Note Title 1/14/2020

Integrating Factor Summary

① Linear First Order DE

$$\frac{dy}{dt} + p(t)y = q(t)$$

② Find $m(t)$ by solving $\frac{d}{dt} m = p\,m$

we can write a formula $m = e^{\int p(t)}$

③ Multiply First Order DE by m

$m\frac{dy}{dt} + m p(t) y$ equals $\frac{d}{dt}(m y)$

④ Integrate both sides of

$$\frac{d}{dt}(m y) = m q(t)$$

and divide by m

$$y(t) = \frac{1}{m}\left(\int m q(t)\right) + \frac{c}{m}$$

to obtain the general solution.

I don't remember most of these formulas.
I just remember that I want

$m\cancel{\frac{dy}{dt}} + m p(t) y$ to be $\frac{d}{dt}(m y) = \cancel{m\frac{dy}{dt}} + \frac{dm}{dt} y$

so $\frac{dm}{dt} = mp$

Example 2 Newton's Law of Cooling

Temperature T of an object changes at a rate proportional to the deviation from ambient temperature. Suppose time is in minutes, ambient temperature is $(20+5e^{-.02t})^\circ C$, the constant of proportionality is 0.01 min^{-1}, and the initial temperature of the object is $40^\circ C$.

(a) Write the IVP.

(b) Solve the IVP.

(IVP) $$\frac{dT}{dt} = 0.01\left(20+5e^{-0.02t} - T\right)$$

$$T(0) = 40$$

Why not

$$\left(T-\left(20+5e^{-0.002t}\right)\right) |$$

?

Solve

$$\frac{dT}{dt} + 0.01\,T = 0.2 + 0.05\,e^{-0.02t}$$

Find integrating factor

Multiply $\frac{dT}{dt} + 0.01T$ by something so that it becomes the derivative of a product.

Answer $e^{0.01t}$

$$e^{0.01t}\frac{dT}{dt} + 0.01\,e^{0.01t}\,T = \frac{d}{dt}\left(e^{0.01t}\,T\right)$$

$$\frac{dT}{dt} + 0.01\,T = 0.2 + 0.05\,e^{-0.02t}$$

Multiply by $e^{0.01t}$

$$e^{0.01t}\frac{dT}{dt} + 0.01\,e^{0.01t}\,T = 0.2\,e^{0.01t} + 0.05\,e^{-0.01t}$$

Recognize product rule

$$\frac{d}{dt}\left(e^{0.01t}\,T\right) = 0.2\,e^{0.01t} + 0.05\,e^{-0.01t}$$

Integrate $\int e^{\alpha t} = \frac{e^{\alpha t}}{\alpha}$

$$e^{0.01t}\,T = \frac{0.2}{0.01}e^{0.01t} - \frac{0.05}{0.01}e^{-0.01t} + C$$

Multiply by $e^{-0.01t}$

$$\boxed{T(t) = 20 - 5\,e^{-0.02t} + C\,e^{-0.01t}}$$

This is the general solution

Find C $\quad 40 = T(0) = 20 - 5 + C$

$$25 = C$$

$$\boxed{T(t) = 20 - 5\,e^{-0.02t} + 25\,e^{-0.01t}}$$

This is the solution to the IVP

Note Title 1/17/2018

What does Linear Mean?

Linear Equations

$$3x + 4y = b$$

You can add solutions

If $3 \cdot 1 + 4 \cdot 0 = 3$

and $3 \cdot 0 + 4 \cdot 1 = 4$

then $3(1+0) + 4(0+1) = (3+4)$

and multiply by constants

If $3 \cdot 1 + 4 \cdot 0 = 3$

then $3(2 \cdot 1) + 4(2 \cdot 0) = (2 \cdot 3)$

Nonlinear means this doesn't work

$$x^2 + y^2 = b^2$$

$$1^2 + 0^2 = 1^2$$

$$0^2 + 1^2 = 1^2$$

But $(1+0)^2 + (0+1)^2 \neq (1+1)^2$

Superposition Principle

For a linear equation, the sum of solutions is a solution.

Example $\frac{dv_1}{dt} + \frac{v_1}{5} = -9.8$

$v_1(t) = -49$ is a solution

$v_2(t) = C e^{-t/5}$ is a solution to

$$\frac{dv_2}{dt} + \frac{v_2}{5} = 0$$

Superposition Principle says that

$v_1(t) + v_2(t)$ solves

$$\frac{d}{dt}(v_1 + v_2) + \frac{1}{5}(v_2 + v_2) = -9.8 + 0$$

so the general solution to

$$\frac{dv}{dt} + \frac{v}{5} = -9.8$$

$$v(t) = v_1 + v_2 = -49 + C e^{-t/5}$$

Question

Find the general solution to:

$$\dot{y} = \frac{-2}{t} y + e^t \qquad \text{(DE)}$$

Solution

write as

$$\frac{dy}{dt} + \frac{2}{t} y = e^t$$

Find Integrating factor

$$\frac{d}{dt} m = \frac{2}{t} m$$

$$\frac{dm}{m} = \frac{2}{t} dt$$

$$\ln m = 2 \ln t = \ln(t^2) + c$$ ↖ choose $c = 0$ for convenience

$$m = t^2$$

Now, multiply DE by t^2

$$t^2 \dot{y} + t^2 \frac{2y}{t} = t^2 e^t$$

$$t^2 \frac{dy}{dt} + 2ty = t^2 e^t$$

$$\frac{d}{dt}(t^2 y) = t^2 e^t$$

$$\frac{d}{dt}(t^2 y) = t^2 e^t$$

<u>Integrate</u> $t^2 y = \int t^2 e^t$

$$t^2 y = t^2 e^t - 2te^t + 2e^t + c$$

multiply by $\frac{1}{t^2}$ $\boxed{y = e^t - \frac{2}{t}e^t + \frac{2}{t^2}e^t + \frac{c}{t^2}}$

General Solution

<u>Aside</u> I remember

$$\int t^2 e^t = t^2 e^t + a t e^t \; b \, e^t$$

$$\downarrow \frac{d}{dt}$$

$$t^2 e^t \overset{?}{=} (t^2 e^t + 2t e^t) + (a t e^t + a e^t) + b e^t$$

$$t^2 e^t \overset{?}{=} t^2 e^t + \underbrace{(2+a)}_{=0} t e^t + \underbrace{(a+b)}_{=0} e^t$$

-2

$a = -2$ $b = -a = 2$

Faster than integration by parts ?
