**Math 307AB** **Practice Midterm 1** **Winter 2020**

Your Name

Your Signature

Section (circle one) AA AB AC BA bB BC

| Problem | Total Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |
| Total | 100 | |

- This exam is closed book. You may use one side of one $8\frac{1}{2} \times 11$ sheet of handwritten notes. You may not share notes.
- No graphing or symbolic calculators are allowed. You may not use cell phones during the exam.
- Show your work. Do not do computations in your head. Instead, write them out on the exam paper.
- **Place a box around YOUR FINAL ANSWER to each question.**
- If you need more room, use the backs of the pages and indicate that you have done so.
- If you are not sure what a question means, raise your hand and ask us.
- The hints are suggestions only.

1 (20 points) Solve the Initial Value Problem:

$$\begin{aligned} y' - 2y &= t \\ y(0) &= -1 \end{aligned}$$

Integrating Factor

$$m = e^{\int -2\,dt} = e^{-2t}$$

$$(e^{-2t}y)' = te^{-2t}$$

$$e^{-2t}y = -\frac{te^{-2t}}{2} - \frac{e^{-2t}}{4} + C$$

$$y = -\left(\frac{t}{2} + \frac{1}{4}\right) + Ce^{2t}$$

Initial Condition

$$-1 = -\frac{1}{4} + C$$

$$C = -\frac{3}{4}$$

$$\boxed{y(t) = -\left(\frac{t}{2} + \frac{1}{4}\right) - \frac{3}{4}e^{2t}}$$

$\boxed{2}$ (20 points) A 1 kg model car is moving so fast that the force of air resistance is proportional to the square of the speed (and opposes the direction of motion). Let $k$ represent the positive proportionality constant. No other forces act on the car. *Assume that the car is always moving forward, so that speed and velocity are always the same.*

(a) Formulate the first order differential equation for the velocity.

$$1\frac{dv}{dt} = -kv^2$$

(b) Suppose the initial velocity is 100 meters/sec and the velocity after 10 seconds is 90 meters/sec. Find $k$.

$$\frac{dv}{v^2} = -k\,dt$$

$$-\frac{1}{v} = -kt + C$$

$$v = \frac{1}{kt - C}$$

$$100 = v(0) = \frac{1}{-C}$$

$$v(t) = \frac{1}{kt + \frac{1}{100}}$$

$$90 = v(10) = \frac{1}{10k + \frac{1}{100}}$$

$$\frac{1}{90} = 10k + \frac{1}{100}$$

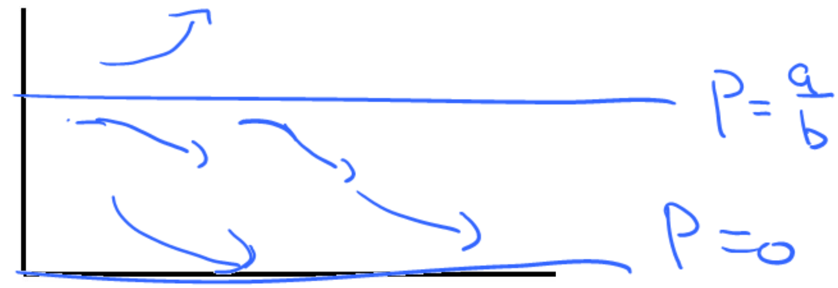
$$\frac{\frac{1}{90} - \frac{1}{100}}{10} = k$$

$$\frac{1}{9000} = k$$

**k** = $\boxed{1/9000}$ kg/

$\boxed{3}$ (20 points) Nurgaliev's Law models the evolution of a fish population as the solution $P(t)$ to the initial value problem below. Suppose that $a$ and $b$ are positive constants. Sketch a direction field. Label all equilibrium solutions and classify them as stable or unstable.

$$\begin{aligned} \frac{dP}{dt} &= bP^2 - aP \\ P(0) &= P_0 \end{aligned}$$




In this model, the population will either grow or die out as time progresses. State conditions under which the population will die out, and under which the population will grow.

If $P_0 > \frac{a}{b}$ population grows $P(t) \uparrow \infty$

If $P_0 < \frac{a}{b}$ population dies out $P(t) \to 0$

4 (20 points) Solve the initial value problem:

$$\frac{dy}{dx} = 4x(y^2 - 9) \qquad y(0) = 0$$

$$\frac{dy}{y^2-9} = 4x\,dx$$

$$\frac{1}{(y-3)(y+3)} = \frac{1/6}{y-3} - \frac{1/6}{y+3}$$

$$\int \frac{dy}{y-3} - \int \frac{dy}{y+3} = 24\int x\,dx$$

$$\ln|y-3| - \ln|y+3| = 12x^2 + C_1$$

$$\ln\left|\frac{y-3}{y+3}\right| = 12x^2 + C_1$$

$$\frac{y-3}{y+3} = C_2 e^{12x^2} \quad \text{where } C_2 = \pm e^{C_1}$$

Initial Condition

$$-1 = \frac{0-3}{0+3} = C_2$$

$$y - 3 = -e^{12x^2}(y+3)$$

$$y(1 + e^{12x^2}) = 3(1 - e^{12x^2})$$

$$y(x) = 3\left(\frac{1-e^{12x^2}}{1+e^{12x^2}}\right) = 3\left(\frac{e^{-12x^2}-1}{e^{-12x^2}+1}\right)$$

What are $\lim_{x\to\infty} y(x)$ and $\lim_{x\to-\infty} y(x)$?

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty} 3\left(\frac{e^{-12x}-1}{e^{-12x^2}+1}\right) = \frac{0-1}{0+1} = \boxed{-3}$$

$$\lim_{x\to-\infty} y(x) = \lim_{x\to-\infty} 3\left(\frac{e^{-12x}-1}{e^{-12x^2}+1}\right) = \frac{0-1}{0+1} = \boxed{-3}$$

5 (20 points) A rocket engine generates a constant thrust (upward force) of 100 newtons. It has a base mass of 8kg, and, initially, it carries 2kg of fuel. The fuel burns at a rate of 2 kg per second. The rocket has a coefficient of air resistance of 2 Newton seconds per meter. Assume the rocket starts from rest, and that up is the positive direction. Take the gravitational acceleration to be -10$m/sec^2$ for simplicity. **Write an initial value problem for the velocity of the rocket during the time the fuel is burning. Then solve the IVP.**

$m \quad \dot{v} = AR + THRUST + Gravity$

$(10-2t)\dot{v} = -2v + 100 - (10-2t)10$

$(10-2t)\dot{v} + 2v = 20t$

$(t-5)\dot{v} - v = -10t$

$\dot{v} - \frac{1}{t-5}v = -10\left(\frac{t}{t-5}\right)$

$\left((t-5)^{-1}v\right)^{\bullet} = -10\left(\frac{t}{(t-5)^2}\right)$

$(t-5)^{-1}v = -10\left(\ln|t-5| - \frac{5}{t-5}\right) + C$

Integrating Factor

$-\int \frac{dt}{t-5} = -\ln(t-5)$

$e^{-\ln(t-5)} = (t-5)^{-1}$

Integral

$\int \frac{t}{(t-5)^2} = \int \frac{u+5}{u^2}$

$= \int \frac{1}{u} + 5\int \frac{1}{u^2}$

$= \ln|u| - \frac{5}{u}$

$= \ln|t-5| - 5/(t-5)$

$v = -10(t-5)\ln|t-5| + 50 + C(t-5)$

$0 = 50\ln|5| + 50 - 5C$

$C = 10\ln|5| + 10$

$$\boxed{v(t) = -10(t-5)\ln|t-5| + 50 + (10\ln|5|+10)(t-5)}$$