Note Title 10/1/2017

Finding A Formula

$\dot{y} = t(4-y)$

$y(0) = 0$

$$\frac{dy}{dt} = t(4-y)$$

$$\frac{dy}{4-y} = t\,dt$$

$$\frac{dy}{y-4} = -t\,dt$$

$$\ln|y-4| = -\frac{t^2}{2} + c$$

$$|y-4| = e^{-t^2/2} \cdot e^{c}$$

$$y-4 = \pm e^{c} e^{-t^2/2}$$

$$y-4 = k e^{-t^2/2}$$

define $k = \pm e^{c}$

$y(0) = 0$ so $-4 = k$

$$\boxed{y = 4\left(1 - e^{-t^2/2}\right)}$$

$$\lim_{t \to \infty} y(t) = 4$$

Justification

$$\frac{1}{4-y}\frac{dy}{dt} = t$$

$$\int \frac{1}{y-4}\frac{dy}{dt}\,dt = -\int t\,dt$$

Substitute $\boxed{u = y}$

$$du = \frac{dy}{dt}\,dt$$

$$\int \frac{du}{u-4} = -\int t\,dt$$

$$\ln|u-4| = -\frac{t^2}{2} + c$$

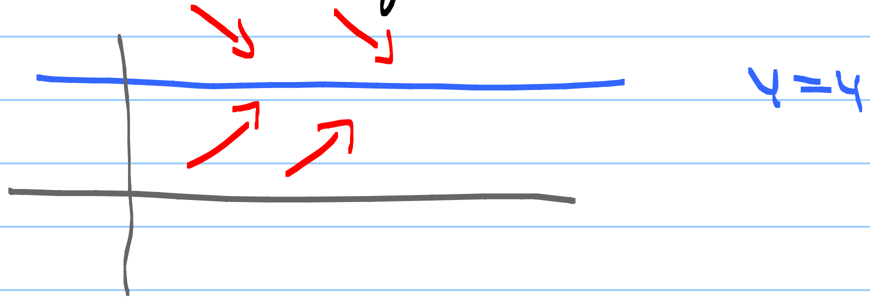
But $u = y$, so

$$\ln|y-4| = -\frac{t^2}{2} + c$$

The solution $y=4$ is a "stable equilibrium".

If $\dot{y} = t(4-y)$, $y(0) = \text{anything}$ then $\lim\limits_{t\to\infty} y(t) = 4$

We can tell that $\lim\limits_{t\to\infty} y(t) = 4$ without finding a formula:



For $t > 0$

If $y(t) > 4$, $\dot{y} < 0$, so $y(t)$ decreases

If $y(t) < 4$, $\dot{y} > 0$, so $y(t)$ increases

If $y(t) = 4$, $\dot{y} = 0$, so $y(t)$ doesn't change

Solving Initial Value Problem with Formulas

Separable Differential Equation

$$\frac{dy}{dt} = F(t)\,G(y) \qquad \text{(DE)}$$

$$y(t_0) = y_0 \qquad \text{(Initial Condition)}$$

Step 1 $\dfrac{dy}{G(y)} = F(t)\,dt$

Step 2 Integrate Both Sides

$$\int \frac{dy}{G(y)} = \int F(t)\,dt + C$$

Step 3 Use (IC) to find C

Step 4 Try to solve for $y(t)$

Sometimes you can

Explicit Solution $y(t) =$

Sometimes you can't

Implicit Solution

Example coming

Problem Find an "implicit" solution

to
$$\frac{dy}{dt} = \frac{y}{1+y^2}$$
$$y(0) = 1$$

Answer
$$\left(\frac{1+y^2}{y}\right) dy = dt$$

$$\frac{dy}{y} + y\,dy = dt$$

$$\ln|y| + \frac{y^2}{2} = t + c$$
$$\frac{1}{2} = c$$
$$\ln|y| + \frac{y^2}{2} = t + \frac{1}{2}$$

We would like to write an "explicit solution" $y = f(t)$

but sometimes we can't.

Example $\frac{dy}{dx} = \frac{-x}{y}$; $y(0)=1$

Step 1 $y\,dy = -x\,dx$

Step 2 $\frac{y^2}{2} = -\frac{x^2}{2} + C$

Step 3 Find C using (IC) $y(0)=1$

$$\frac{1^2}{2} = -\frac{0^2}{2} + C$$

$$C = \frac{1}{2}$$

Implicit Solution

$$\frac{y^2}{2} = -\frac{x^2}{2} + \frac{1}{2}$$

or $\boxed{y^2 + x^2 = 1}$

Explicit Solution

$$y^2 = 1 - x^2$$

$$y = \pm\sqrt{1-x^2}$$

Step 4 $y(0)=1$ so $\boxed{y = \sqrt{1-x^2}}$

(IVP) $\frac{dy}{dx} = \frac{-x}{y}$; $y(0) = 1$

Solution

$$\boxed{y = \sqrt{1-x^2}}$$

Notice - Formula only makes sense for $-1 \leq x \leq 1$

Recall Theorem - There is exactly one solution to the IVP.

The solution is a function $y(x)$

To be discussed ~~later~~ now

① There are conditions:

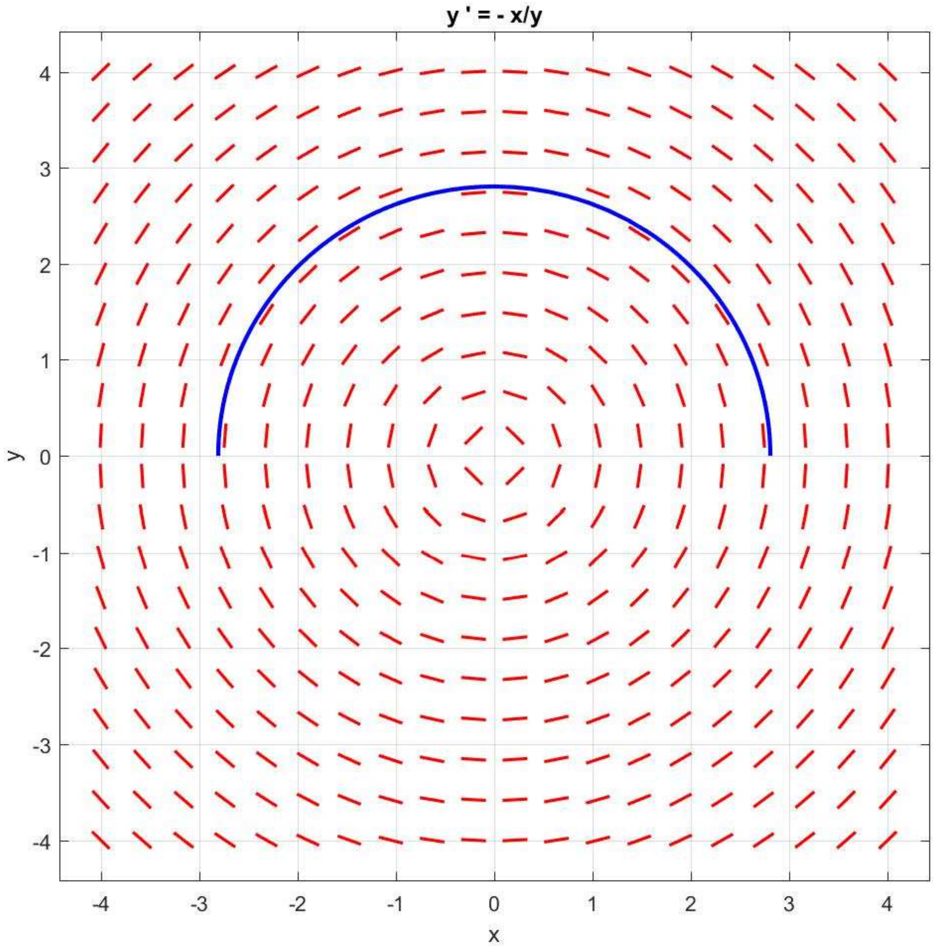
$f(x, y)$ = differentiable function

② Solution may not last forever.

" There is a unique solution defined in some interval about x_0. "

$\frac{-x}{y} = f(x,y)$ is a differentiable function as long as $y \neq 0$, so the solution may "stop" if $y \to 0$, which happens as $x \to \pm 1$

I don't expect you to be able to see from the DE that this will happen



$$y = \frac{-x}{y}$$

Computer generated solution stops at $x = \pm 1$

* Parametric Equations could find the whole circle

$$\frac{dy}{dx} = \frac{\frac{dy}{ds}}{\frac{dx}{ds}} = \frac{-x}{y}$$

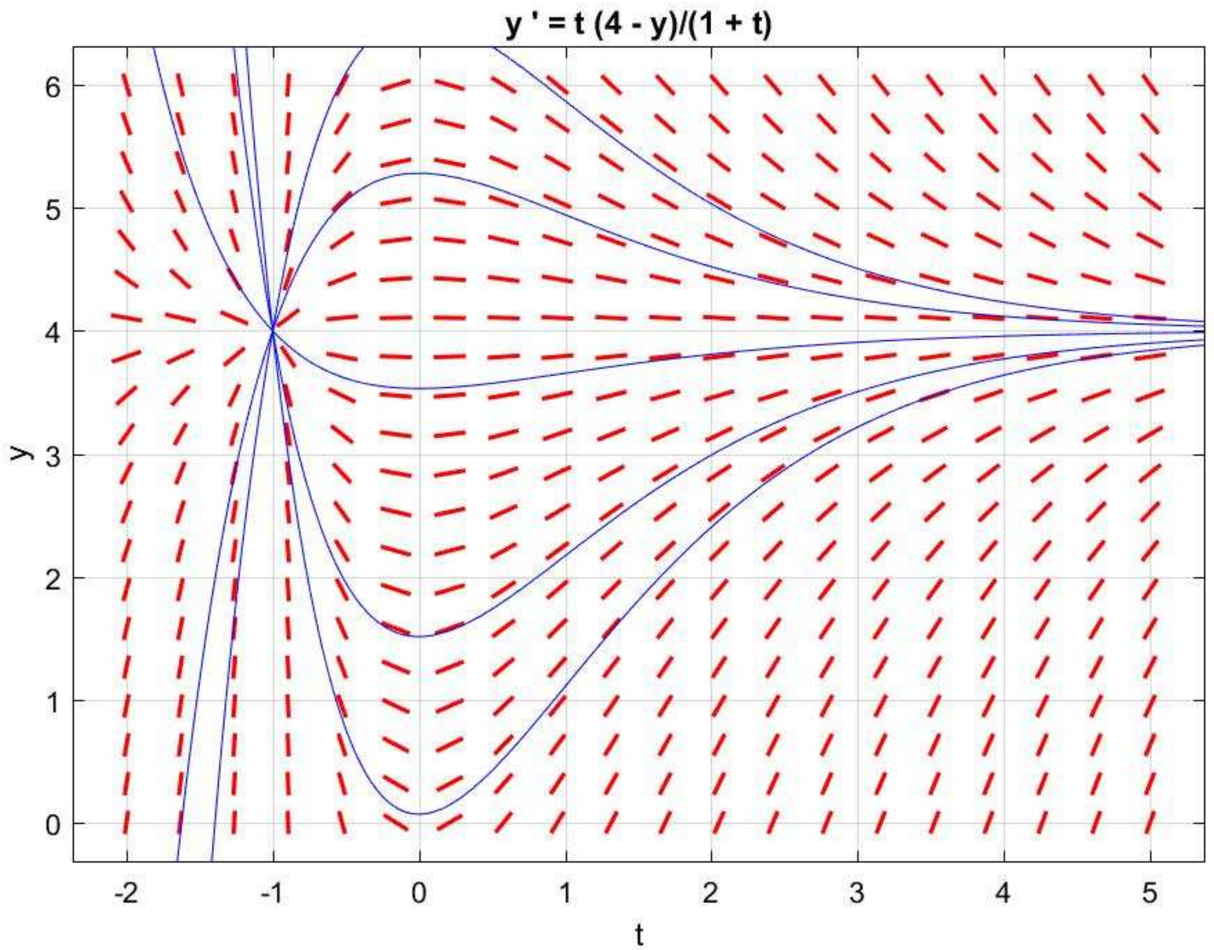
$$\frac{dy}{ds} = -x \qquad \frac{dx}{ds} = y$$

Solution:

$$x(s) = -\cos s$$
$$y(s) = \sin s$$

* I won't test you on this

$$\dot{y} = \frac{t(4-y)}{1+t}$$



What's wrong with this picture?

Does it contradict theorem that says IVP has unique solution? *

* I won't test you on this.

Problem A 100 kg sky diver drops from a great height. The force of air resistance is proportional to his velocity and always opposes the motion. The constant of proportionality is 20 kg/sec.
Take "UP" to be the positive direction.

(a) Formulate the Initial Value Problem

(b) Find the skydiver's "terminal velocity".

Solution:

(a) mass · acceleration = gravitational force + force of air resistance

$$100\dot{v} = -100g \pm 20 \cdot v$$

? $-$

$$\dot{v} = -9.8 - \frac{20}{100} v$$

"drops" means $v(0) = 0$

IVP $\quad \frac{dv}{dt} = -9.8 - \frac{v}{5} \qquad v(0) = 0$

$$\frac{dv}{dt} = -9.8 - \frac{v}{5}$$

Equilibrium Solution (Constant Solution)

$$v = -9.8 \cdot 5 = -49$$

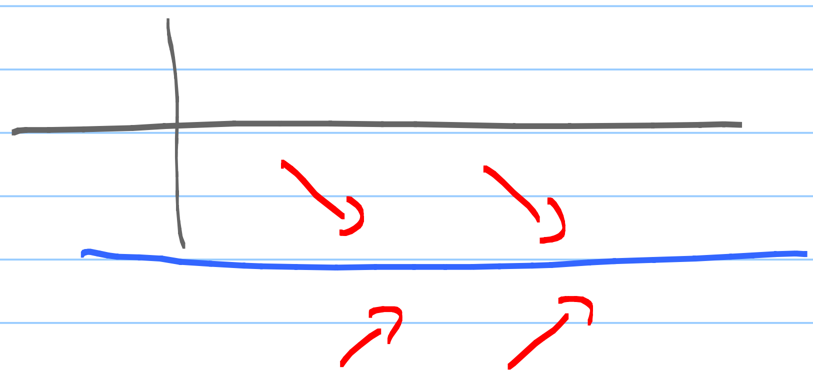
Check $\quad 0 = \frac{dv}{dt} = -9.8 - \frac{(-49)}{5} = 0$ ✓

Is $v = -49$ a stable equilibrium?

Answer: Yes because

If $v(t) > -49$, then $\frac{dv}{dt} < 0$, so $v(t) \downarrow$

If $v(t) < -49$, then $\frac{dv}{dt} > 0$, so $v(t) \uparrow$



$v = -49$