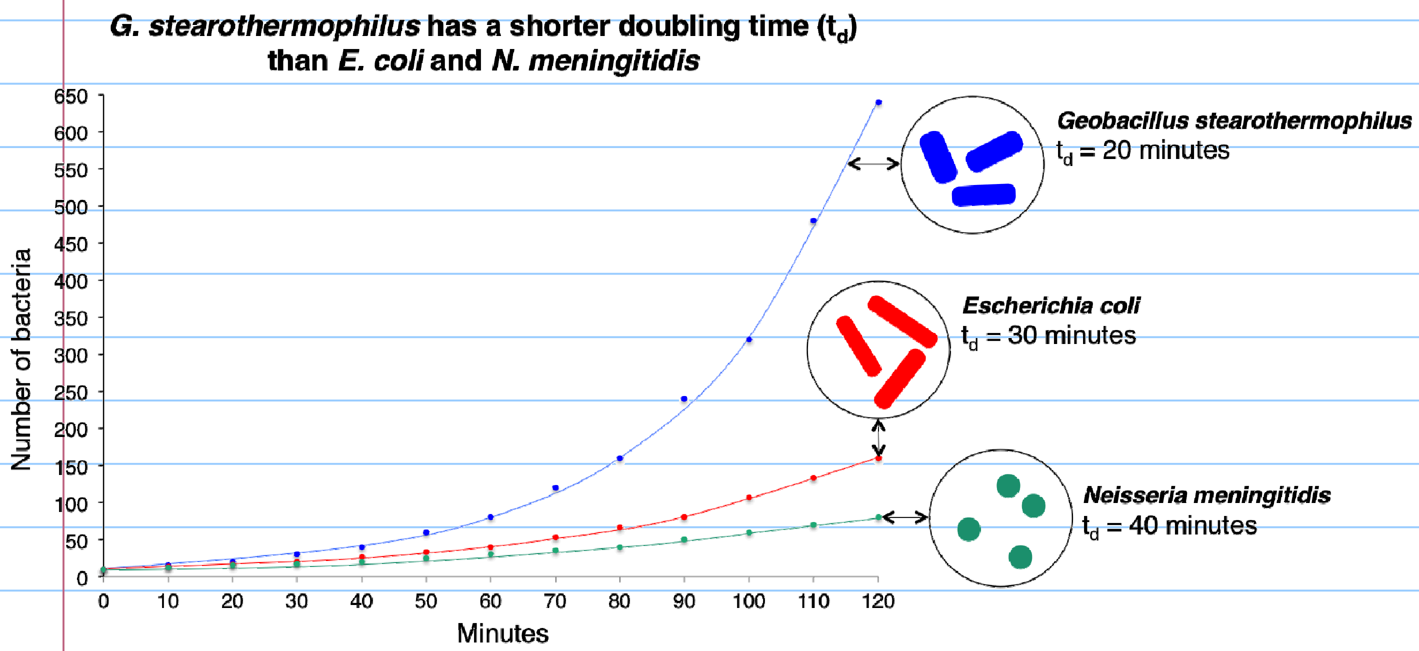
Lecture 06

Note Title 10/6/2017

Population Models

Unlimited Resources (Exponential Growth)



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$P(t)$ = population at time t

r = growth rate = birth rate − death rate

$$\left.\begin{aligned} \frac{dP}{dt} &= rP \\ P(0) &= P_0 \end{aligned}\right\} \quad \text{Initial Value Problem}$$

Question Suppose the doubling time is 30 minutes. Find r.

r is sometimes called the "proportionality constant

$$\left.\begin{aligned} \frac{dP}{dt} &= rP \\ P(0) &= P_0 \end{aligned}\right\} \text{ Initial Value Problem}$$

<u>Question</u> Suppose the doubling time is 30 minutes. Find r

<u>Answer</u>

$$\frac{dP}{P} = r\,dt$$

$$\ln P = rt + C$$

$$P = k e^{rt}$$

$$P_0 = P(0) = k$$

$$P(t) = P_0 e^{rt}$$

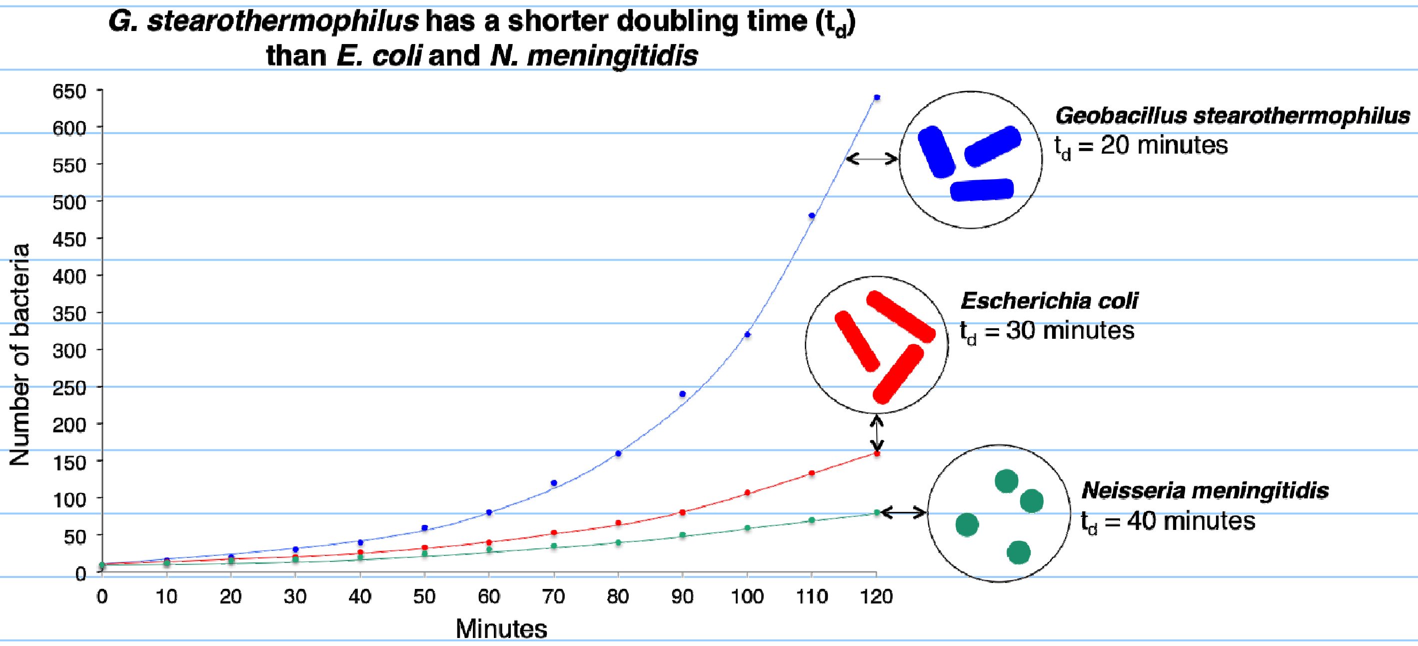
Calculate r

$$2\cancel{P_0} = P(30) = \cancel{P_0} e^{r \cdot 30}$$

$$2 = e^{30r}$$

$$\boxed{\frac{\ln 2}{30} = r}$$

<u>Note:</u> The value of P_0 didn't matter for this problem.



Remark - Its important to work with "letters" (r, P_0) rather than just numbers.

In real applications, we almost never measure parameters like r directly.

Warning: You will see many word problems in textbooks or online where its not clear if they are telling you r or some data from which you should determine r.

Limited Resources (Logistic Growth Model)

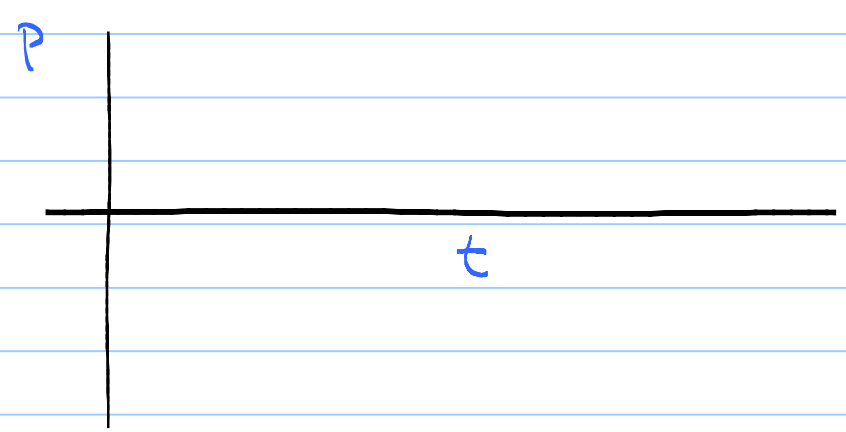
$$\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right) \qquad \text{(DE)}$$

r = growth rate

k = carrying capacity

$k = \begin{cases} \text{maximum population that} \\ \text{available resources (e.g. food supply)} \\ \text{can sustain} \end{cases}$

Question: Sketch direction field for the DE, label equilibrium solutions and classify as stable or unstable.



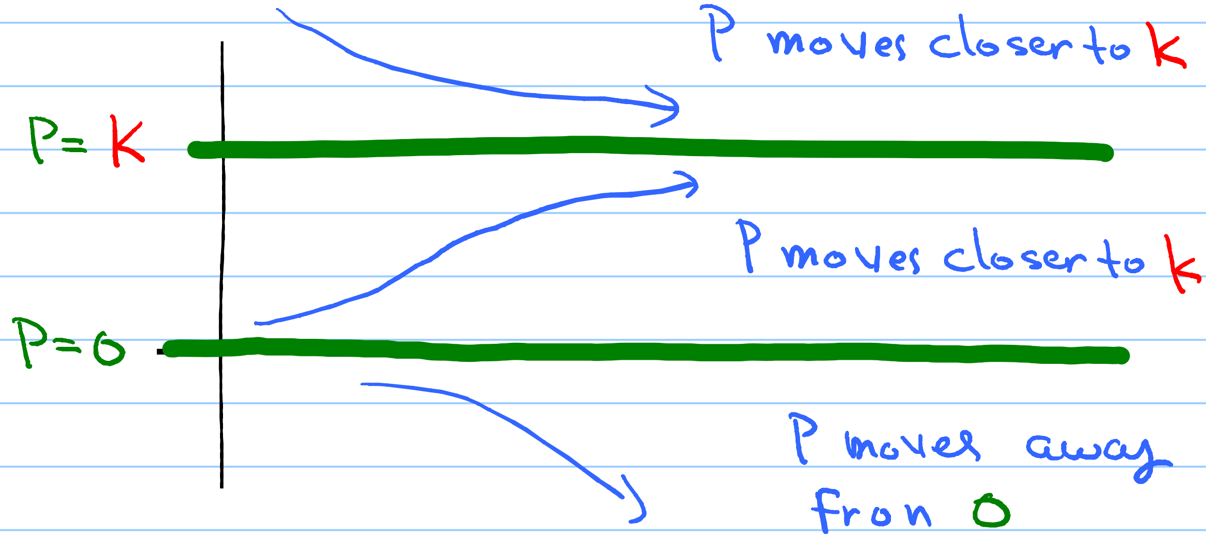
what are the two equilibrium solutions

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right)$$

$P < 0$ $\quad rP(1-\frac{P}{k}) < 0$

$0 < P < k$ $\quad rP(1-\frac{P}{k}) > 0$

$k < P$ $\quad rP(1-\frac{P}{k}) < 0$



<u>Recall</u> – Theorem says IVP has a unique (exactly one) solution, so curves cannot cross.

$P = k$ is a stable equilibrium

$P = 0$ is an unstable equilibrium or a threshhold

Question $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$

Suppose $r = 1$ and $K = 10$

and $P(0) = 1$. Solve the IVP.

(DE) $\frac{dP}{dt} = P\left(1 - \frac{P}{10}\right)$ (IC) $P(0) = 1$

Solve

$$\frac{dP}{P\left(1 - \frac{P}{10}\right)} = dt$$

Separate variables

$$\frac{dP}{P(10-P)} = \frac{dt}{10}$$

Make it look a little neater

$$\frac{1}{10}\frac{dP}{P} + \frac{1}{10}\frac{dP}{10-P} = \frac{dt}{10}$$

partial fractions (details not shown)

Integrate both sides,

Note the minus sign

$$\frac{1}{10}\ln|P| - \frac{1}{10}\ln|10-P| = \frac{1}{10}t + c$$

$$\ln\left|\frac{P}{10-P}\right| = t + c$$

$$\left|\frac{P}{10-P}\right| = c_2 e^t$$

$$\left|\frac{P}{10-P}\right| = C_2 e^t$$

Now, worry about absolute values

$$\frac{P}{10-P} = \pm C_2 e^t$$

Fortunately, its easy, set $C_3 = \pm C_2$

The $\pm$ just changes the sign of the constant

$$\frac{P}{10-P} = C_3 e^t$$

Initial Condition:

$$\frac{1}{9} = \frac{1}{10-1} = C_3$$

$$\frac{P}{10-P} = \frac{1}{9} e^t$$

Now solve for P

$$P = \frac{1}{9}(10-P) e^t$$

$$P = \frac{10e^t}{9} - P\frac{e^t}{9}$$

$$9P + e^t P = 10 e^t$$

$$P(9 + e^t) = 10 e^t$$

$$\boxed{P(t) = \frac{10e^t}{9 + e^t}}$$

Reminder

Partial Fractions - cover up method

$$\frac{1}{(x-2)(x-3)(x-4)} = \frac{A}{(x-2)} + \frac{B}{(x-3)} + \frac{C}{x-4}$$

Find A, B, C

To find A Cover up $(x-2)$ & set $x=2$

$$A = \left.\frac{1}{\cancel{(x-2)}(x-3)(x-4)}\right|_{x=2} = \frac{1}{(2-3)(2-4)}$$

To find B Cover up $(x-3)$ & set $x=3$

$$B = \left.\frac{1}{(x-2)\cancel{(x-3)}(x-4)}\right|_{x=3} = \frac{1}{(2-3)(2-4)}$$

Justification - multiply both sides by $(x-2)$

$$\frac{1\,\cancel{(x-2)}}{\cancel{(x-2)}(x-3)(x-4)} = \frac{A\cancel{(x-2)}}{\cancel{(x-2)}} + \frac{B(x-2)}{(x-3)} + \frac{C(x-2)}{x-4}$$

Set $x=2$

$$\frac{1}{(2-3)(2-4)} = A + 0 + 0$$

$$\frac{1}{(2-3)(2-4)} = A \qquad \frac{1}{(3-2)(3-4)} = B \qquad \frac{1}{(4-2)(4-3)} = C$$