Note Title 1/8/2020

From Last Lecture (3)

Problem A 100 kg sky diver drops from a great height. The force of air resistance is proportional to his velocity and always opposes the motion. The constant of proportionality is 20 kg/sec.
Take "UP" to be the positive direction.

(a) Formulate the Initial Value Problem

(b) Find the skydiver's "terminal velocity".

Solution:

(a) mass · acceleration = gravitational force + force of air resistance

$$100\,\dot{v} = -100g \pm 20\cdot v$$

?

$$\dot{v} = -9.8 - \frac{20}{100}v$$

"drops" means $v(0)=0$

IVP $\quad \frac{dv}{dt} = -9.8 - \frac{v}{5} \qquad v(0) = 0$

$$\frac{dv}{dt} = -9.8 - \frac{v}{5}$$

Equilibrium Solution (Constant Solution)

$$0 = \frac{dv}{dt} = -9.8 - \frac{(v)}{5} = 0$$

$$v = -9.8 \cdot 5 = -49$$

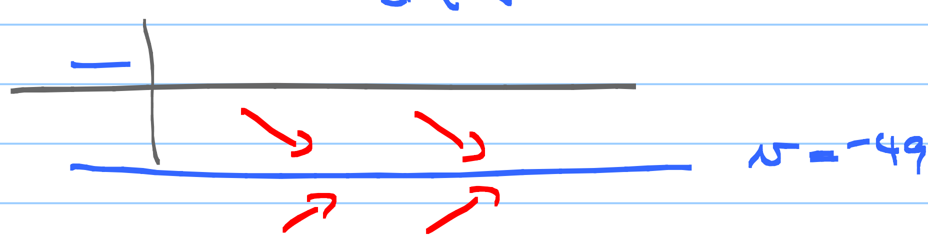
Is $v = -49$ a stable equilibrium?

Answer: Yes because

If $v(t) > -49$, then $\frac{dv}{dt}\left(= \frac{-1}{5}(v+49)\right) < 0$ so $v(t) \downarrow$

If $v(t) < -49$, then $\frac{dv}{dt}\left(= \frac{-1}{5}(v+49)\right) > 0$ so $v(t) \uparrow$

DField



If he starts to fall at any speed < 49 m/s he speeds up to (almost) 49 m/s but he never falls faster than 49 m/s.

If he starts falling faster than 49 m/s, he slows down to 49 m/s

For this reason, we call the stable equilibrium the terminal velocity.

Find a Formula for the "general solution"

to $\frac{dv}{dt} = -9.8 - \frac{v}{5}$

Solution: This is a separable DE, but we will use a different method, called an "integrating factor".

Rewrite as: $\frac{dv}{dt} + \frac{1}{5}v = -9.8$

Multiply by $e^{t/5}$ ← I haven't told you why.

$$e^{t/5}\frac{dv}{dt} + \frac{1}{5}e^{t/5}v = -9.8e^{t/5}$$

Recognize the product rule

$$\frac{d}{dt}\left[e^{t/5}v\right] = -9.8e^{t/5}$$

Integrate both sides

$$e^{t/5}v(t) = -49e^{t/5} + C$$

Solve for $v(t)$

$$v(t) = -49 + Ce^{-t/5}$$

This is the "general solution". It always has an arbitrary constant.

Note Title 1/12/2020

First Order Linear DE's

Examples

$$y' = -ty + \cos t$$

y and derivatives of y OK

$$y' = 3y + e^{-2t}$$

$$y' = y\cos t + \sin t$$

General Example

$$y' = p(t)\,y + g(t)$$

Non-Examples These DE's are NOT linear

$$y' = ty^2 + \sin t$$

other functions of y or derivative NOT OK

$$(y')^2 = ty + \cos t$$

$$y' = \sin y$$

Method of Integrating factors

$$\dot{y} + 4y = 1$$

$$y(0) = 0$$

Multiply both sides by e^{4t} ↖ Not telling you why yet.

$$e^{4t}\dot{y} + 4e^{4t}y = e^{4t}$$

Recognize product rule

$$(e^{4t}y)^{\cdot} = e^{4t}$$

Integrate

$$e^{4t}y = \frac{e^{4t}}{4} + C$$

$$y = \frac{1}{4} + Ce^{-4t}$$ ⟵ This is the "general solution"

$$0 = y(0) = \frac{1}{4} + C$$

$$C = -1/4$$

$$y(t) = \frac{1}{4} - \frac{1}{4}e^{-4t}$$ ↙ This is the solution to the Initial Value Problem

How did we find the integrating Factor?
Call it $m(t)$

$$\dot{y} + 4y = 1$$

Multiply by a function $m(t)$

$$m\dot{y} + 4my = m(t)\cdot\text{something}$$

We want to choose m so that

$$\frac{d}{dt}(my) = m\frac{dy}{dt} + 4my$$

$$\cancel{m\frac{dy}{dt}} + \frac{dm}{dt}y = \cancel{m\frac{dy}{dt}} + 4ym$$

so we need

$$\frac{dm}{dt} = 4m$$

which is separable

$$\frac{dm}{m} = 4\,dt$$

$$\ln|m| = 4t + C$$

$$m = e^{4t}\cdot k$$

k doesn't matter so we set $k=1$

Example 2

(DE) $\frac{dy}{dt} = -\frac{1}{t}y + t^2$

(IC) $y(1) = 1$

How to solve?

(DE) $\frac{dy}{dt} + \frac{1}{t}y = t^2$

$\frac{dm}{dt} = \frac{m}{t}$

$\frac{dm}{m} = \frac{dt}{t}$

$\ln|m| = \ln|t|$

$m = \pm t$

$\pm$ doesn't matter

$m = t$

Multiply both sides by t

(DE) $t\frac{dy}{dt} + y = t^3$

Recognize the "product rule"

$t\frac{dy}{dt} + y = \frac{d(ty)}{dt}$

so

(DE) $\frac{d}{dt}(ty) = t^3$

Now integrate both sides using

$ty = \frac{t^4}{4} + C$

Divide by t

$y(t) = \frac{t^3}{4} + \frac{C}{t}$

This is called the "general solution"

Use the initial condition

$1 = y(1) = \frac{1}{4} + C$ so $C = \frac{3}{4}$

so $\boxed{y(t) = \frac{t^3}{4} + \frac{3}{4t}}$