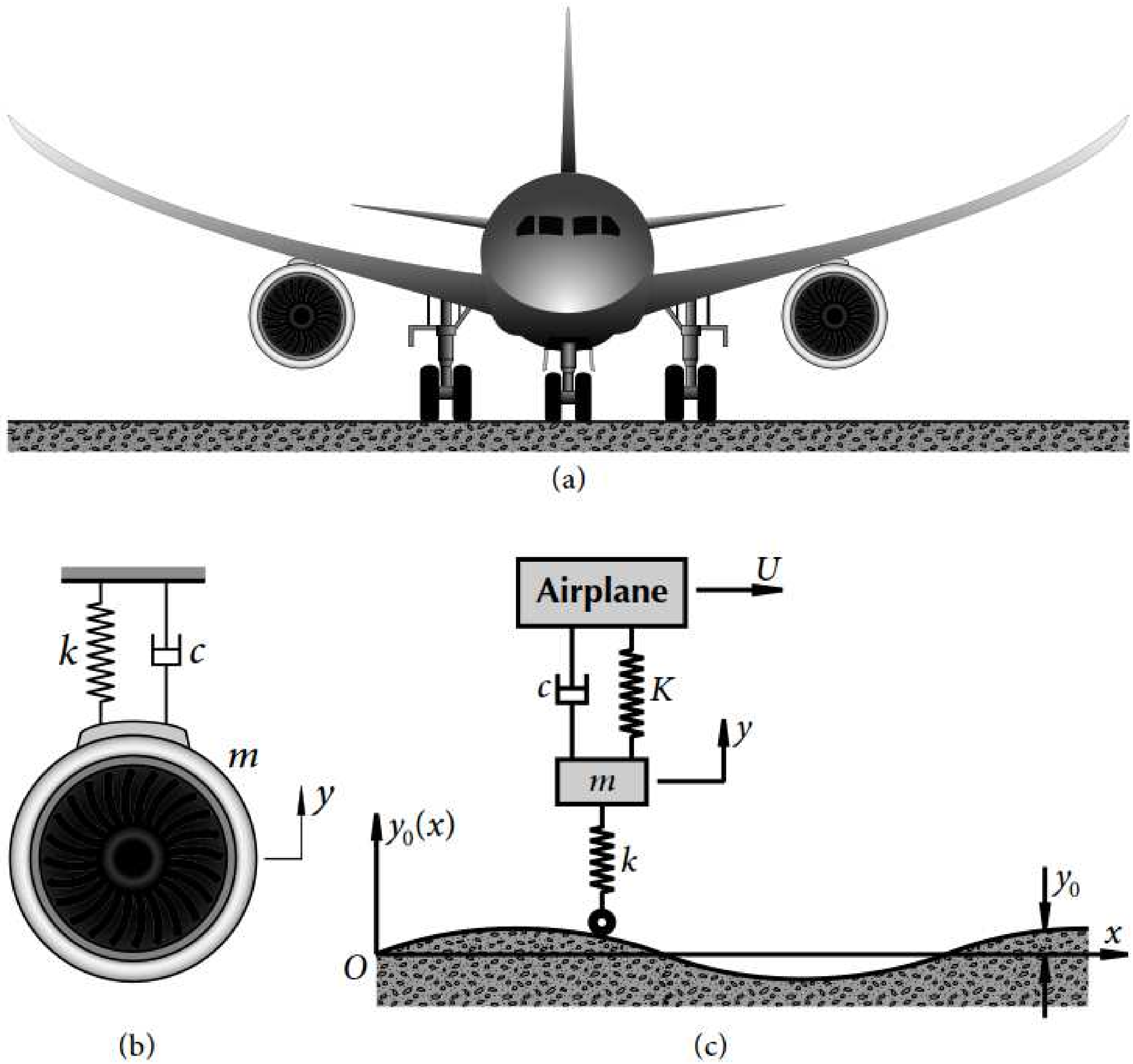


Figure 5.8 Mathematical modeling of jet engine and landing gear.