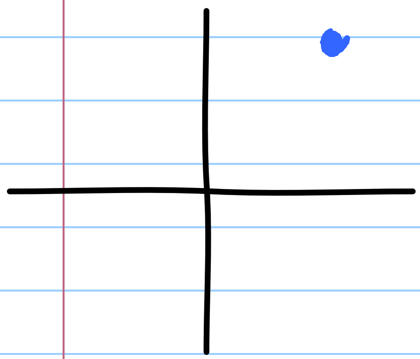
Complex Numbers



$z = x + iy$

These are just the x, y coordinates

Multiplication

There is no natural way to multiply $x\,y$ coordinates

but there is a way to multiply complex numbers.

$$(a+ib)\cdot(c+id) = ac + iad + ibc + i^2 bd$$

$$\boxed{i^2 = -1} \qquad = (ac - bd) + i(ad + bc)$$

We write $z = a + ib$ and say

a is the **real part** of z and
b is the **imaginary part** of z

We also write $\bar{z} = a - ib$ and call

$\bar{z}$ the **complex conjugate** of z

spoken as "zee-bar"

Let $z = a + ib$

$|z| = |a+ib|$ is the length (or modulus) of z.

We calculate $|z|$ using the formula

$$|z|^2 = z \cdot \bar{z} = (a+ib)(a-ib)$$
$$= a^2 + b^2$$

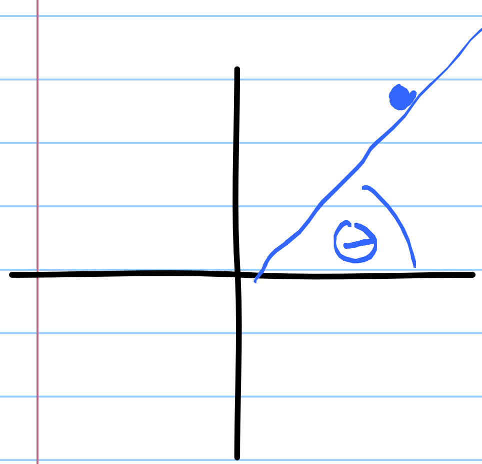
Reciprocals and Division

$$\frac{1}{a+ib} = \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2}$$
$$= \frac{a}{a^2+b^2} + i \, \frac{-b}{a^2+b^2}$$

This can also be written as

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} \qquad \left[= \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} \right]$$

Polar Representation



$z = x + iy$

Polar Coordinates

$x = r\cos\theta$

$y = r\sin\theta$

so $\quad z = r\cos\theta + i r\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$= r(e^{i\theta})$

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Polar Representation of a Complex Number
(also called phasor representation)
(the same as polar coordinates)

What is the polar representation of $1+\sqrt{3}i$?

This means, find r and Θ so that

$$1+\sqrt{3}i = re^{i\Theta}$$

$$= r\cos\Theta + i r\sin\Theta$$

so we require

and solve

$$1 = r\cos\Theta$$

$$\sqrt{3} = r\sin\Theta$$

Solve for r

$$1^2+(\sqrt{3})^2 = r^2\cos^2\Theta + r^2\sin^2\Theta = r^2(\cos^2\Theta+\sin^2\Theta) = r^2$$

$$4 = r^2$$

Solve for Θ

$$\frac{\sqrt{3}}{1} = \tan\Theta$$

$$\Theta = \operatorname{atan}(\sqrt{3}) = \frac{\pi}{3}$$

so $\boxed{1+\sqrt{3}i = 2e^{i\frac{\pi}{3}}}$

Question

Write the real and imaginary parts of

$$(1+\sqrt{3}\,i)\cdot e^{i\left(2t+\frac{\pi}{3}\right)}.$$

$$(1+\sqrt{3}\,i)\cdot e^{i\left(2t+\frac{\pi}{3}\right)} =$$

$$2e^{i\frac{\pi}{3}}\cdot e^{i\left(2t+\frac{\pi}{3}\right)} = 2e^{i\left(2t+\frac{2\pi}{3}\right)}$$

$$= \underbrace{2\cos\left(2t+\frac{2\pi}{3}\right)}_{\text{Real part}} + i\,\underbrace{2\sin\left(2t+\frac{2\pi}{3}\right)}_{\text{Imaginary part}}$$

Justifying Euler's Identity

$$e^{it} = \cos t + i \sin t$$

Characterization of e^t

Theorem If $f(t+s) = f(t)f(s)$

then ① $f'(t) = f(t) \cdot f'(0)$ and $f(0) = 1$

② $f(t) = e^{f'(0)t}$

Proof

①
$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

$$= \lim_{h \to 0} \frac{f(t)\,f(h) - f(t)}{h}$$

$$= f(t) \lim_{h \to 0} \frac{f(h) - 1}{h}$$

$$= f(t)\,f'(0)$$

② $f'(t) = f(t) \cdot b \qquad b := f'(0)$

$$\frac{df}{f} = b\,dt$$

$$\ln|f| = bt + C_1$$

$$f = C_2 e^{bt}$$

$$1 = f(0) = C_2$$

$$f(t) = e^{bt} \qquad \blacksquare$$

Let $f(t) = \cos t + i \sin t$

$$f'(t) = -\sin t + i \cos t$$

$$= i(\cos t + i \sin t)$$

$$f'(t) = i\, f(t)$$

$$f(0) = 1$$

So $f(t) = e^{it}$ $\blacksquare$

Sum of angle Formulas

$$\cos(t+s) + i\sin(t+s) = e^{i(t+s)}$$

$$= e^{it} \cdot e^{is}$$

$$= (\cos t + i\sin t) \cdot (\cos s + i \sin s)$$

$$= \cos t \cos s + i \cos t \sin s$$
$$+ i \sin t \cos s + i^2 \sin t \sin s$$

$$= (\cos t \cos s - \sin t \sin s)$$
$$+ i(\cos t \sin s + \sin t \cos s)$$