Note Title 10/1/2017

Initial Value Problem

Example

(DE) $\dot{y} \quad f(t, y)$ $\qquad \dot{y} = t(4-y)$

(IC) $y(t_0) = y_0$ $\qquad y(0) = 0$

DE = Differential Equation

IC = Initial Condition

Theorem - There is exactly one solution to the IVP.

The solution is a function $y(t)$

To be discussed later

① There are conditions:

$f(t, y)$ = differentiable function

② Solution may not last forever.

" There is a unique solution defined in some interval about t_0. "

3 Topics

Direction Fields

- Graphical representation of a Differential Equation that helps us to understand the behavior of solutions.

Separable Differential Equations

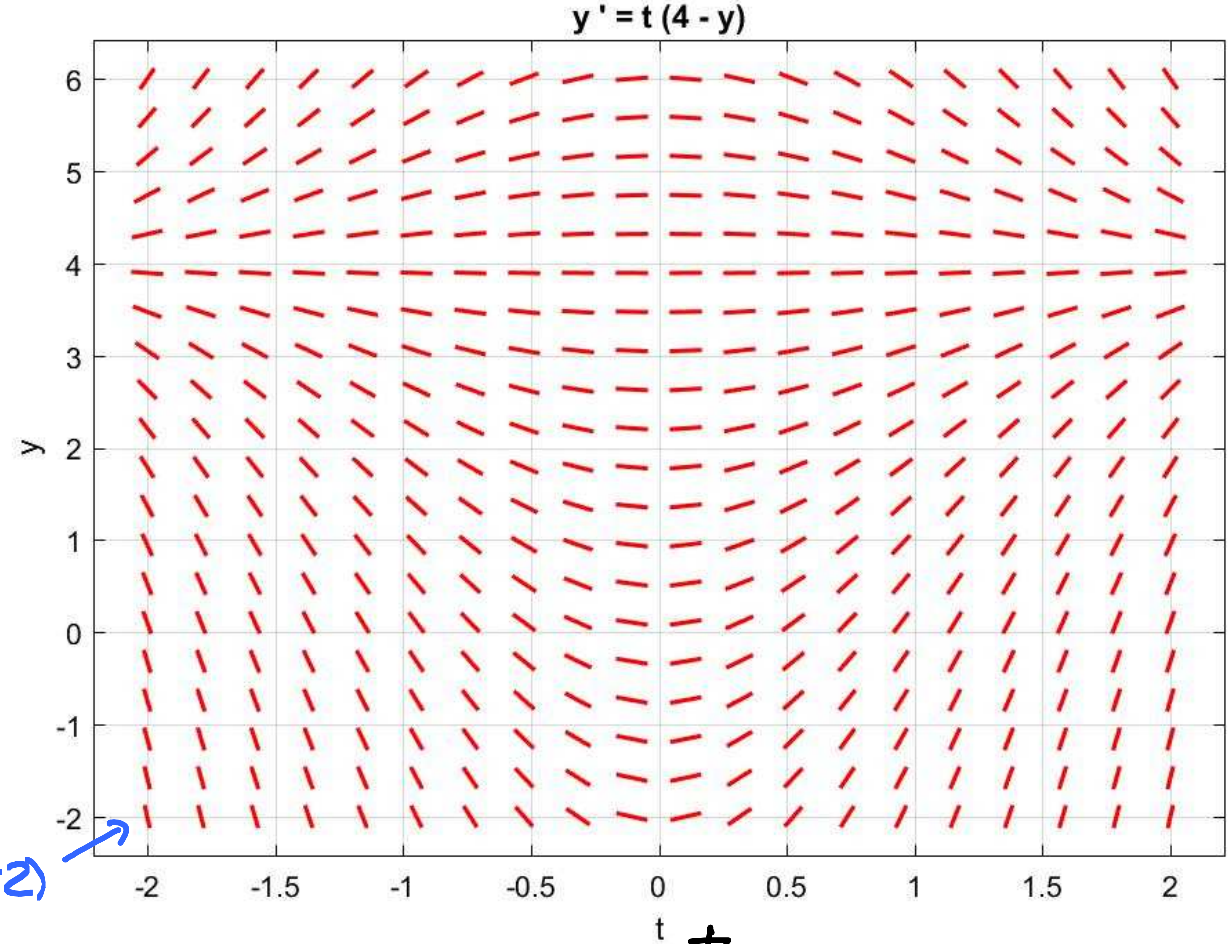
method for deriving a formula for the solution to one type of IVP.

Euler's Method

Method for writing a computer program for solving an IVP.

Direction Field $\dot{y} = t(4-y)$

Y



(-2,-2)

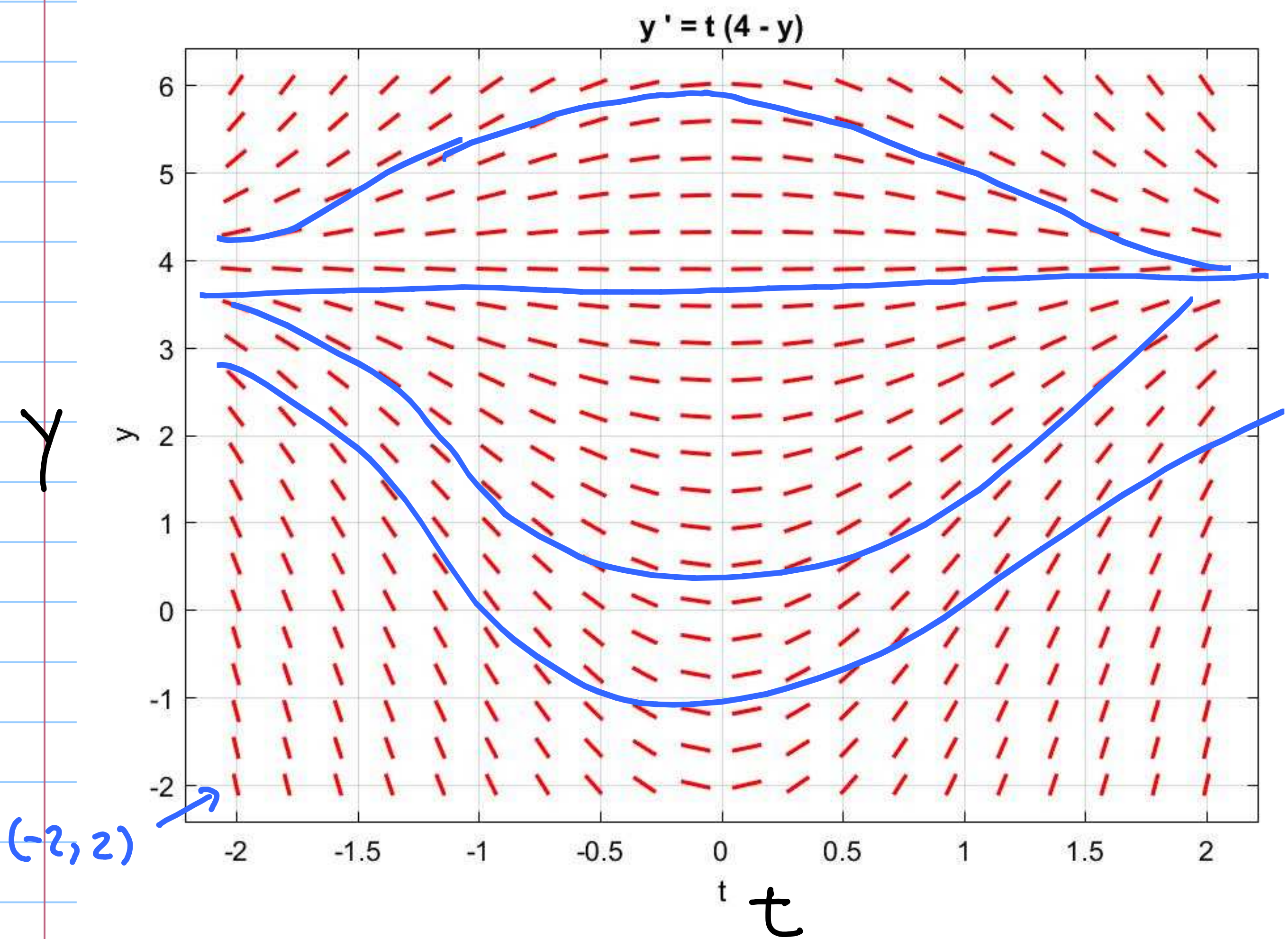
t

The red line at the point (t, y) has slope $t(4-y)$

Ans

- All the lines at $(t, 4)$ have slope ? 0
- All the lines at $(0, y)$ have slope ? 0
- The line at $(-2,-2)$ has slope $-2(4+2) = -12$

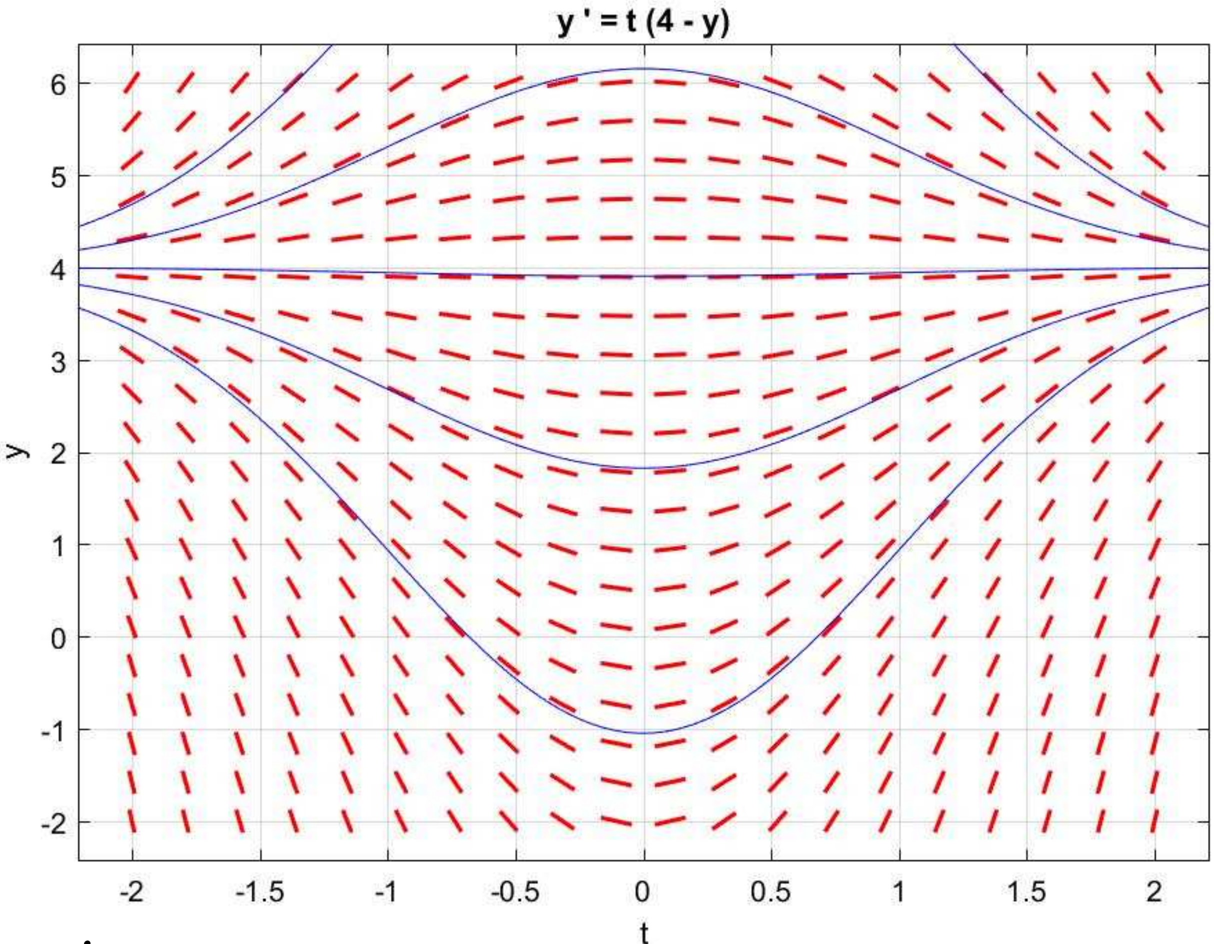
Direction Field $\dot{y} = t(4-y)$



The red line at the point (t, y) has slope $t(4-y)$

Curves that are tangent to the lines are solutions to the DE.

Draw Some Solutions



What important properties of solutions to the DE can we see from this picture?

Solutions that start below 4, ? stay below 4

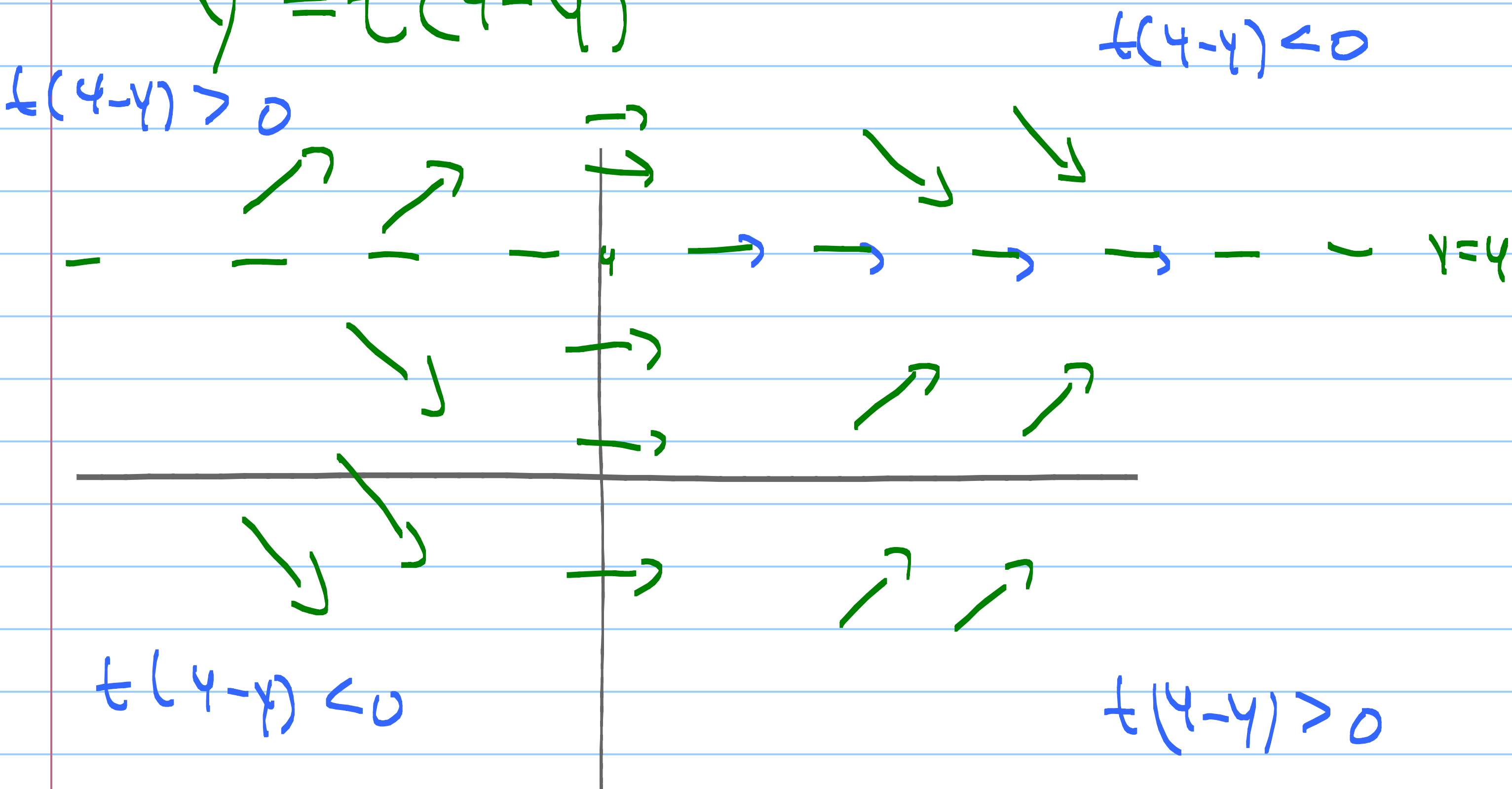
Solutions that start above 4, ? stay above 4

$\lim_{t \to \infty} y(t) = ?\, 4$ $\lim_{t \to -\infty} y(t) = ?\, 4$

The solution $y=4$ is a "stable equilibrium".

Draw DField by hand

$$\dot{y} = t(4-y)$$



Euler's Method

Note Title 10/1/2017

$\dot{y} = f(t, y)$

$y(0) = 0$

Example

$\dot{y} = t(4 - y)$

$y(0) = 0$

Problem Approximate y on the interval $[0, 2]$ using a step size of $h = \frac{1}{2}$.

Linear Approximation

For small h

$$y(t+h) \approx y(t) + h\,\dot{y}(t)$$

$$\approx y(t) + h\, f(t, y(t))$$

For $h = \frac{1}{2}$

$$y(0) = 0$$

$$y(0+\tfrac{1}{2}) \approx 0 + \tfrac{1}{2} f(0, y(0))$$

$$y(1) \approx y(\tfrac{1}{2}) + \tfrac{1}{2} f(\tfrac{1}{2}, y(\tfrac{1}{2}))$$

Example

$f(t, y) = t(4 - y)$

$y(t) + h\, t(4 - y)$

$y(0) = 0$

$y(\tfrac{1}{2}) \approx 0 + \tfrac{1}{2}(0 \cdot (4 - 0))$

$= 0$

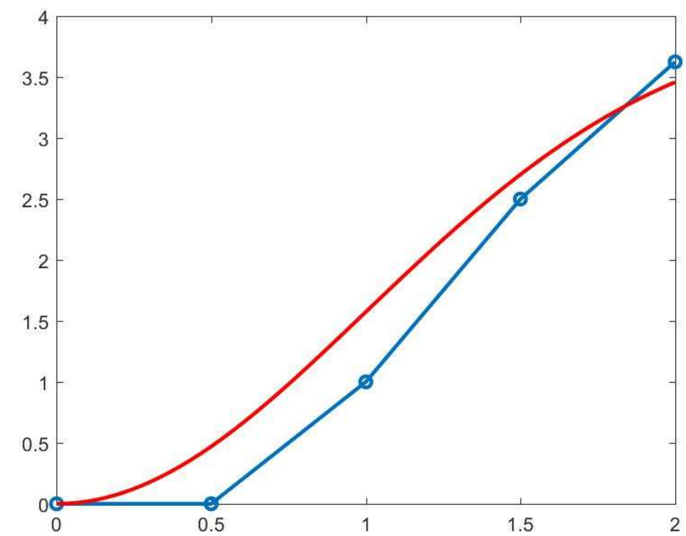
$y(1) \approx 0 + \tfrac{1}{2}(\tfrac{1}{2}(4 - 0))$

$= 1$

Problem Approximate y on the interval $[0,2]$ using a step size of $h=\frac{1}{2}$

$$\dot{y} = t(4-y)$$

t	0	$1/2$	1	$3/2$	2
y					
$f(t,y)$					



$$t_{New} = t_{old} + h$$

$$Y_{New} = Y_{old} + f(t_{old}, Y_{old})\, h$$

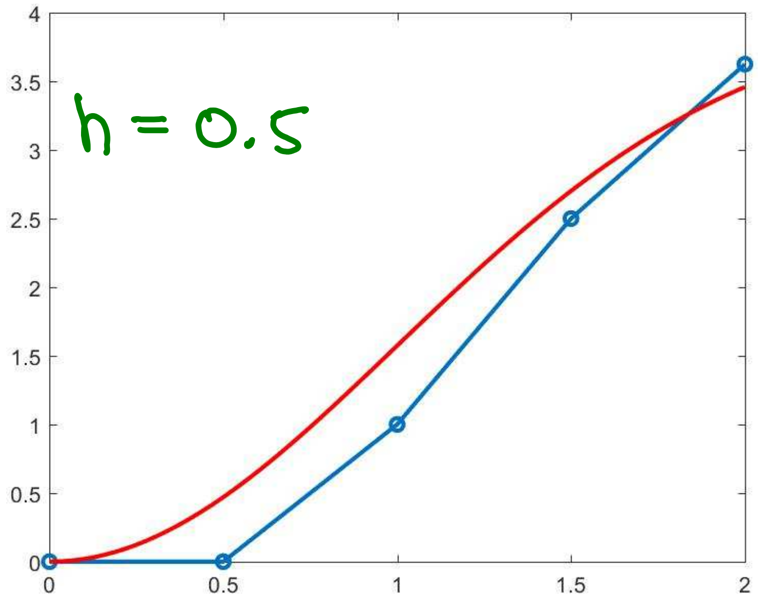
$$y(0) = 0$$

$$y\left(\tfrac{1}{2}\right) = 0 + \tfrac{1}{2}\left[0(4-0)\right] = 0$$

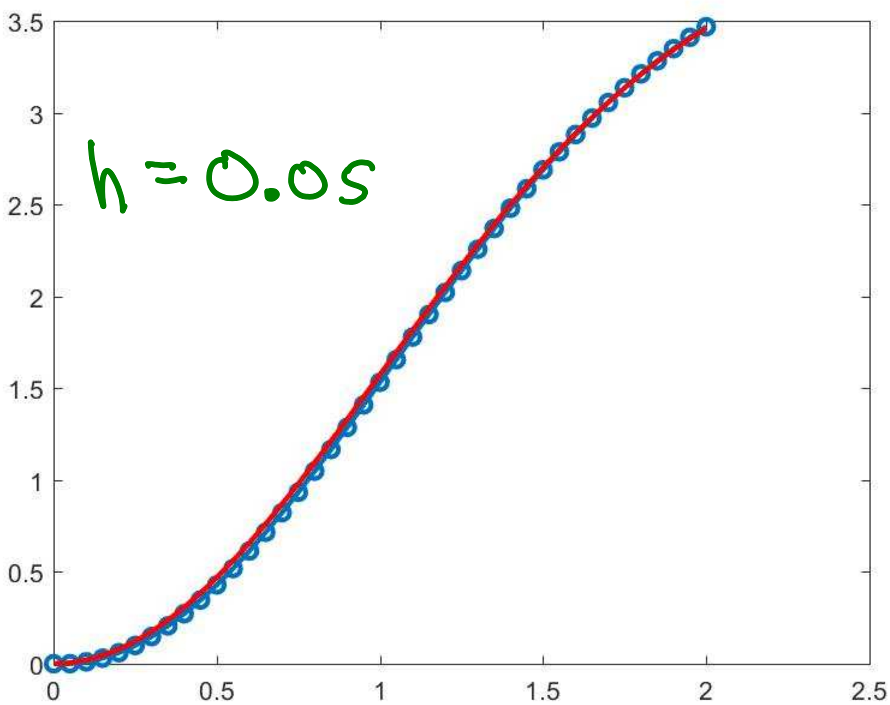
$$y(1) = 0 + \tfrac{1}{2}\left[\tfrac{1}{2}(4-0)\right] = 1$$

$$y\left(\tfrac{3}{2}\right) = 1 + \tfrac{1}{2}\left[1(4-1)\right] = \tfrac{5}{2}$$

$$y(2) = \tfrac{5}{2} + \tfrac{1}{2}\left[\tfrac{3}{2}\left(4-\tfrac{5}{2}\right)\right] = \tfrac{29}{8}$$



t	0	$1/2$	1	$3/2$	2	
y	0	0	1	$5/2$	$\frac{29}{8}$	
$f(t,y)$	0	2	3	$9/4$		



We usually choose h much smaller than in the example above and write a program to calculate y and $f(t,y)$

```matlab
%% Euler's method

% define the DE
f = @(t,y) t*(4-y);

%%
t(1) = 0; %initial time
y(1) = 0; % initial condition

% define h
h = 0.05
% how many steps
numsteps = round(2/h);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% This is Euler's method
for m = 1:numsteps
    t(m+1) = t(m)+h;
    y(m+1) = y(m) + h*f(t(m),y(m));
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% plot the results
figure(2)
plot(t,y,'marker','o')

t = linspace(0,2);
ytrue = 4.*(1-exp(-t.^2/2));
line(t,ytrue,'color','r')
```

$t_{new} = t_{old} + h$

$y_{new} = y_{old} + h f(t_{old}, y(t_{old}))$