The Dirac-delta function

Question Does the impulse response satisfy a DE?

or

$g(t)$ is the system response to what input?

Example

$\ddot{y} + 25y \Longleftrightarrow$ system (e.g. mass-spring)

$$G(s) = \frac{1}{s^2+25} = \text{transfer Function}$$

$$g(t) = \frac{\sin 5t}{5} = \text{impulse response}$$

$\ddot{g} + 25g = f(t) \qquad g(0) = 0 \quad \dot{g}(0) = 0$

What is $f(t)$?

Its easy to see what $F(s)$ is.

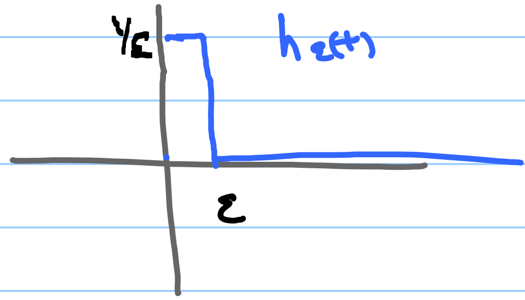
$$s^2 G(s) + 25 G(s) = F(s)$$

$$G(s) = \frac{F(s)}{s^2+25} \quad \text{so} \quad F(s) = 1$$

$\ddot{g} + 25g = f(t)$ where $f(t)$ is the function whose Laplace transform is 1

What function $f(t)$ has Laplace transform $= 1$?

A high, narrow "Top Hat Function" comes close



$$h_\varepsilon(t) = \begin{cases} 1/\varepsilon & t \le \varepsilon \\ 0 & \varepsilon < t \end{cases}$$

$$h_\varepsilon(t) = \frac{1 - u_\varepsilon(t)}{\varepsilon}$$

$$\mathcal{L}\{h_\varepsilon(t)\} = \frac{1}{\varepsilon}\left(\frac{1}{s} - \frac{e^{-\varepsilon s}}{s}\right)$$

$$\mathcal{L}\{h_\varepsilon(t)\} = H_\varepsilon(s) = \frac{1 - e^{-\varepsilon s}}{\varepsilon s}$$

High means $1/\varepsilon$ is big, and narrow means ε is small.

As $\varepsilon \to 0$, $H_\varepsilon(s) \longrightarrow 1$. We use L'Hopital's rule:

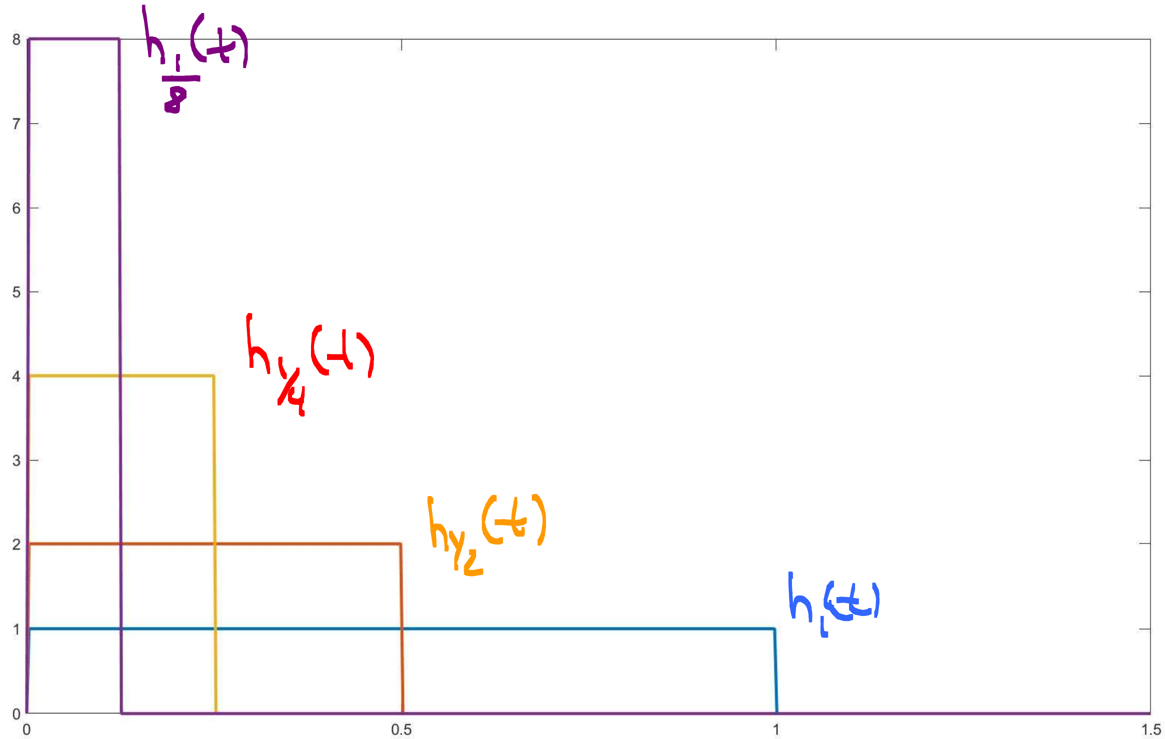
$$\lim_{\varepsilon \to 0^+} H_\varepsilon(s) = \text{"}\frac{0}{0}\text{"} = \frac{\frac{d}{d\varepsilon}\left(1 - e^{-\varepsilon s}\right)}{\frac{d}{d\varepsilon}\varepsilon s}$$

$$= \lim_{\varepsilon \to 0^+} \frac{\cancel{s}\, e^{-\varepsilon s}}{\cancel{s}}$$

$$= 1$$

$$\lim_{\varepsilon \to 0^+} H_\varepsilon(s) = 1$$

What about $\lim_{\varepsilon \to 0^+} h_\varepsilon(t)$?



$$\lim_{\varepsilon \to 0^+} h_\varepsilon(t) = \delta_0(t)$$

$$\int_0^\infty h_\varepsilon(t)\,dt = \int_0^\varepsilon \frac{1}{\varepsilon}\,dt = 1$$

so $\displaystyle\int_0^\infty \delta_0(t)\,dt = 1$

Also $\lim_{\varepsilon \to 0^+} h_\varepsilon(t_0) = 0$ for every $t_0 > 0$!!!

$\delta_0(t)$ is not a "real" function

$$\delta_0(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \int_0^\infty \delta_0(t)\,dt = 1 \end{cases}$$

$$\delta_c(t) = \lim_{\varepsilon \to 0^+} h_\varepsilon(t - c)$$

$$\delta_c(t) = \delta_0(t - c)$$

$$\mathcal{L}\{\delta_c(t)\} = e^{-cs}\,\mathcal{L}\{\delta_0(t)\}$$

$$\mathcal{L}\{\delta_c(t)\} = e^{-cs}$$

The delta function at c, is the "generalized function" whose Laplace transform $= e^{-cs}$. The impulse response $g(t)$ satisfies:

$$m\ddot{g} + \gamma\dot{g} + kg = \delta_0(t)$$

You can stop here!! I am including some info below but won't test you on it.

The (Dirac) delta function was introduced by Paul Dirac to model a point mass

https://en.wikipedia.org/wiki/Dirac_delta_function

It was "anticipated" by Fourier in conjunction with the Fourier transform which is similar to the Laplace Transform

Interpretation of the delta

Physical - A force that is applied during a time interval so short that that your equipment can't resolve it, but delivers a unit impulse*, i.e. $\int \delta_c(t)\,dt = 1$

* Impulse here has its correct physical meaning. Impulse is a change in momentum, and calculated by integrating force with respect to time.

Mathematical Definition of the delta

Fact - Suppose two functions f_1 and f_2 satisfy $\int f_1(t)\, g(t)\, dt = \int f_2(t)\, g(t)\, dt$ for every continuous function $g(t)$. Then $f_1(t) = f_2(t)$.

So we can identify a "generalized" function with how it integrates against conventional continuous functions.

The delta function is defined by

$$\int \delta_a(t)\, g(t)\, dt = g(a)$$

for every continuous function $g(t)$

It can also be defined as the "derivative" of the Heaviside function.

$$\frac{d}{dt} u_a(t) = \delta_a(t)$$