Transfer function, Impulse Response, and Convolution

$$m\ddot{y} + \gamma\dot{y} + ky = f(t)$$

$$y(0) = 0 \qquad \dot{y}(0) = 0$$

This choice of initial conditions makes the calculation simpler, and doesn't change the steady state solution

$$(ms^2 + \gamma s + k)Y(s) = F(s)$$

$$Y(s) = \frac{1}{ms^2 + \gamma s + k} \cdot F(s)$$

The Laplace transform of the solution $Y(s)$ is the Laplace transform of the forcing function $F(s)$ times $\frac{1}{ms^2+\gamma s+k}$ (this part comes from the DE)

$$Y(s) = G(s)F(s) \qquad G(s) = \frac{1}{ms^2 + \gamma s + k}$$

In control theory, signal processing, and engineering

$F(s)$ is called the input

$Y(s)$ is called the output or system response

$G(s)$ is called the transfer function

$g(t) = \mathcal{L}^{-1}\{G(s)\}$ is called the impulse response

Convolution Theorem

The solution to $m\ddot{y} + \gamma\dot{y} + ky = f(t)$

$y(0) = 0 \quad \dot{y}(0) = 0$

is $y(t) = \int_0^t g(t-\tau) f(\tau) d\tau$

where $g(t) = \mathcal{L}^{-1}\left\{\frac{1}{ms^2+\gamma s+k}\right\}$ is the impulse response

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$\int_0^t g(t-\tau) f(\tau) d\tau$ is called the convolution

of g and f and denoted by $g * f$

i.e. $g*f = \int_0^t g(t-s) f(s) ds$

<u>Example</u> $\ddot{y} + 25y = \cos t$

$y(0) = 0 \quad \dot{y}(0) = 0$

$$G = 1/(s^2+25)$$

$$g = \mathcal{L}^{-1}\{G\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\} = \frac{\sin 5t}{5}$$

$$y(t) = \int_0^t \frac{\sin 5(t-\tau)}{5} \cos\tau \, d\tau \quad *$$

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* Lets not worry about calculating the integral for now

Calculating $\frac{1}{5}\int_0^t \sin(5t-5\tau)\, e^{-\tau}\, d\tau$ *

① Let $s = 5t-5\tau$ so $\tau = t - \frac{s}{5}$

$$= \frac{1}{25}\int_{5t}^{0} \sin s\; e^{-(t-\frac{s}{5})}\left(\frac{-ds}{5}\right)$$

$$= \frac{e^{-t}}{25}\int_0^{5t} \sin s\; e^{s/5}\, ds$$

Calculating this integral

$$A \sin s\, e^{s/5} + B\cos s\, e^{s/5}$$

$$\Big\downarrow \frac{d}{ds}$$

$$\left(\frac{A}{5} - B\right)\sin s\, e^{s/5} + \left(A + \frac{B}{5}\right)\cos s\, e^{s/5}$$

$$\left.\begin{aligned} \frac{A}{5} - B &= 1 \\ A + \frac{B}{5} &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= \frac{5}{26} \\ B &= \frac{-25}{26} \end{aligned}$$

$$= \frac{e^{-t}}{25}\left[\frac{5}{26}\sin s\, e^{s/5} - \frac{25}{26}\cos s\, e^{s/5}\right]_0^{5t}$$

$$= e^{-t}\left[\frac{1}{130}\sin 5t\, e^{t} - \frac{1}{26} e \cos 5t\, e^{t} + \frac{1}{26}\right]$$

$$= \frac{1}{130}\sin 5t - \frac{1}{26}\cos 5t + \frac{1}{26}e^{-t}$$

Comment: It is much easier to solve

$$\ddot{y} + 25y = e^{-t}$$

using the method of undetermined coefficients. set $y = Ae^{-t}$, then $Ae^{-t} + 25Ae^{-t} = e^{-t}$ gives $A = \frac{1}{26}$

The convolution formula is valuable because we can use it, and Riemann sums, to calculate $y(t)$ when the forcing function is measured data. It is an important theoretical tool as well.

* I want you to focus on writing the convolution integral, rather than calculating the integral. The example above is there to show you that you can do the calculation.

Question

① what is the impulse responce of the system

$$\ddot{y} + 4\dot{y} + 5y$$

② Use the impulse response to write the solution to

$$\ddot{y} + 4\dot{y} + 5y = f(t)$$

$$y(0) = 0 \quad \dot{y}(0) = 0$$

Answer

The tranfer function is $G(s) = \dfrac{1}{s^2 + 4s + 5}$

$$= \frac{1}{(s+2)^2 + 1}$$

so the impulse response is

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{-2t}\sin t$$

②

$$y(t) = \int_0^t e^{-2(t-\tau)} \sin(t-\tau)\, f(\tau)\, d\tau$$

Remark - If $f(t)$ is a measured input, rather than a formula, ② can be evaluated numerically. It is much easier to use than Euler's method.

If $f(t)$ is given by a formula, then ② gives the same answer as the other methods we know.

Two Homework Problems on Convolution

Express the solution as a convolution integral:

① $y'' + \omega^2 y = 2e^{-t}\cos 2t$

$y(0) = 0 \quad y'(0) = 0$

② $y'' + 2y' + 2y = \sin t$

$y(0) = 0 \quad y'(0) = 0$

Answers

① $y(t) = \int_0^t \frac{\sin \omega(t-\tau)}{\omega} e^{-\tau} \cos 2\tau \, d\tau$

② $y(t) = \int_0^t e^{-(t-\tau)} \sin(t-\tau) \sin\tau \, d\tau$

Convolution and Laplace Transform

Thm $\mathcal{L}^{-1}\{G(s)F(s)\} = \int_0^t g(t-\tau)f(\tau)\,d\tau$

equivalently, [because Laplace transform is invertible]

$$\mathcal{L}\left\{\int_0^t g(t-\tau)f(\tau)\,d\tau\right\} = F(s)\,G(s)$$

Proof

$$F(s)\,G(s) = \int_0^\infty e^{-st}f(t)\,dt \int_0^\infty e^{-s\tau}g(\tau)\,d\tau$$

$$= \int_0^\infty \int_0^\infty e^{-s(t+\tau)} f(t)\,g(\tau)\,dt\,d\tau$$

Change Variabl in dt integral

$\omega = t+\tau$

$d\omega = dt$

$$= \int_0^\infty \int_\tau^\infty e^{-s\omega} f(\omega-\tau)\,g(\tau)\,d\omega\,d\tau$$

Change the order of integration

$$= \int_0^\infty \int_0^\omega e^{-s\omega} f(\omega-\tau)\,g(\tau)\,d\tau\,d\omega$$

$$= \int_0^\infty e^{-s\omega}\left[\int_0^\omega f(\omega-\tau)\,g(\tau)\,d\tau\right] d\omega$$

$$= \mathcal{L}\left\{\int_0^\omega f(\omega-\tau)\,g(\tau)\,d\tau\right\}$$

Convolution Theorem

The solution to

$$m\ddot{y} + \gamma\dot{y} + ky = f(t)$$

$$y(0) = 0 \quad \dot{y}(0) = 0$$

is $$y(t) = \int_0^t g(t-\tau) f(\tau) d\tau$$

Proof

$$ms^2 Y + \gamma s Y + kY = F(s)$$

$$(ms^2 + \gamma s + k)Y = F(s)$$

$$Y(s) = \frac{1}{ms^2 + \gamma s + k} F(s)$$

$$Y(s) = G(s) F(s)$$

So $$y(t) = \int_0^t g(t-\tau) f(\tau) d\tau$$