Note Title 1/4/2020

Problem

DE $\dot{y} = y(y-6)(y-4)$

1. Find the equilibrium solutions

2. Sketch the direction field

3. Sketch the phase line

4. Label equilibrium solutions as stable or unstable

Implicit Solution

$$\left[(y-4)^3 = K e^{-24t} y(y-6)^2\right]$$

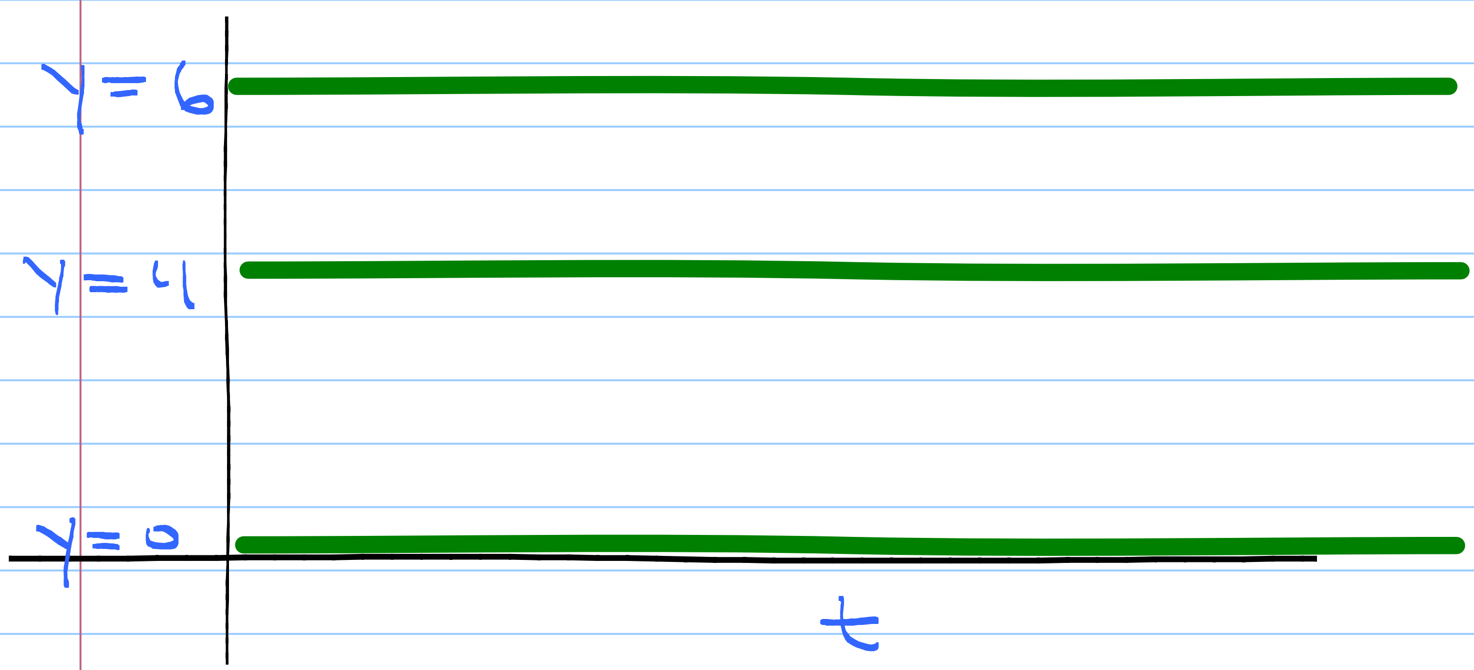
Equilibrium Solutions are $y = \text{constant}$

If $y = \text{constant}$, $\dot{y} = 0$, so we must have $0 = \dot{y} = y(y-4)(y-6)$

So equilibrium solutions are

$$y = 0 \; ; \; y = 4 \; ; \; y = 6$$

Draw them:



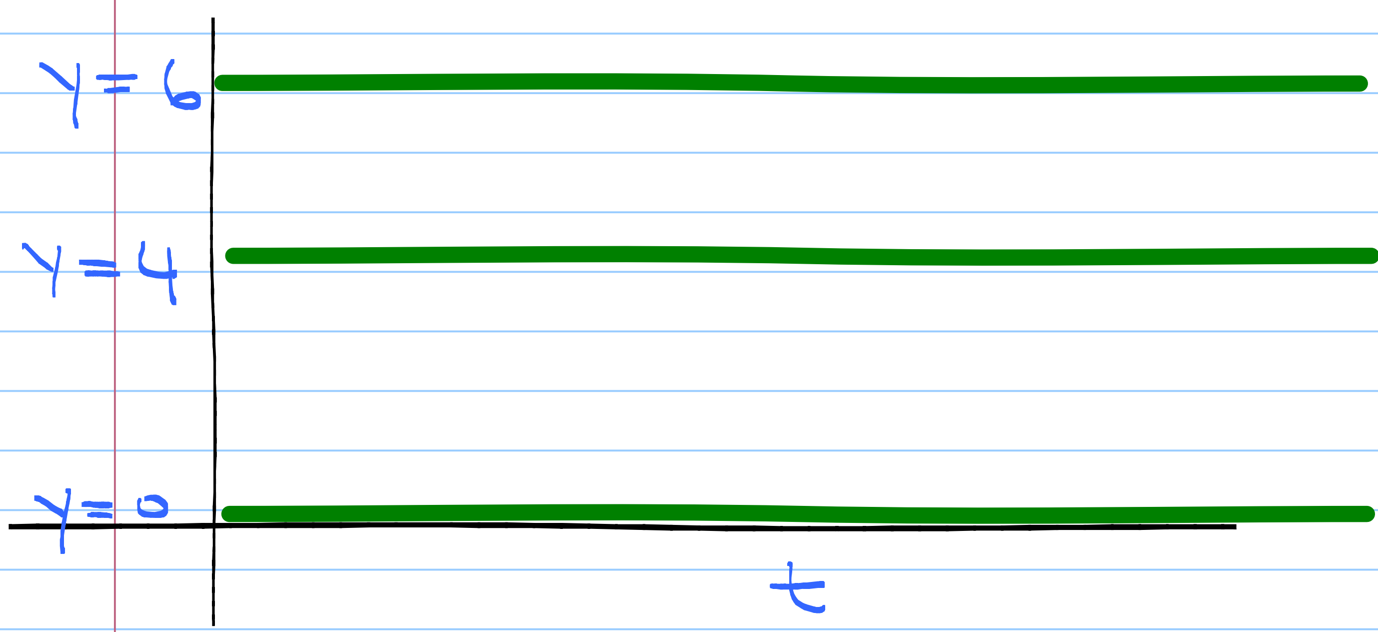
Step ① Sketch D field

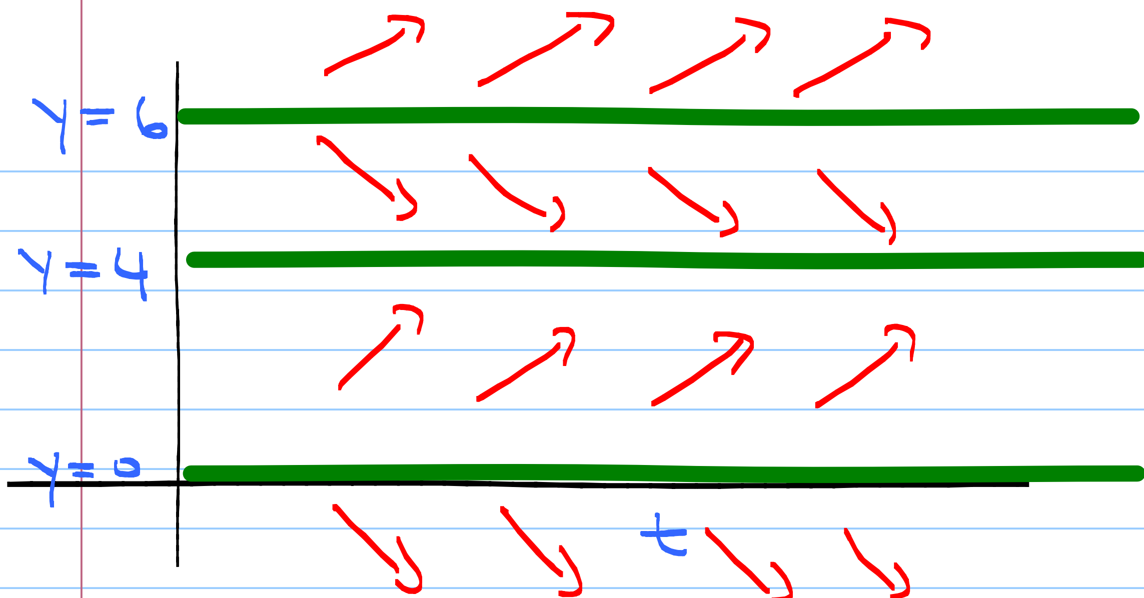
If $y < 0$ $\quad \dot{y} = y(y-6)(y-4) < 0$

If $0 < y < 4$ $\quad \dot{y} = y(y-6)(y-4) > 0$

If $4 < y < 6$ $\quad \dot{y} = y(y-6)(y-4) < 0$

If $6 < y$ $\quad \dot{y} = y(y-6)(y-4) > 0$

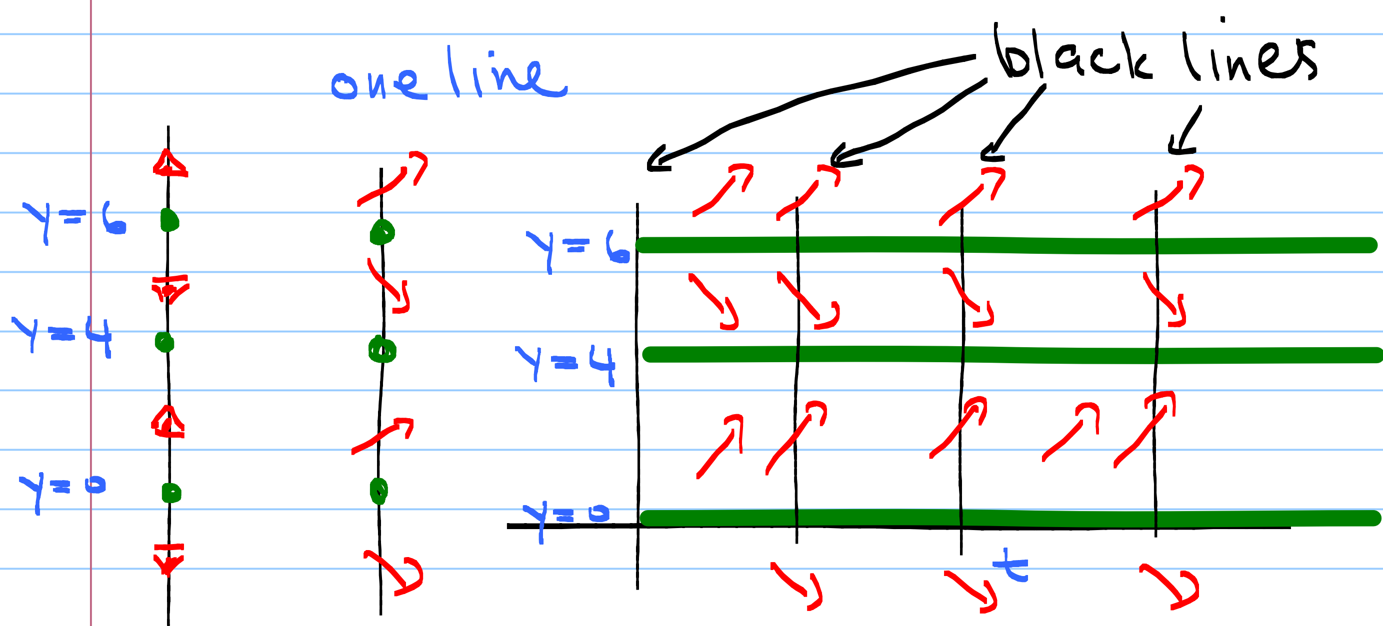




Sketch the Phase Line

Notice that $F(y)$ doesn't involve t.

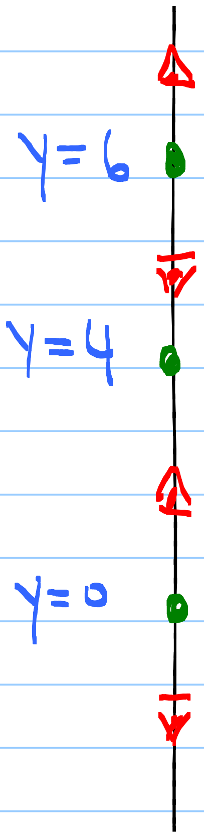
The Direction Field looks the same on each of the black lines below so we can make one line that summarizes all the DField information.



The Phase line

But we usually draw it differently

Label equilibrium solutions as stable or unstable

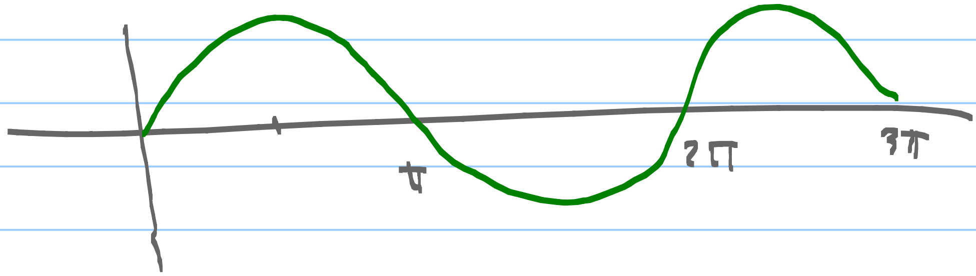


← solutions near y=6 move away from 6
Unstabe

← solutions near y=4 move towards 4
Stabe

← solutions near y=0 move away from 0
Unstabe

Problem $\dot{y} = \sin y$ Draw the Phase Line

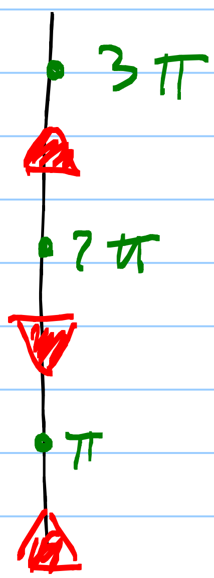


because sine is positive

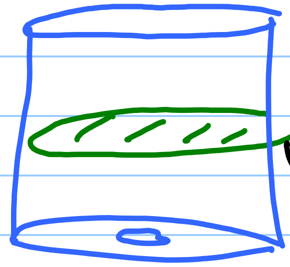
because sine is negative

because sine is positive

Now turn it sideways



Toricelli's Law (problem 7 HW3)



ylindrical Tank
Water leaks out hole

Determine $h(t)$ = height of water at time t

Motivation – Build a water clock by marking $h(t)$ on the sides of the cylinder.

Two Equations

h = height of water
v = speed of water exiting

Water Out = decrease in Volume

$$A_{hole} \cdot v = A_{\substack{\text{cylinder}\\ \text{cross section}}} \cdot \frac{dh}{dt} \qquad (T1)$$

Increase in kinetic energy = $\left\{ \begin{array}{l} \text{Decrease in} \\ \text{potential energy} \end{array} \right.$

$$\frac{1}{2} \Delta m \, v^2 = \Delta m \, g \, h \qquad (T2)$$

$\Delta m = A_{hole} \rho v \, dt$ = mass that exits in time dt

Find DE for h(t)

$$\frac{1}{2} \cancel{A} m v^2 = \cancel{A} m g h \qquad (I2)$$

$$v = \pm\sqrt{2gh}$$

$$A_{cyl} \cdot \frac{dh}{dt} = \pm A_{hole}\, v \qquad (I1)$$

$$A_{cyl} \frac{dh}{dt} = \pm A_{hole} \sqrt{2gh}$$

$$\frac{dh}{dt} = \pm \underbrace{\frac{\pi r_{hole}^2}{\pi r_{cyl}^2}}_{\text{constant}} \cdot \sqrt{2000}\; h^{1/2}$$

Now I know what sign to choose.
h(t) must decrease so choose minus sign.

Question Find DE for $v(t)$

Answer

minus, because up is positive

$$v^2 = 2gh \Rightarrow v = -\sqrt{2g}\,\sqrt{h}$$

$$\cancel{2} v \frac{dv}{dt} = \cancel{2} g \frac{dh}{dt}$$

$$= g \frac{r_{hole}^2}{r_{cyl}^2} \cdot \sqrt{2g} \quad h^{1/2}$$

$$\cancel{v} \frac{dv}{dt} = g \frac{r_{hole}^2}{r_{cyl}^2} \cancel{\sqrt{2g}} \left(\frac{-\cancel{v}}{\cancel{\sqrt{2g}}} \right)$$

$$\frac{dv}{dt} = -g \frac{r_{Hole}^2}{r_{cylinder}^2}$$

A pond has an initial volume of 10,000 m^3. Two streams flow in and one stream flows out.

<u>Stream A</u>
- $\frac{500 \, m^3}{day}$ influx
- water contains $\frac{5 \, kg}{1000 \, m^3}$ salt

<u>Stream B</u>
- $\frac{750 \, m^3}{day}$ influx
- No salt

<u>Stream C</u> $\frac{1300 \, m^3}{day}$ outflux

Find the differential Equation for S = total amount of salt in the pond

$$\frac{dS}{dt} = \begin{matrix}\text{rate}\\\text{of}\\\text{salt}\\\text{influx}\end{matrix} - \begin{matrix}\text{rate}\\\text{of}\\\text{salt}\\\text{outflux}\end{matrix}$$

$$= \frac{500\ m^3}{\text{day}} \cdot \frac{5\ \text{kg}}{1000\ m^3} - \frac{1300\ m^3}{\text{day}} \cdot \begin{matrix}\text{concentration}\\\text{of}\\\text{salt}\end{matrix}$$

$$\begin{matrix}\text{concentration}\\\text{of}\\\text{Salt}\end{matrix} = \frac{S}{\begin{matrix}\text{Volume}\\\text{of}\\\text{lake}\end{matrix}} = \frac{S}{10{,}000 - 50t}$$

$$\text{Volume} = \underbrace{10{,}000}_{\text{initial volume}} + \underbrace{500t}_{\text{stream A in}} + \underbrace{750t}_{\text{stream B in}} - \underbrace{1300t}_{\text{stream C out}}$$

$$\frac{dS}{dt} = \frac{5}{2}\,\frac{\text{kg}}{\text{day}} - \frac{1300}{10{,}000 - 50t}\,S$$

small times

$$\frac{dS}{dt} = \frac{5}{2} - \frac{13}{100}S$$

$$S = 3\left(1 - e^{-\frac{1}{100}t}\right)$$

$$\frac{dS}{dt} - \frac{1300}{50t - 10000}\,S = \frac{5}{2}$$

$$\boxed{\frac{dS}{dt} - \frac{26}{(t-200)}\,S = \frac{5}{2} \qquad S(0) = 10{,}000}$$

$$\frac{dS}{dt} - \frac{26}{(t-200)} S = \frac{5}{2} \qquad S(0) = 0$$

No salt at time = 0

Find Integrating Factor

$$\frac{dm}{dt} = \frac{-26}{(t-200)} m$$

$$\frac{dm}{m} = \frac{-26}{t-200} dt$$

$$\ln|m| = -26 \ln|t-200| + c$$

$$= \ln|(t-200)^{-26}| + c$$

$$m = (t-200)^{-26}$$

Multiply Both sides by m

$$(t-200)^{-26}\frac{dS}{dt} - 26(t-200)^{-27} = \frac{5}{2}(t-200)^{-26}$$

$$\frac{d}{dt}\left[(t-200)^{-26} S\right] = \frac{5}{2}(t-200)^{-26}$$

Integrate

$$(t-200)^{-26} S = \frac{5}{2} \cdot \frac{(t-200)^{-25}}{-25} + c$$

$$S = -\frac{1}{10}(t-200) + c(t-200)^{26}$$

$$S = 20 - \frac{t}{10} + c(t-200)^{26}$$

Find C $\qquad 0 = 20 + c(200)^{26}$

$$\boxed{S = 20 - \frac{t}{10} - \frac{20}{(200)^{26}}(t-200)^{26}}$$